5 SEQUENCES AND SERIES

Mixed practice 5

1. $u_4 = 17 \Rightarrow u_1 + 3d = 17 \quad \cdots (1)$

$S_{20} = 990 \Rightarrow \frac{20}{2}[2u_1 + 19d] = 990$

$\Rightarrow 2u_1 + 19d = 99 \quad \cdots (2)$

From (1), $u_1 = 17 - 3d$; substituting this into (2) gives

$2(17 - 3d) + 19d = 99$

$\Leftrightarrow 34 + 13d = 99$

$\Leftrightarrow d = 5$

So $u_1 = 17 - 3d = 17 - 3 \times 5 = 2$.

2. The fourth, tenth and thirteenth terms of the geometric sequence are:

$u_4 = u_1 r^3$

$u_{10} = u_1 r^9$

$u_{13} = u_1 r^{12}$

As these form an arithmetic sequence:

$u_{10} - u_4 = u_{13} - u_{10}$

$\Rightarrow u_1 r^9 - u_1 r^3 = u_1 r^{12} - u_1 r^9$

$\Rightarrow r^{12} - 2r^9 + r^3 = 0 \quad (\text{as } u_1 \neq 0)$

$\Rightarrow r^9 - 2r^6 + 1 = 0 \quad (\text{as } r \neq 0)$

$\Rightarrow r = 1,\ 1.17,\ -0.852 \quad$ (from GDC to 3 SF)

$\therefore\ r = -0.852 \quad$ (for sum to infinity to exist, $|r| < 1$)

3. $\sum_{r=1}^{12} 4r + \left(\frac{1}{3}\right)^r = 4\sum_{r=1}^{12} r + \sum_{r=1}^{12}\left(\frac{1}{3}\right)^r$

The first sum is an arithmetic series, and the second sum is a geometric series. So, using the formulae:

$$4\left[\frac{12}{2}(2\times 1 + (12-1)\times 1)\right] + \frac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^{12}\right]}{1-\frac{1}{3}} = 312.5 \text{ (4 SF)}$$

4. For 20 terms of this series, i.e. with $n = 20$:

$\ln x + \ln x^4 + \ln x^7 + \ln x^{10} + \ldots = \ln x + 4\ln x + 7\ln x + 10\ln x + \ldots$

$= \ln x(1 + 4 + 7 + 10 + \ldots)$

$= \ln x\left(\frac{20}{2}(2 + 19\times 3)\right)$

$= 590 \ln x$

$= \ln x^{590}$

5. We know that the total length of the pieces is 300, i.e. $S_n = 300$:

$S_n = \frac{n}{2}(u_1 + u_n)$

$300 = \frac{n}{2}(1 + 19)$

$\Leftrightarrow 300 = 10n$

$\Leftrightarrow n = 30$

So $u_{30} = 19$:

$u_n = u_1 + (n-1)d$

$19 = 1 + 29d$

$\Leftrightarrow d = \frac{18}{29}$ metres

6. The amount Aaron has in his account for the first few months is:

1st: 100
2nd: 100 + 110
3rd: 100 + 110 + 120

His monthly balance forms an arithmetic series with $a = 100$ and $d = 10$. So after n months he will have:

$S_n = \frac{n}{2}[2\times 100 + (n-1)10]$

$= \frac{n}{2}[200 + 10n - 10]$

$= \frac{n}{2}[190 + 10n]$

$= 5n^2 + 95n$

The amount Blake has in his account for the first few months is:

1st: 50
2nd: 50×1.05
3rd: 50×1.05^2

His monthly balance forms a geometric series with $a = 50$ and $r = 1.05$. So after n months he will have:

$S_n = \frac{50(1.05^n - 1)}{1.05 - 1}$

$= 1000(1.05^n - 1)$

Therefore, Blake will have more in his account than Aaron does when:

$1000(1.05^n - 1) > 5n^2 + 95n$

$\therefore\ n = 73$ months (from GDC)

Going for the top 5

1. (a) (i) Writing out the sum from the first term, a, to the nth term, ar^{n-1}:

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \quad (1)$

Multiplying through by r:

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \quad (2)$

We can see that (1) and (2) have many terms in common:

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \quad (1)$

$rS_n = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \quad (2)$

So (2) − (1) gives:

$rS_n - S_n = ar^n - a$

$\Rightarrow S_n(r-1) = a(r^n - 1)$

$\Rightarrow S_n = \dfrac{a(r^n-1)}{r-1}$

(ii) If $-1 < r < 1$ (or equivalently $|r| < 1$), then as $n \to \infty$, $r^n \to 0$. So

$S_n \to \dfrac{a(0-1)}{r-1} = \dfrac{-a}{r-1} = \dfrac{a}{1-r}$

If $|r| \geq 1$, then r^n has no limit as $n \to \infty$, and so there is no finite limit for S_n.

Thus, $S_\infty = \dfrac{a}{1-r}$ only for $|r| < 1$.

(b) The sum of the first n terms is $S_n = \dfrac{u_1(1-r^n)}{1-r}$.

The sum of the next n terms is given by

$$S_{2n} - S_n = \frac{u_1(1-r^{2n})}{1-r} - \frac{u_1(1-r^n)}{1-r}$$
$$= \frac{u_1(1-r^{2n}-1+r^n)}{1-r}$$
$$= \frac{u_1(r^n - r^{2n})}{1-r}$$
$$= \frac{u_1 r^n(1-r^n)}{1-r}$$

So the ratio of the sum of the first n terms to the sum of the next n terms is

$$\frac{u_1(1-r^n)}{1-r} : \frac{u_1 r^n(1-r^n)}{1-r} = 1 : r^n$$

(c) (i) $u_7 + 4u_5 = u_8$

i.e. $u_1 r^6 + 4u_1 r^4 = u_1 r^7$

$\Rightarrow r^2 + 4 = r^3$ (since $u_1, r \neq 0$)

$\Rightarrow r^3 - r^2 - 4 = 0$

To factorise this, substitute small positive and negative integers into $f(r) = r^3 - r^2 - 4$ until you find an r such that $f(r) = 0$:

$f(1) = 1^3 - 1^2 - 4 = -4$

$f(-1) = (-1)^3 - (-1)^2 - 4 = -6$

$f(2) = 2^3 - 2^2 - 4 = 0$

Therefore, by the factor theorem, $(x-2)$ is a factor and so by long division or equating coefficients we get:

$r^3 - r^2 - 4 = (r-2)(r^2 + r + 2)$

For the quadratic $r^2 + r + 2$,

$\Delta = 1^2 - 4 \times 1 \times 2 = -7 < 0$

Hence there are no real roots for this quadratic factor, and so the only root of $f(r)$ is $r = 2$.

Then, from part (b), the ratio is

$1 : r^{10} = 1 : 2^{10} = 1 : 1024$

(ii) No, because $r > 1$ in this case, and as shown in part (a)(ii), the condition for a sum to infinity to exist is that $|r| < 1$.

2. The integers from 1 to 1000 form an arithmetic sequence with $u_1 = 1$, $u_n = 1000$ and $n = 1000$. So

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$= \frac{1000}{2}(1 + 1000)$$
$$= 500\,500$$

The multiples of 7 between 1 and 1000 form an arithmetic sequence with $u_1 = 7$, $u_n = 994$ and $n = \dfrac{994}{7} = 142$. So

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$= \frac{142}{2}(7 + 994)$$
$$= 71\,071$$

Therefore the sum of the integers between 1 and 1000 that are not divisible by 7 is

$500\,500 - 71\,071 = 429\,429$.

6 TRIGONOMETRY

Mixed practice 6

1. $A = \dfrac{1}{2}ab\sin C$

$12 = \dfrac{1}{2}x(x+2)\sin 30°$

$\Rightarrow 12 = \dfrac{1}{2}x(x+2)\dfrac{1}{2}$

$\Rightarrow 48 = x^2 + 2x$

$\Rightarrow x^2 + 2x - 48 = 0$

$\Rightarrow (x+8)(x-6) = 0$

$\therefore x = 6$ (as $x > 0$)

2. By the sine rule:

$\dfrac{\sin\theta}{5} = \dfrac{\sin 2\theta}{7}$

$\Rightarrow 7\sin\theta = 5\sin 2\theta$

$\Rightarrow 7\sin\theta = 5(2\sin\theta\cos\theta)$

$\Rightarrow 7\sin\theta - 10\sin\theta\cos\theta = 0$

$\Rightarrow \sin\theta(7 - 10\cos\theta) = 0$