In this chapter you will learn:

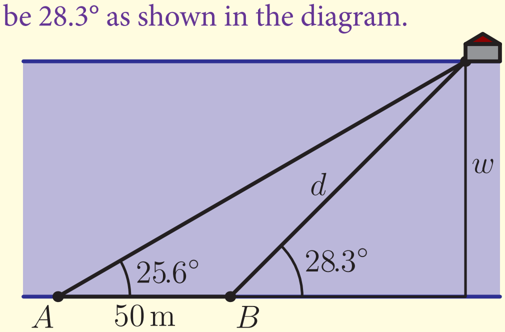
- a reminder of the use of trigonometry in right angled triangles
- how to use the sine rule to find sides and angles of any triangle
- how to use the cosine rule to find sides and angles of any triangle
- an alternative formula for the area of a triangle
- to use techniques to solve problems in two and three dimensions
- how to calculate the length of an arc of a circle
- how to calculate the area of a sector of a circle
- to apply trigonometry to solving problems involving circles and triangles.

The first steps in developing trigonometry were made by Babylonian astronomers as early as the 2nd millennium BCE. It is thought that Egyptians used trigonometric calculations when building the pyramids. These were further developed by Greek and Indian mathematicians. Some of the most significant contributions to trigonometry were made by Islamic mathematicians in the second half of the 1st millennium BCE.

11 Geometry of triangles and circles

Introductory problem

Two observers are trying to measure the width of a river. There is no bridge across the river, but they have instruments for measuring lengths and angles. They stand at A and B, 50 m apart, on the opposite side of the river to a tower. The person at A measures the angle between the line (AB) and the line from A to the tower as 25.6°. The observer at B similarly measures the corresponding angle to be 28.3° as shown in the diagram.



Can they use this information to calculate the width of the river?

The problem above involves finding lengths and angles in triangles. Such problems can be solved using trigonometric functions. In fact, trigonometry was first used to solve similar problems in land measurement, building and astronomy. The word *trigonometry* means 'measuring triangles'.

In this chapter we will use what we already know about trigonometric functions and also develop some new results to enable us to calculate lengths and angles in triangles.

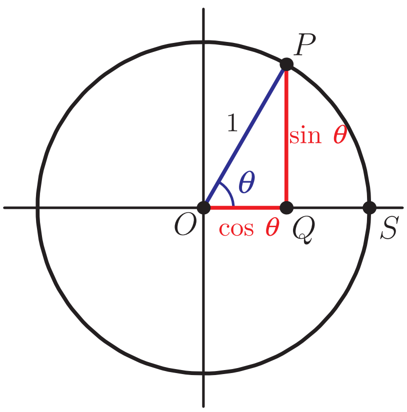
11A Right-angled triangles

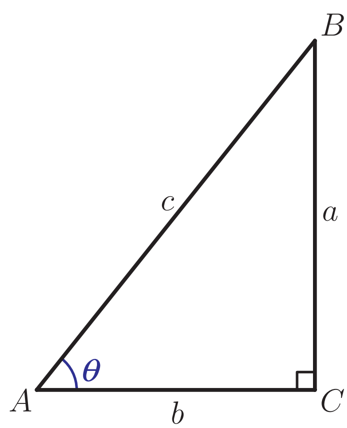
In previous studies, you may have already seen definitions of sine, cosine and tangent functions in terms of the sides of a right-angled triangle. In this section we will briefly discuss how they are related to the definitions we introduced in the previous chapter.

Remember the definition of sine and cosine functions using the unit circle. Let $0° < \theta < 90°$, and let P be the point on the unit circle such that SOP = θ. Then $PQ = \sin\theta$ and $OQ = \cos\theta$.

Trigonometric functions were defined in Section 9B.

Consider now a right-angled triangle ABC with right angle at C and $B\hat{A}C = \theta$.





Triangles OPQ and ABC have the same angles, so they are similar. Therefore:

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = \frac{c}{1}$$

From this we find that the ratios of sides in a right-angled triangle are trigonometric functions of angle θ.

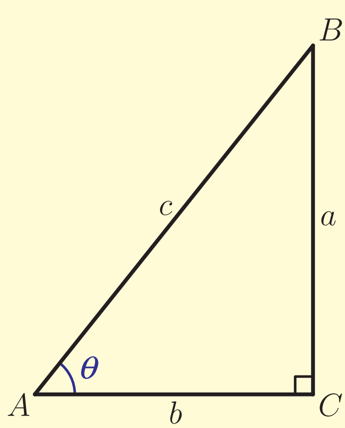
KEY POINT 11.1

In a right-angled triangle:

$$\frac{a}{c} = \sin\theta$$

$$\frac{b}{c} = \cos\theta$$

$$\frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

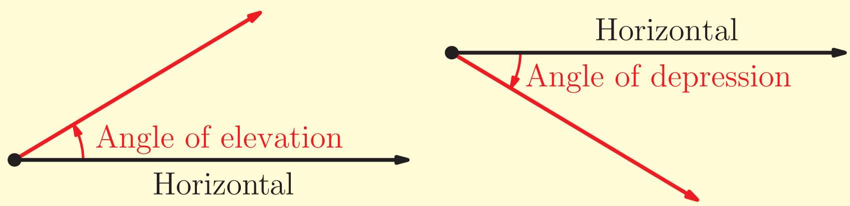


See Prior learning Section U on the CD-ROM for practice questions on right-angled triangles.

These equations only apply to right-angled triangles, so the angles are always acute. In the remainder of this chapter we shall deal with triangles which can also have obtuse angles. In those cases we need to use our definitions of trigonometric functions from the unit circle.

Two terms which you will see frequently in the context of trigonometry are **angle of elevation** and **angle of depression**.

KEY POINT 11.2



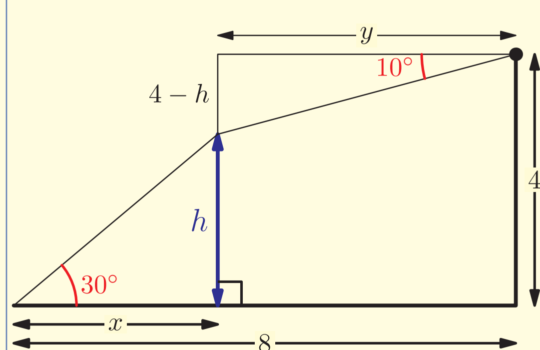
The angle of elevation is the angle above the horizontal.

The angle of depression is the angle below the horizontal.

Worked example 11.1

Daniel and Theo are trying to find the height of a birds' nest in their garden. From Theo's bedroom window, which is 4 m above the ground, the angle of depression of the nest is 10°. From the end of the flat garden, 8 m away from the house, the angle of elevation is 30°. Find the height of the nest.

Sketch a diagram



EXAM HINT

If a diagram is not given it is always a good use of time to sketch one.

Apply trigonometry to the right-angled triangles

$$y = \frac{4-h}{\tan 10°}$$

$$x = \frac{h}{\tan 30°}$$

continued . . .

From the diagram we can find an equation linking x and y

$x + y = 8$

$\therefore \frac{4-h}{\tan 10°} + \frac{h}{\tan 30°} = 8$

Evaluate tan 10 and tan 30 on the calculator

$\Rightarrow \frac{4-h}{0.176} + \frac{h}{0.577} = 8$

$\Rightarrow 22.69 - 5.67h + 1.73h = 8$

$\Rightarrow 14.69 = 3.94h$

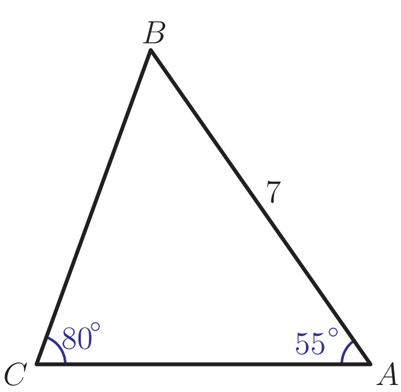
$\Rightarrow h = 3.73\text{m (3SF)}$

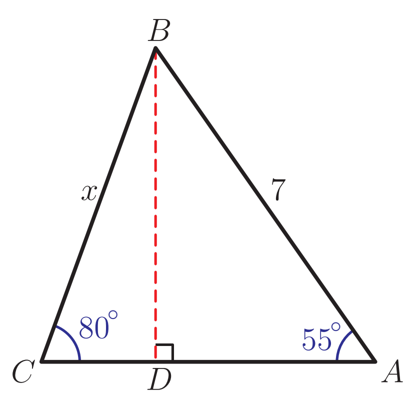
EXAM HINT

In a question with several steps like this, it is important not to round intermediate values. Use the answer button on your calculator and round at the end. However, you do not have to write down all the digits on your calculator display.

11B The sine rule

Can we use trigonometry to calculate sides and angles in triangles that do not have a right angle? The answer is yes, and one way to do this is called the **sine rule**. In the triangle shown on the left below, $AB = 7$, $B\hat{A}C = 55°$ and $A\hat{C}B = 80°$. Can we find the length of BC?





There are no right angles in the diagram, but we can create some by drawing the line $[BD]$ perpendicular to $[AC]$ as shown in the second diagram.

We now have two right-angled triangles. In triangle ABD,

$\frac{BD}{7} = \sin 55°$, so $BD = 7\sin 55°$.

In triangle BCD, $\frac{BD}{x} = \sin 80°$, so $BD = x\sin 80°$.

Comparing the two expressions for BD, we get:

$$x\sin 80° = 7\sin 55° \qquad (*)$$

And rearranging gives:

$$x = \frac{7\sin 55°}{\sin 80°} = 5.82$$

Notice that we did not actually need to calculate the length of BD but can go straight to equation marked (*). This equation is more commonly written as:

$$\frac{x}{\sin 55°} = \frac{7}{\sin 80°}$$

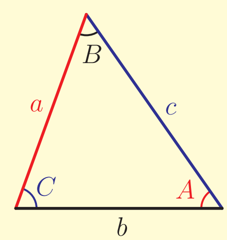
This last equation is an example of the sine rule. Notice that the length of each side is divided by the sine of the angle *opposite* that side. We can repeat the same process to obtain a general equation.

You may think that the sum of angles in a triangle being 180° is an absolute fact, but it is only the case on a flat plane. If you draw a triangle on a sphere, the sum of the angles will be greater than 180°. The study of these spherical triangles is vital to navigation and astronomy. There are even analogues of the cosine and sine rules for these triangles.

KEY POINT 11.3

The sine rule

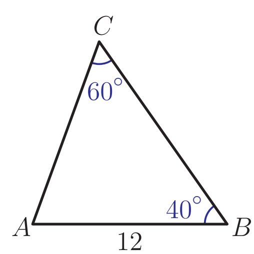
$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$



To use the sine rule you need to have both an angle and its opposite side. When using the sine rule you will normally use only two of the three ratios. To decide which ones, you need to look at what information is given in the question.

Worked example 11.2

Find the length of side AC.



continued . . .

We are given the angles opposite sides AB and AC, so use the sine rule with those two sides

$$\frac{12}{\sin 60°} = \frac{AC}{\sin 40°}$$

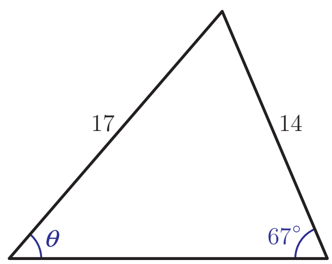
$$\Rightarrow AC = \frac{12\sin 40°}{\sin 60°}$$

$$= 8.91 \text{ (3SF)}$$

We can also use the sine rule to find angles.

Worked example 11.3

Find the size of the angle marked θ.



Use the sine rule with the two given sides, as we have one of the opposite angles and want to find the other one

$$\frac{17}{\sin 67°} = \frac{14}{\sin\theta}$$

$$\Rightarrow \sin\theta = \frac{14\sin 67°}{17} = 0.758$$

$$\therefore \theta = \arcsin 0.758 = 49.3°$$

You should remember from your work on trigonometric equations that there is more than one value of θ with $\sin\theta = 0.758$. So does that mean that the last question has more than one possible answer?

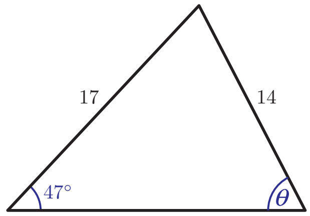
See Section 10A for solving trigonometric equations.

Another solution of the equation $\sin\theta = 0.758$ is $180 - 49.3 = 130.7°$. However, as one of the other angles is 67°, this value is impossible, because all three angles of the triangle must add up to 180°, and $130.7 + 67 = 190.7 > 180$. All other possible values of θ are outside the interval [0°, 180°], so cannot be angles of a triangle. In this example, there is only one possible value of angle θ.

The next example shows that this is not always the case.

Worked example 11.4

Find the size of the angle marked θ, giving your answer to the nearest degree.



Use the sine rule with the two given sides

$$\frac{17}{\sin\theta} = \frac{14}{\sin 47°}$$

$$\Rightarrow \sin\theta = \frac{17\sin 47°}{14} = 0.888$$

Find the two possible values of θ

$\arcsin 0.888 = 62.6°$

$\Rightarrow \theta = 62.6°$ or $180 - 62.6 = 117.4°$

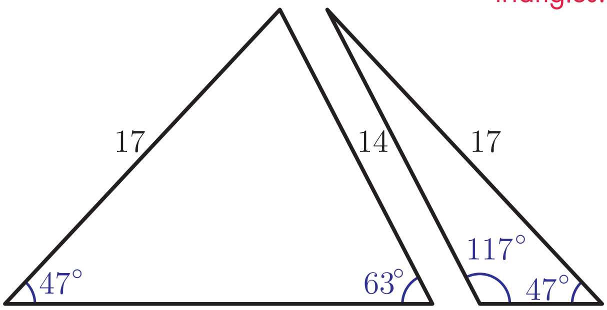
Check whether both solutions are possible: do the two angles add up to less than 180°?

$62.6 + 47 = 109.6 < 180$

$117.4 + 47 = 164.4 < 180$

Both solutions are possible.
The diagram below shows the two possible triangles.

$\therefore \theta = 63°$ or $117°$



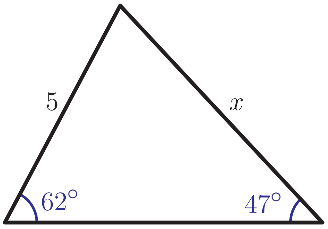
EXAM HINT

In an examination, a question will often alert you to look for two possible answers. However, if it doesn't, you should check whether the second solution is possible by finding the sum of the known angles.

Exercise 11B

1. Find the lengths of sides marked with letters.

(a) (i)



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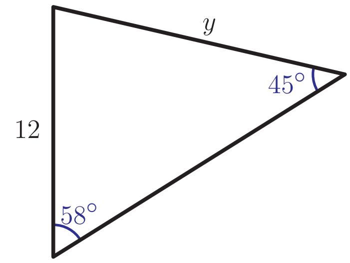
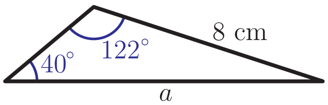


(b) (i)



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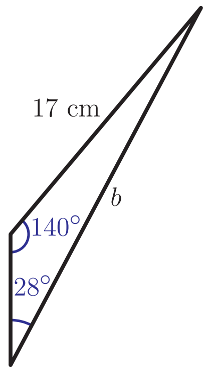
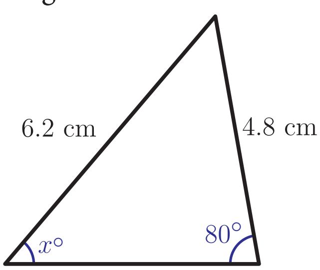


2. Find the angles marked with letters, checking whether there is more than one solution.

(a) (i)



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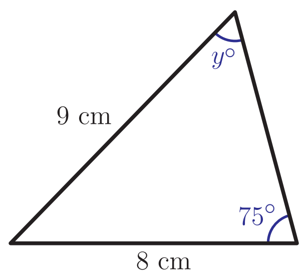
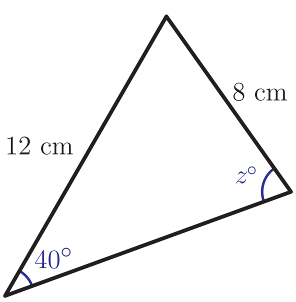


(b) (i)



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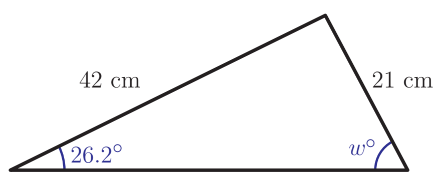
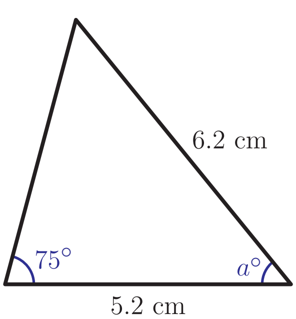
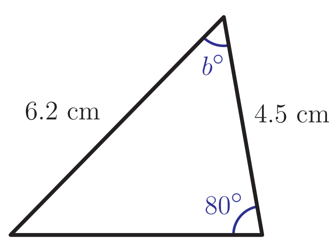


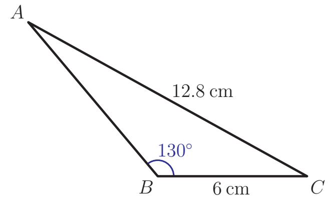
(c) (i)



(ii)

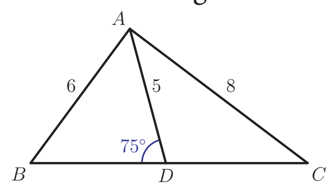


3. Find all the unknown sides and angles of triangle ABC.



4. In triangle ABC, $AB = 6\,\text{cm}$, $BC = 8\,\text{cm}$, $A\hat{C}B = 35°$. Show that there are two possible triangles with these measurements and find the remaining side and angles for each. *[4 marks]*

5. In the triangle shown in the diagram, $AB = 6$, $AC = 8$, $AD = 5$ and $A\hat{D}B = 75°$. Find the length of the side BC.



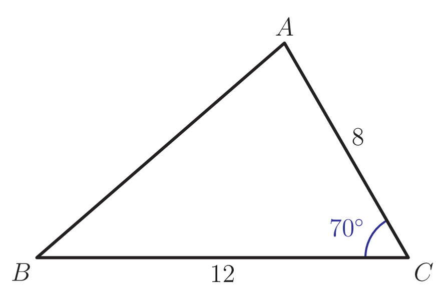
[5 marks]

6. Show that it is impossible to draw a triangle ABC with $AB = 12\,\text{cm}$, $AC = 8\,\text{cm}$ and $A\hat{B}C = 47°$. *[5 marks]*

11C The cosine rule

Not all information about triangles can be found using the sine rule. In particular, if we have two sides and the angle between them or all three sides we can find further information using a new rule called the **cosine rule**.

Can we find the length of the side AB in the triangle shown?



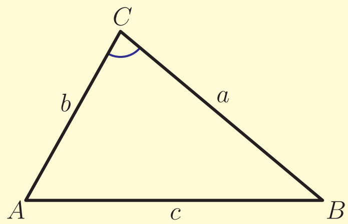
The sine rule for this triangle gives $\frac{AB}{\sin 70°} = \frac{8}{\sin \hat{B}} = \frac{12}{\sin \hat{C}}$. But we do not know either of the angles B or C, so it is impossible to find AB from this equation. We need a different strategy.

As before, by creating two right-angled triangles we can derive a formula for the length of AB (See Fill-in proof 8 'Cosine rule' for proof on the CD-ROM).

KEY POINT 11.4

The Cosine rule

$c^2 = a^2 + b^2 - 2ab\cos\hat{c}$



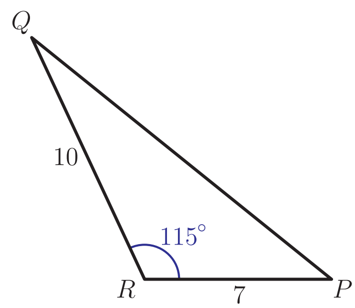
EXAM HINT

This is the form of the cosine rule given in the Formula booklet. However, you can change the names of the variables to anything you like as long as the angle corresponds to the side given on the left-hand side of the equation.

The cosine rule can be used to find the third side of the triangle when we know the other two sides and the angle between them.

Worked example 11.5

Find the length of the side PQ.



The question involves two sides and an angle, so use the cosine rule. The known angle is opposite PQ

$PQ^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos 115°$

$\Rightarrow PQ^2 = 208.2$

$\Rightarrow PQ = \sqrt{208.2} = 14.4$

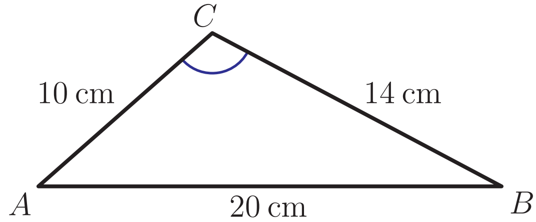
We can also use the cosine rule to find an angle if we know all three sides of a triangle. To help with this there is a rearrangement of the cosine rule:

KEY POINT 11.5

The cosine rule: $\cos\hat{C} = \dfrac{a^2 + b^2 - c^2}{2ab}$

Worked example 11.6

Find the size of the angle $A\hat{C}B$ correct to the nearest degree.



The question involves three sides and an angle, so use the cosine rule. Side AB is opposite the angle we want

$$\cos\hat{C} = \frac{196+100-400}{2\times 14\times 10}$$
$$= -\frac{104}{280}$$
$$= -0.371$$

Use inverse cosine to find the angle

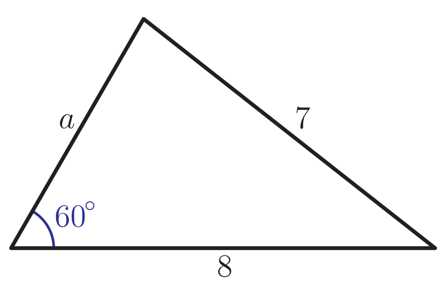
$$\hat{C} = \arccos(-0.371) = 112°$$

Notice that in the last two examples the angle is obtuse and its cosine is negative. Notice also that when using the cosine rule to find an angle, there is no second solution. This is because the second possibility is (360° – first solution) and so it will always be greater than 180°.

It is possible to use the cosine rule even when the given angle is not opposite the required side, as illustrated in the next example. This example also reminds you that there are some exact values of trigonometric functions you should know.

Worked example 11.7

Find the possible lengths of the side marked a.



continued . . .

The question involves three sides and an angle, so use the cosine rule. The known angle is opposite the side marked 7

$7^2 = a^2 + 8^2 - 2a \times 8\cos 60°$

Use the fact that $\cos 60° = \frac{1}{2}$

$\Rightarrow 49 = a^2 + 64 - 8a$

We recognise that this is a quadratic equation

$\Rightarrow a^2 - 8a + 15 = 0$

$\Rightarrow (a-3)(a-5) = 0$

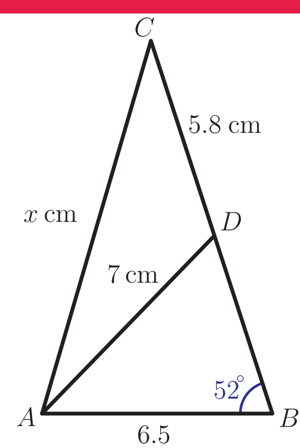
$\Rightarrow a = 3 \text{ or } 5$

It is also possible to answer this question using the sine rule twice, first to find the angle opposite the side marked 8, and then to find side a. Try to see if you can get the same answers.

The next example illustrates how to select which of the two rules to use. For both sine and cosine rule, we need to know three measurements in a triangle to find a fourth one.

Worked example 11.8

In the triangle shown in the diagram, $AB = 6.5$ cm, $AD = 7$ cm, $CD = 5.8$ cm, $A\hat{B}C = 52°$ and $AC = x$. Find the value of x correct to one decimal place.



The only triangle in which we know three measurements is ABD. We know two sides and a non-included angle (i.e. the angle does not lie between two given sides), so we can use the sine rule to find $A\hat{D}B$

Sine rule in triangle ABD; let $A\hat{D}B = \theta$:

$\frac{6.5}{\sin\theta} = \frac{7}{\sin 52°}$

$\Rightarrow \sin\theta = \frac{6.5\sin 52°}{7} = 0.7317$

$\arcsin 0.7317 = 47°$

continued . . .

Are there two possible solutions?

$180 - 47 = 133, \; 133 + 52 > 180$

There is only one solution.

$\therefore \theta = 47^\circ$ so, $A\hat{D}B = 47°$

In triangle ADC, we know two sides and want to find the third. We can find $A\hat{D}C$ and then use the cosine rule.

$A\hat{D}C = 180 - 47 = 133^\circ$

Cosine rule in triangle ADC:

$x^2 = 7^2 + 5.8^2 - 2 \times 7 \times 5.8 \cos 133^\circ$

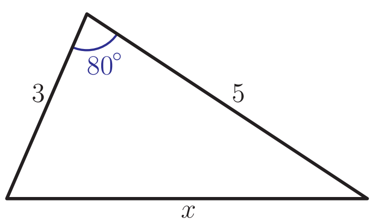
$\Rightarrow x^2 = 137.99$

$\Rightarrow x = \sqrt{137.99} = 11.7\text{cm}$

Exercise 11C

1. Find the lengths of the sides marked with letters.

(a) (i)



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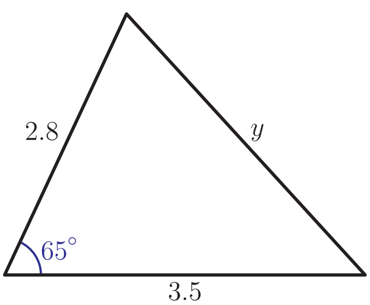
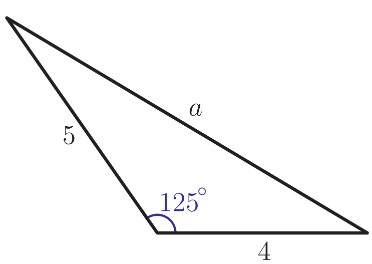


(b) (i)



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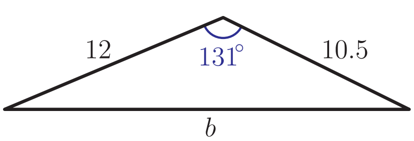
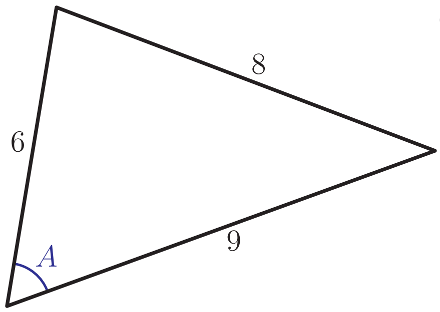


2. Find the angles marked with letters.

(a) (i)



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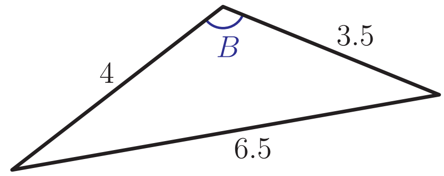
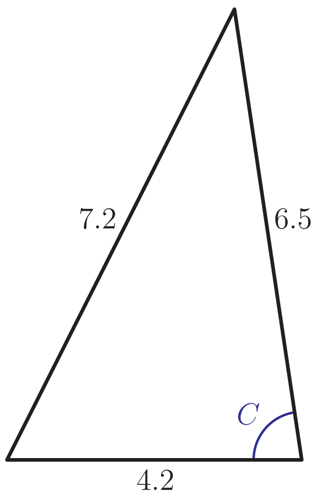
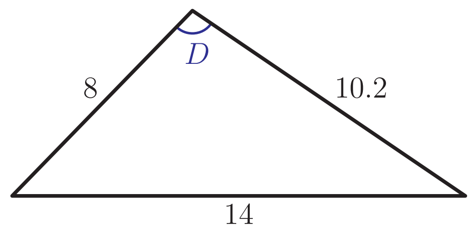


(b) (i)



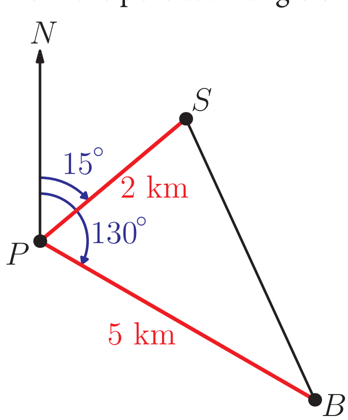
(ii)



3. (i) Triangle PQR has sides $PQ = 8$ cm, $QR = 12$ cm, $RP = 7$ cm. Find the size of the largest angle.

(ii) Triangle ABC has sides $AB = 4.5$ cm, $BC = 6.2$ cm, $CA = 3.7$ cm. Find the size of the smallest angle.

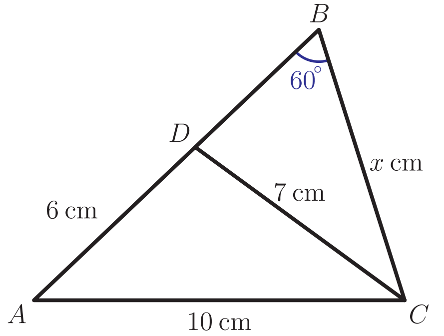
4. Ship S is 2 km from the port at an angle of 15° from North and boat B is 5 km from the port at an angle of 130° from North.



Find the distance between the ship and the boat. *[6 marks]*

Land surveying and astronomy were two areas which motivated the development of trigonometry. Supplementary sheet 4 'Coordinate systems and graphs' on the CD-ROM shows you some examples of how the sine and cosine rule can be applied in these areas.

5. Find the value of x in the diagram below.



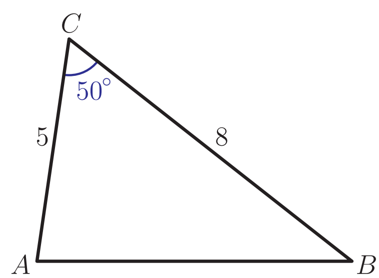
[6 marks]

6. In triangle ABC, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $B\hat{A}C = 60°$. Find the value of x. *[6 marks]*

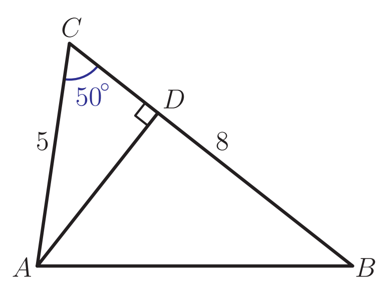
7. In triangle KLM, $KL = 4$, $LM = 7$ and $L\hat{K}M = 45°$. Find the exact length of KM. *[6 marks]*

11D Area of a triangle

Now that we know how to calculate sides and angles in a triangle, we can ask how to calculate the area. We know that the formula for the area of a triangle is $\frac{1}{2} \times \text{base} \times \text{perpendicular height}$ and we can use this to find the area of the triangle shown in the diagram:



We need to find a height of the triangle in order to calculate the area. For example, we can draw the line (AD) perpendicular to (BC), as in the second diagram:



Then $AD = 5\sin 50°$, so the area of the triangle is:

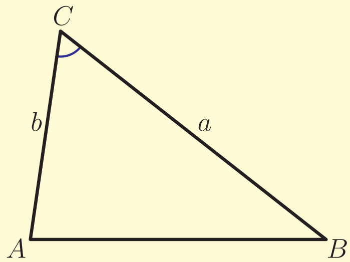
$$\frac{1}{2}BC \times AD = \frac{1}{2} \times 8 \times 5\sin 50° = 15.2 \text{ cm}^2$$

This method can be applied to any triangle so the general formula for the area is:

KEY POINT 11.6

The area of the triangle is given by:

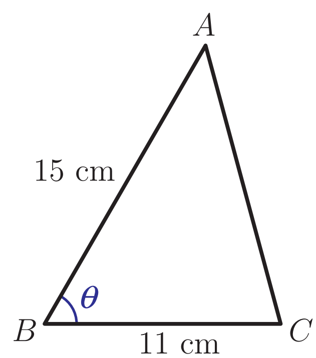
$$\text{area} = \frac{1}{2}ab\sin\hat{C}$$



There is also a formula for the area of a triangle if you know all three of its sides. It is called *Heron's formula*.

Worked example 11.9

The area of the triangle shown in the diagram is 52 cm². Find the two possible values of $A\hat{B}C$, correct to one decimal place.



Use the formula for the area of a triangle with known angle $\hat{B}$: area $= \frac{1}{2}ac\sin\hat{B}$

$$\frac{1}{2}(11 \times 15)\sin\theta = 52$$

$$\Rightarrow \sin\theta = \frac{2 \times 52}{11 \times 15} = 0.6303$$

$$\Rightarrow \arcsin 0.6303 = 39.07°$$

$$\therefore \theta = 39.1° \text{ or } 180 - 39.1 = 140.9°$$

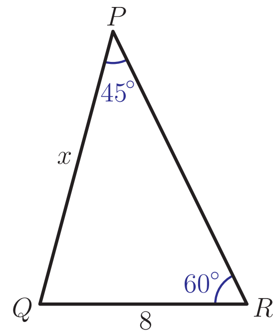
The next example combines the area of the triangle with the sine rule and working with exact values.

Worked example 11.10

For triangle PQR shown in the diagram:

(a) Calculate the exact value of x.

(b) Find the area of the triangle.



The question involves two angles and two sides, so we can use the sine rule

(a) $$\frac{8}{\sin 45°} = \frac{x}{\sin 60°}$$

We need the exact value of x, so use the exact values for $\sin 45°$ and $\sin 60°$

$$\Rightarrow \frac{8}{\sqrt{2}/2} = \frac{x}{\sqrt{3}/2}$$

$$\Rightarrow \frac{16}{\sqrt{2}} = \frac{2x}{\sqrt{3}}$$

$$\Rightarrow x = \frac{16\sqrt{3}}{2\sqrt{2}}$$

$$\Rightarrow x = 4\sqrt{6}$$

To use the formula for the area of the triangle, we need $P\hat{Q}R$

(b) $$P\hat{Q}R = 180 - 60 - 45 = 75°$$

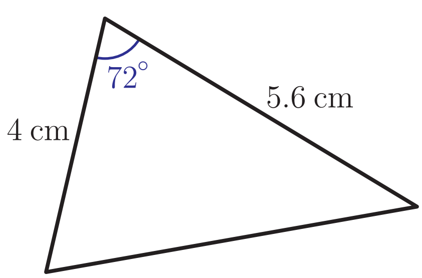
$$\therefore \text{area} = \frac{1}{2}\left(8 \times 4\sqrt{6}\right)\sin 75°$$

$$= 37.9 \text{ (3SF)}$$

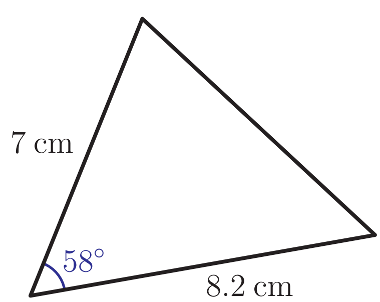
Exercise 11D

1. Calculate the areas of these triangles.

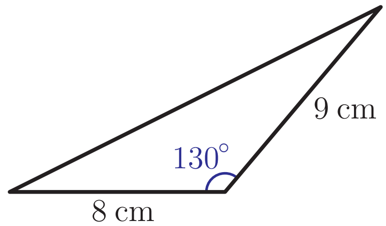
(a) (i)



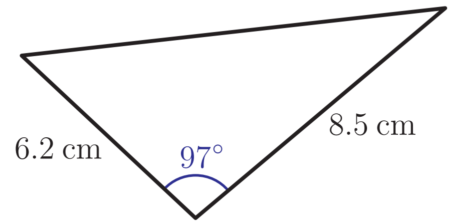
(ii)



(b) (i)

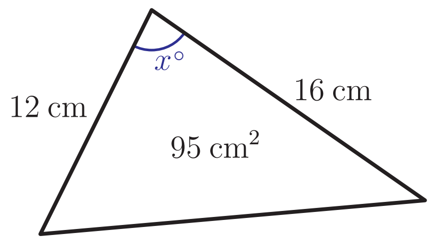


(ii)

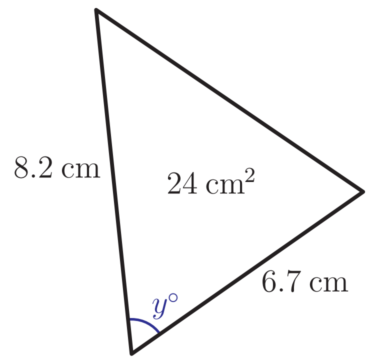


2. Each triangle has the area shown. Find two possible values of each marked angle.

(a)



(b)



3. In triangle LMN, $LM = 12$ cm, $MN = 7$ cm and $L\hat{M}N = 135°$.
Find the length of the side LN and the area of the triangle.

[6 marks]

4. In triangle PQR, $PQ = 8$ cm, $RQ = 7$ cm and $R\hat{P}Q = 60°$.
Find the exact difference in areas between the two possible triangles.

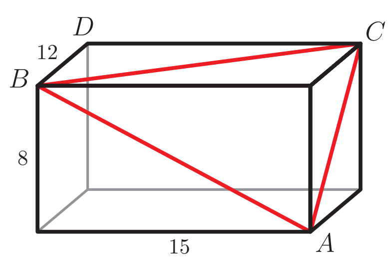
[6 marks]

11E Trigonometry in three dimensions

In many applications we work with three-dimensional objects. Examples in this section show you how to apply trigonometry in three dimensions. The general strategy is to identify a suitable triangle and then apply one of the rules from the previous sections.

Worked example 11.11

A cuboid has sides of length 8 cm, 12 cm and 15 cm. Diagonals of three of the faces are drawn as shown.



continued . . .

(a) Find the lengths of AB, BC and CA.

(b) Find the size of the angle $A\hat{C}B$.

(c) Calculate the area of the triangle ABC.

(d) Find the length AD.

AB is the diagonal of the front face, so it is a hypotenuse of a right-angled triangle with sides 8 and 12

(a) $AB^2 = 15^2 + 8^2 = 289$

$AB = \sqrt{298} = 17 \text{ cm}$

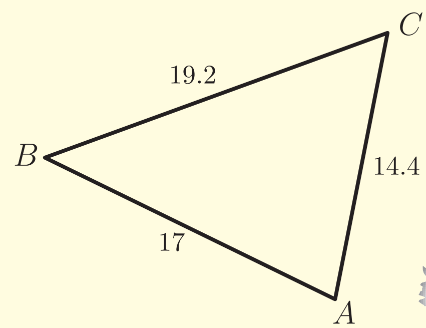
Find BC and CA in a similar way

$BC = \sqrt{12^2 + 15^2} = 19.2 \text{ cm}$

$CA = \sqrt{12^2 + 8^2} = 14.4 \text{ cm}$

We now look at triangle ABC. Draw the triangle if it helps

(b)



We know all three sides and want to find the angle, so we use the cosine rule

$$\cos\hat{C} = \frac{14.4^2 + 19.2^2 - 17^2}{2 \times 14.4 \times 19.2} = 0.519$$

$$C = \arccos 0.519 = 58.7°$$

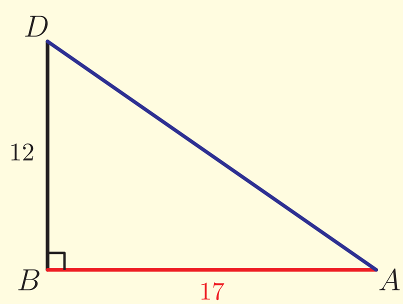
Use the formula for the area, with the angle we have just found

(c) $\text{area} = \frac{1}{2}(14.4 \times 19.2)\sin 58.7°$

$= 118 \text{ cm}^2$

ABD is a right-angled triangle

(d)



$AD^2 = 12^2 + 17^2$

$AD = \sqrt{433} = 20.8 \text{ cm}$

Part (d) illustrates a very useful fact about the diagonal of a cuboid.

KEY POINT 11.7

The diagonal of a $p \times q \times r$ cuboid has length $\sqrt{p^2 + q^2 + r^2}$.

In chapter 13 you will meet vectors which can also be used to solve three-dimensional problems.

The key to solving three-dimensional problems is spotting right angles. This is not always easy, as diagrams are drawn in perspective, but there are some common configurations to look for, such cross-sections and bisectors of isosceles triangles.

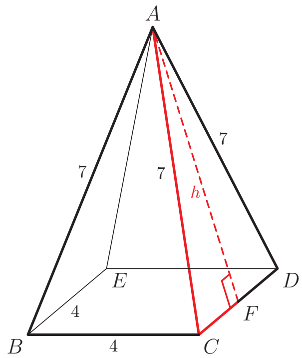
Worked example 11.12

The base of a pyramid is a square of side 4 cm. The other four faces are isosceles triangles with sides 7 cm. The height of one of the side faces is labelled h.

(a) Find the exact value of h.

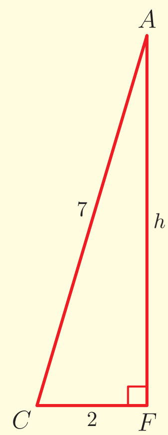
(b) Find the exact height of the pyramid.

(c) Calculate the volume of the pyramid, correct to 3 significant figures.



Triangle AFC is right angled. Draw it separately and label known sides

(a)



Use Pythagoras to find h

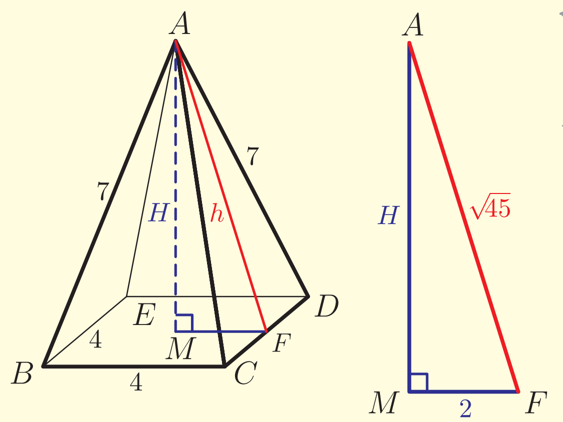
$h^2 = 7^2 - 2^2$

$h = \sqrt{45}$ cm

continued . . .

Add the height to the diagram. It is perpendicular to the base, so it makes a right-angle with MF. M is in the centre of the base, so $MF = 2$ cm

(b)



Look at triangle AMF

Use Pythagoras to find H

$$H^2 = \left(\sqrt{45}\right)^2 - 2^2$$

$$H = \sqrt{41}\text{ cm}$$

From the Formula booklet, the formula for the volume of a pyramid is $\frac{1}{3}$(area of the base × height)

(c) $V = \frac{1}{3}(4^2 \times \sqrt{41})$

$$V = 34.1\text{ cm}^3$$

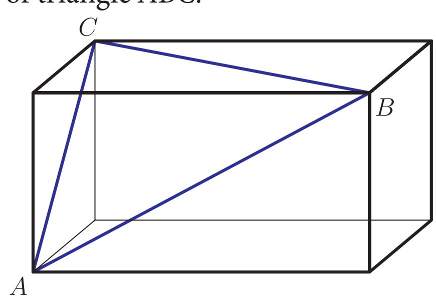
See Prior learning Section V on the CD-ROM for volumes of three-dimensional shapes.

Exercise 11E

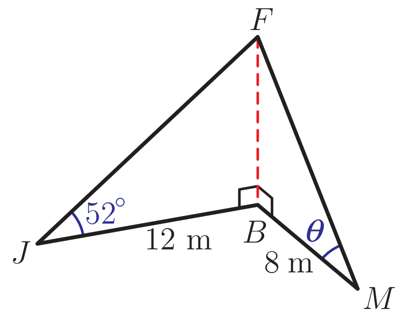
1. Find the length of the diagonal of the following cuboids:

   (a) 3 cm × 5 cm × 10 cm (b) 4 cm × 4 cm × 8 cm

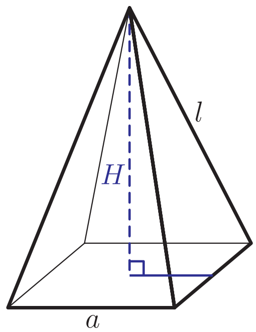
2. A cuboid has sides 12.5 cm, 10 cm and 7.3 cm. It is intersected by a plane passing through vertices A, B and C. Find the angles and the area of triangle ABC. *[8 marks]*



3. Johann stands 12 m from the base of a flagpole and sees the top at the angle of elevation of 52°. Marit stands 8 m from the flagpole. At what angle of elevation does she see the top? *[6 marks]*



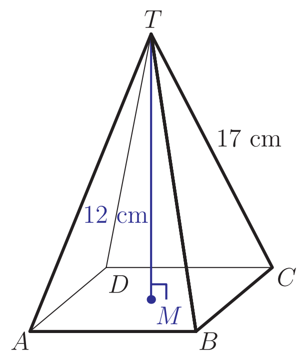
4. A square-based pyramid has a base of side $a = 8$ cm and a height $H = 12$ cm. Find the length of the sloping side, l. *[6 marks]*



5. The base of a pyramid $TABCD$ is a square. The height of the pyramid $TM = 12$ cm and the length of a sloping edge $TC = 17$ cm.

(a) Calculate the length of MC.

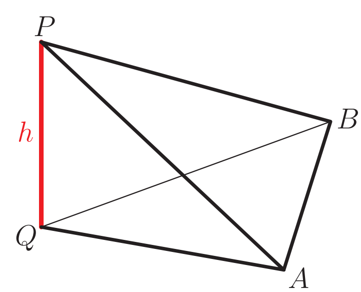
(b) Find the length of a side of the base. *[6 marks]*



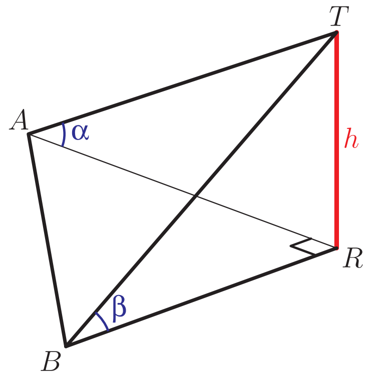
6. The diagram shows a vertical tree PQ and two observers at A and B, standing on horizontal ground. $AQ = 25$ m, $Q\hat{A}P = 37°$, $Q\hat{B}P = 42°$ and $A\hat{Q}B = 75°$.

(a) Find the height of the tree, h.

(b) Find the distance between the two observers. *[8 marks]*



7. Annabelle and Berta are trying to measure the height, h, of a vertical tree RT. They stand on horizontal ground, a distance d apart so that ARB is a right-angle. From where Annabelle is standing, the angle of elevation of the top of the tree is α, and from where Berta stands the angle of elevation of the top of the tree is β.

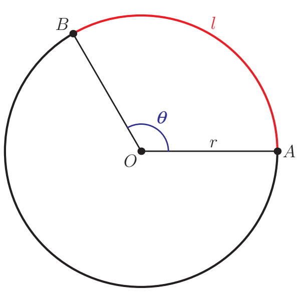


(a) Find expressions for RA and RB in terms of h, α and β, and hence show that:

$$h^2\left(\frac{1}{\tan^2 \alpha}+\frac{1}{\tan^2 \beta}\right)=d^2$$

(b) Given that $d = 26\,\text{m}$, $\alpha = 45°$ and $\beta = 30°$, find the height of the tree. *[8 marks]*

11F Length of an arc



The diagram shows a circle with centre O and radius r, and points A and B on its circumference. The part of the circumference between points A and B is called an **arc** of a circle. You can see that there are in fact two such parts; the shorter one is called the **minor arc**, and the longer one the **major arc**. We say that the minor arc AB **subtends** angle θ at the centre of the circle.

You know that a measure of angle θ in radians is equal to the ratio of the length of the arc AB, l, to the circumference of the circle; in other words, $\theta = \frac{l}{r}$.

Radian measure was introduced in Section 9A.

This gives us a very simple formula to calculate the length of an arc of a circle if we know the angle it subtends at the centre.

KEY POINT 11.8

Length of an arc of a circle

$l = \theta r$

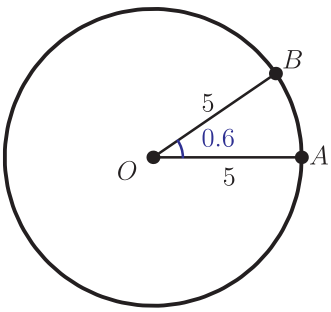
where r is the radius of the circle and θ is the angle subtended at the centre, measured in radians.

Worked example 11.13

Arc AB of a circle with radius 5 cm subtends an angle of 0.6 at the centre, as shown on the diagram.

(a) Find the length of the minor arc AB.

(b) Find the length of the major arc AB.



We know the formula for the length of an arc

(a) $l = r\theta$

$l = 5 \times 0.6$

$= 3$ cm

The angle subtended by the major arc is equal to a full turn minus the smaller angle. A full turn is 2π radians

(b) $\theta_1 = 2\pi - 0.6$

$= 5.683$

$l = r\theta_1 = 5 \times 5.683$

$= 28.4$ cm (3SF)

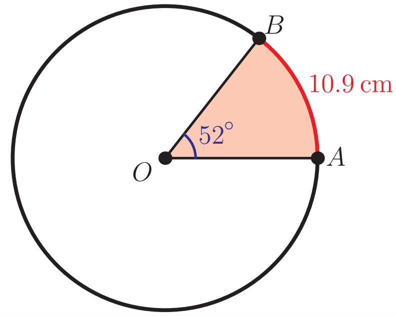
We could also have done the second part differently, by finding the length of the whole circumference and then taking away the minor arc: circle circumference is $2\pi r = 2\pi \times 5 = 31.42$, so the length of the major arc is $31.42 - 3 = 28.4$ cm (3 SF).

If the angle is given in degrees, we need to convert to radians before using the formula for the arc length.

Worked example 11.14

Two points, A and B, lie on the circumference of a circle of radius r cm. The arc AB has length 10.9 cm and subtends an angle of 52° at the centre of the circle.

(a) Find the value of r.
(b) Calculate the perimeter of the shaded region.



We know the arc length and the angle, so we can find the radius

(a) $l = r\theta \Rightarrow r = \frac{l}{\theta}$

To use the formula $l = r\theta$, the angle must be in radians

$\theta = 52 \times \frac{\pi}{180} = 0.908$

$\therefore r = \frac{10.9}{0.908} = 12.0\text{ cm}$

The perimeter is made up of two radii and the arc

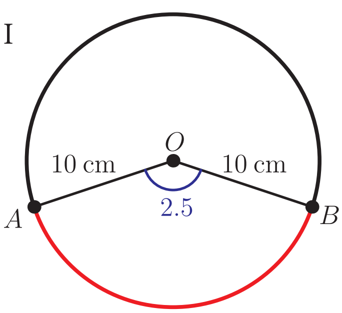
(b) $p = 2r + l$

$= 2 \times 12.0 + 10.9$

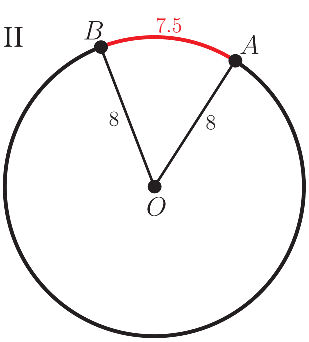
$= 34.9\text{ cm}$

Exercise 11F

I



II



1. Calculate the length of the minor arc subtending an angle of θ radians at the centre of a circle of radius r cm.

(a) $\theta = 1.2, \quad r = 6.5$ (b) $\theta = 0.4, \quad r = 4.5$

2. Points A and B lie on the circumference of a circle with centre O and radius r cm.

Angle $A\hat{O}B$ is θ radians. Calculate the length of the major arc AB.

(a) $r = 15, \quad \theta = 0.8$ (b) $r = 1.4, \quad \theta = 1.4$

3. Calculate the length of the minor arc AB in the upper diagram (I). *[4 marks]*

4. In the lower diagram (II), the radius of the circle is 8 cm and the length of the minor arc AB is 7.5 cm. Calculate the size of the angle $A\hat{O}B$:

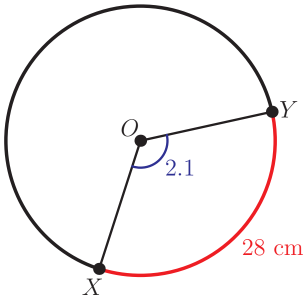
(a) in radians
(b) in degrees. *[5 marks]*

5. Points M and N lie on the circumference of a circle with centre C and radius 4 cm. The length of the *major* arc MN is 15 cm. Calculate the size of the *smaller* angle $M\hat{C}N$. *[4 marks]*

6. Points P and Q lie on the circumference of a circle with centre O. The length of the minor arc PQ is 12 cm and $P\hat{O}Q = 1.6$. Find the radius of the circle. *[4 marks]*

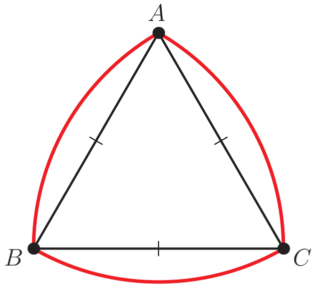
7. In the diagram, the length of the major arc XY is 28 cm. Find the radius of the circle.

[4 marks]



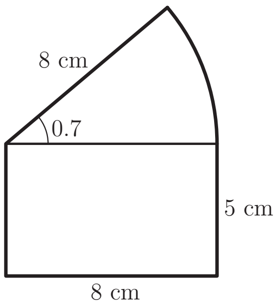
8. The figure below shows an equilateral triangle ABC with side $a = 5$ cm, and three arcs of circles with centres at the vertices of the triangle. Calculate the perimeter of the figure.

[5 marks]



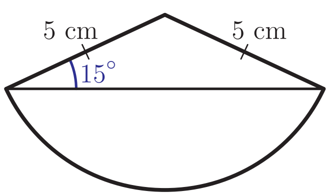
9. Calculate the perimeter of this figure:

[6 marks]



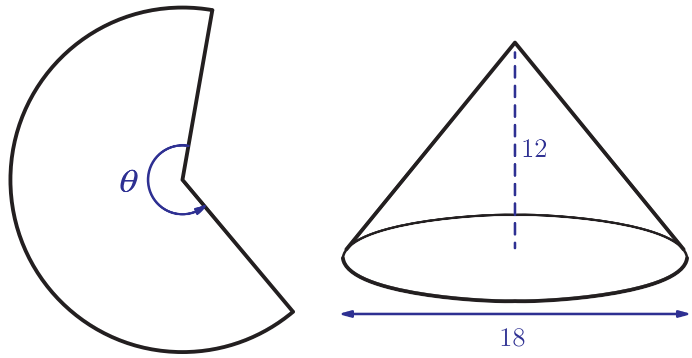
10. Find the exact perimeter of the figure shown:

[6 marks]



11. A sector of a circle has perimeter $p = 12$ cm and angle at the centre $\theta = 0.4$. Find the radius of the circle. *[5 marks]*

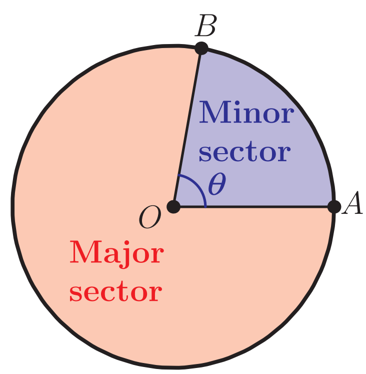
12. A cone is made by rolling a piece of paper as shown in the diagram.



If the cone is to have height 12 cm and base diameter 18 cm, find the size of the angle marked θ. *[6 marks]*

11G Area of a sector

A sector is a part of a circle bounded by two radii. As with arcs, we distinguish between a **minor sector** and a **major sector**.



The ratio of the area of the sector to the area of the whole circle is the same as the ratio of angle θ to a full turn. If A is the area of the sector, and angle θ is measured in radians, this means that $\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$. Rearranging this equation gives the formula for the area of the sector.

KEY POINT 11.9

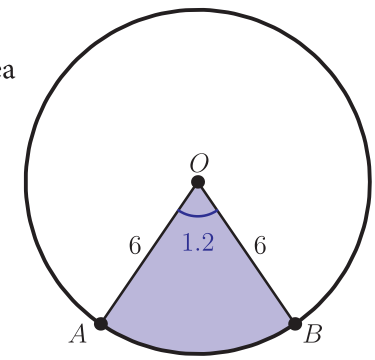
The area of a sector of a circle

$$A = \frac{1}{2} r^2 \theta,$$

where r is the radius of the circle and θ is the angle subtended at the centre, measured in radians.

Worked example 11.15

The diagram shows a circle with centre O and radius 6 cm, and two points on its circumference such that $A\hat{O}B = 1.2$. Find the area of the minor sector AOB.



Formula for the area of a sector

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2} \times 6^2 \times 1.2 = 21.6 \text{ cm}^2$$

We may also have to use the formulae for area and arc length in reverse.

Worked example 11.16

A sector of a circle has perimeter $p = 12$ cm and angle at the centre $\theta = 50°$.

Find the area of the sector.

If we are going to use the formula for the area of a sector, $A = \frac{1}{2}r^2\theta$, we first need to find r

We are given the perimeter, and we know that it consists of arc length $l = r\theta$ and two radii

$$p = r\theta + 2r$$

The angle needs to be in radians

$$\theta = 50 \times \frac{\pi}{180} = 0.873$$

Substitute all the values into the formula for perimeter to find r

$$12 = 0.873r + 2r = 2.873r$$

$$r = \frac{12}{2.873} = 4.177$$

Substitute θ and r into the formula for sector area

$$A = \frac{1}{2}(4.177)^2(0.873)$$

$$A = 7.62 \text{ cm}^2 \text{ (3SF)}$$

Exercise 11G

1. Points M and N lie on the circumference of a circle, centre O and radius r cm, with $M\hat{O}N = \alpha$. Calculate the area of the minor sector MON.

   (a) $r = 5, \alpha = 1.3$ (b) $r = 0.4, \alpha = 0.9$

2. Points A and B lie on the circumference of a circle with centre C and radius r cm. The size of the angle ACB is θ radians. Calculate the area of the major sector $A\hat{C}B$.

   (a) $r = 13, \quad \theta = 0.8$ (b) $r = 1.4, \quad \theta = 1.4$

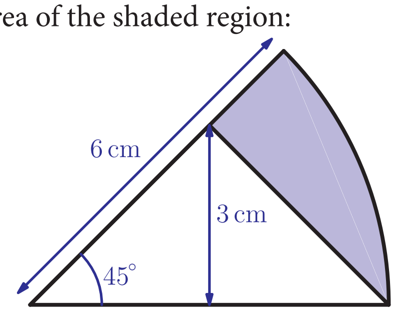
3. A circle has centre O and radius 10 cm. Points A and B lie on the circumference of the circle so that the area of the minor sector AOB is 40 cm^2. Calculate the size of the smaller angle $A\hat{O}B$. *[5 marks]*

4. Points P and Q lie on the circumference of a circle with radius 21 cm. The area of the *major* sector POQ is 744 cm^2. Find the size of the *smaller* angle $P\hat{O}Q$ in *degrees*. *[5 marks]*

5. A sector of a circle with angle 1.2 radians has area 54 cm^2. Find the radius of the circle. *[4 marks]*

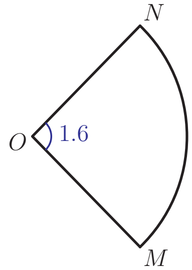
6. A sector of a circle with angle 162° has area 180 cm^2. Find the radius of the circle. *[4 marks]*

7. Find the area of the shaded region:



[6 marks]

8. The perimeter of the sector shown in the diagram is 28 cm. Find its area.



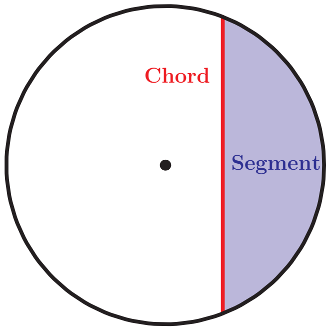
[5 marks]

9. A sector of a circle has perimeter 7 cm and area 3 cm². Find the possible values of the radius of the circle. *[6 marks]*

10. Points P and Q lie on the circumference of the circle with centre O and radius 5 cm. The difference between the areas of the major sector POQ and the minor sector POQ is 15 cm². Find the size of the angle $P\hat{O}Q$. *[5 marks]*

11H Triangles and circles

You already know how to calculate lengths of arcs and areas of sectors of circles. In this section we will look at two other important parts of circles: **chords** and **segments.**

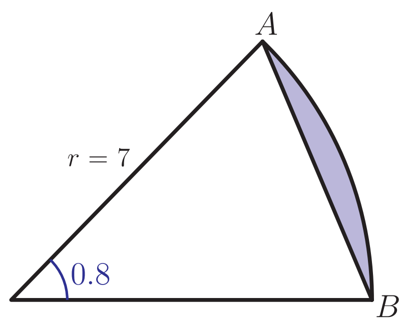


Worked example 11.17

The diagram shows a sector of a circle of radius 7 cm and the angle at the centre 0.8 radians. Find:

(a) the perimeter of the shaded region

(b) the area of the shaded region



The perimeter is made up of the arc AB and the chord $[AB]$

(a) $p = \text{arc} + \text{chord}$

The formula for the length of the arc is $l = r\theta$

$l = 7 \times 0.8 = 5.6$ cm

The chord $[AB]$ is the third side of the triangle ABC. As we know the other two sides and the angle between them, we can use the cosine rule. Remember that the angle is in radians

Cosine rule:

$AB^2 = 7^2 + 7^2 - 2 \times 7 \times 7 \cos 0.8$

$AB^2 = 29.7$

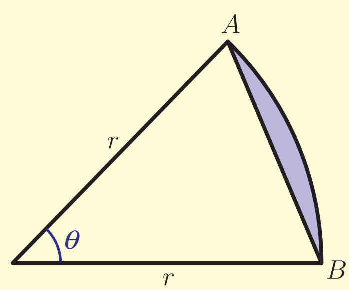
$AB = \sqrt{29.7} = 5.45$ cm

continued . . .

We can now find the perimeter	$\therefore p = 5.6 + 5.45 = 11.1$ cm
We know how to calculate the area of a sector. If we subtract the area of triangle *ABC*, we are left with the area of the segment	(b) area = sector − triangle
The formula for the area of a sector is $\frac{1}{2}r^2\theta$	sector $= \frac{1}{2}(7^2 \times 0.8) = 19.6$ cm^2
The formula for the area of a triangle is $\frac{1}{2}ab \sin C$	triangle $= \frac{1}{2}(7 \times 7)\sin 0.8 = 17.58$ cm^2
We can now find the area of the segment	$\therefore$ area $= 19.6 - 17.6 = 2.02$ cm^2

We can follow the same method to derive general formulae for the length of a chord and area of a segment.

KEY POINT 11.10



The length of a chord of a circle is given by:

$$AB^2 = 2r^2(1 - \cos\theta)$$

and **the area of the shaded segment** is

$$\frac{1}{2}r^2(\theta - \sin\theta),$$

where angle θ is measured in radians.

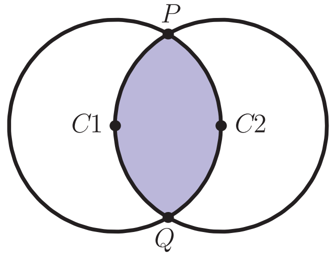
EXAM HINT

These formulae are not given in the Formula booklet, so you need to know how to derive them.

The next example shows how we can solve more complex geometry problems by splitting up the figure into basic shapes such as triangles and sectors.

Worked example 11.18

The diagram shows two equal circles of radius 12 such that the centre of one circle is on the circumference of the other.

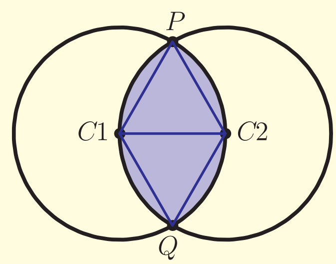


(a) Find the exact size of angle $P\hat{C}_1Q$ in radians.

(b) Calculate the exact area of the shaded region.

The only thing we know is the radius of the circle, so draw all the lengths which are equal to the radius

(a)



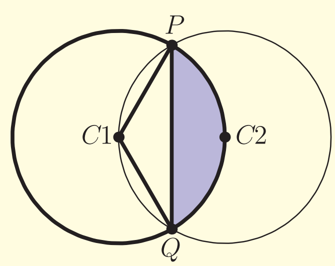
The lengths C_1P, C_2P and C_1C_2 are all equal to the radius of the circle. Therefore triangle PC_1C_2 is equilateral

$P\hat{C}_1C_2 = \frac{\pi}{3}$

$\therefore P\hat{C}_1Q = \frac{2\pi}{3}$

The shaded area is made up of two segments, each with angle at the centre $\frac{2\pi}{3}$

(b)



We can find area of one segment using the formula. Remember to use the exact value of $\sin\left(\frac{2\pi}{3}\right)$

area of one segment $= \frac{1}{2}(12^2)\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$

$= 72\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= 48\pi - 36\sqrt{3}$

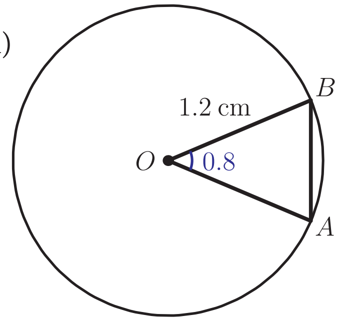
The shaded area consists of two segments

$\therefore$ shaded area $= 96\pi - 72\sqrt{3}$

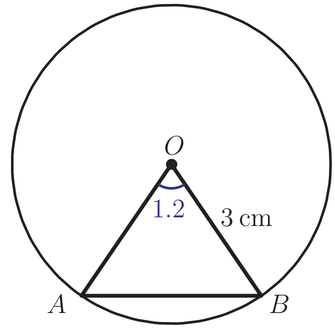
Exercise 11H

1. Find the length of the chord AB:

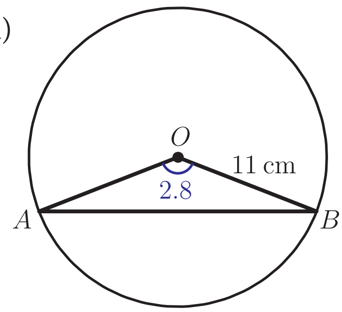
(a) (i)



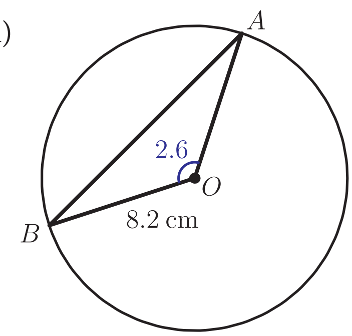
(ii)



(b) (i)



(ii)



2. Find the perimeters of the minor segments from Question 1.

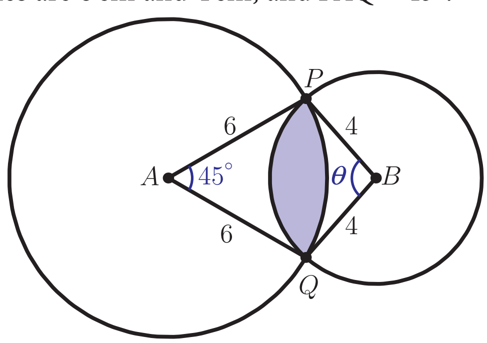
3. Find the areas of minor segments from Question 1.

4. A circle has centre O and radius 5 cm. Chord PQ subtends angle θ at the centre of the circle.

   (a) Write down an expression for the area of the minor segment.

   (b) Given that the area of the minor segment is 15 cm², find the value of θ. *[6 marks]*

5. Two circles, with centres A and B, intersect at P and Q. The radii of the circles are 6 cm and 4 cm, and $P\hat{A}Q = 45°$.

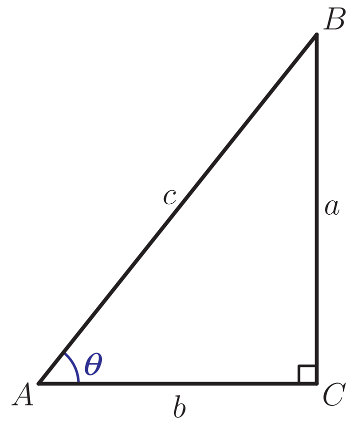


   (a) Show that $PQ = 6\sqrt{2-\sqrt{2}}$.

   (b) Find the size of $P\hat{B}Q$.

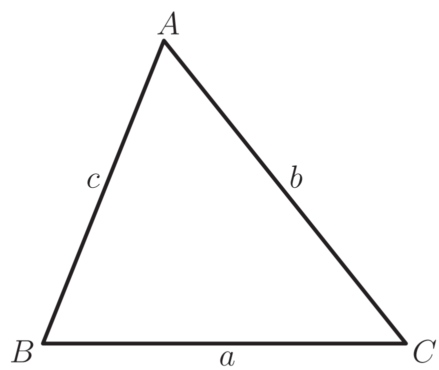
   (c) Find the area of the shaded region. *[9 marks]*

Summary



- In a right-angled triangle: $\frac{a}{c} = \sin\theta, \frac{b}{c} = \cos\theta, \frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$
- The **angle of elevation** is the angle above the horizontal.
- The **angle of depression** is the angle below the horizontal.
- To find a side when two angles and a side are given, or an angle when two sides and a non-included angle are given, we can use the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



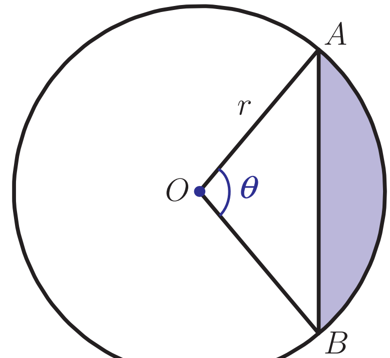
- To find a side when two sides and an angle are given, or an angle when all three sides are given, we can use the cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\text{or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

- An alternative formula for the area of a triangle:

$$\text{area} = \frac{1}{2}ab\sin C$$

- To solve problems in three dimensions, find a suitable triangle and apply one of the above rules. Look out for right angles.
- The diagonal of a $p \times q \times r$ cuboid has length $\sqrt{p^2 + q^2 + r^2}$.
- In a circle of radius r and an angle θ in radians subtended at the centre:
  - Arc length $l = r\theta$
  - Area of sector $A = \frac{1}{2}r^2\theta$



- You need to know how to derive the formulae for the length of a chord and the area of a segment:

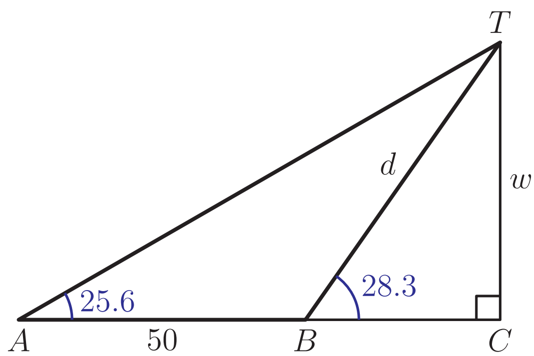
$$AB^2 = 2r^2(1 - \cos\theta)$$

$$\text{area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

where θ is measured in radians.

Introductory problem revisited

Two observers are trying to measure the width of a river. There is no bridge across the river, but they have instruments for measuring lengths and angles. They stand at *A* and *B*, 50 m apart, on the opposite side of the river to a tower. The person at *A* measures that the angle between the line (*AB*) and the line from *A* to the tower as 25.6°.
The observer at *B* similarly measures the corresponding angle to be 28.3° as shown in the diagram.
Can they use this information to calculate the width of the river?



We want to find the width *w*. There are two right-angled triangles, *ACT* and *BCT*. We do not know the lengths of any of their sides, but in triangle *ABT* we know two angles and one side, so we can calculate the remaining sides. In particular, knowing the length of *BT* would allow us to find *w* from triangle *BCT*.

In order to use the sine rule in triangle *ABT*, we need to know the size of $A\hat{T}B$. Since $A\hat{B}T = 180° - 28.3° = 151.7°$, we can find that $A\hat{T}B = 2.7°$.

We can now use the sine rule:

$$\frac{d}{\sin 25.6°} = \frac{50}{\sin 2.7°}$$

$$d = \frac{50 \sin 25.6°}{\sin 2.7°} = 458.6\,\text{m}$$

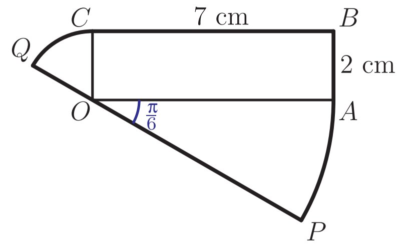
Finally, we can use the right-angled triangle *BCT* to find the width of the river:

$$w = d \sin 28.3° = 217\,\text{m}\,(3\text{SF})$$

Mixed examination practice 11

Short questions

1. In the diagram, $OABC$ is a rectangle with sides 7 cm and 2 cm. PQ is a straight line. AP and CQ are circular arcs, and $A\hat{O}P = \frac{\pi}{6}$.



(a) Write down the size of $C\hat{O}Q$.

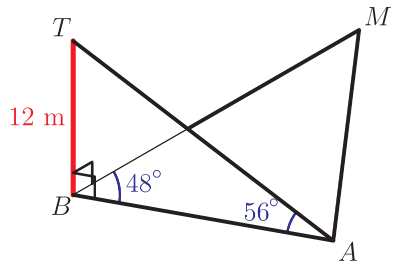
(b) Find the area of the whole shape.

(c) Find the perimeter of the whole shape. *[9 marks]*

2. A sector has perimeter 36 cm and radius 10 cm. Find its area. *[6 marks]*

3. In triangle ABC, $AB = 6.2$ cm, $CA = 8.7$ cm and $A\hat{C}B = 37.5°$. Find the two possible values of $A\hat{B}C$. *[6 marks]*

4. A vertical tree of height 12 m stands on horizontal ground. The bottom of the tree is at the point B. Observer A, standing on the ground, sees the top of the tree, T, at an angle of elevation of 56°.

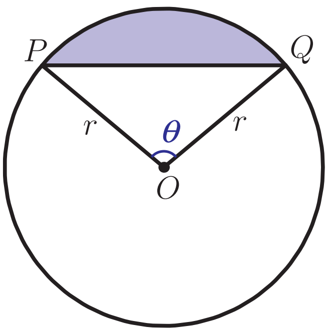


(a) Find the distance of A from the bottom of the tree.

Another observer, M, stands the same distance away from the tree, and $A\hat{B}M = 48°$.

(b) Find the distance AM. *[6 marks]*

5. The diagram shows a circle with centre O and radius $r = 7$ cm. The chord PQ subtends angle $\theta = 1.4$ radians at the centre of the circle.



Find:

(a) the area of the shaded region

(b) the perimeter of the shaded region. *[6 marks]*

6. In triangle ABC, $AB = 2\sqrt{3}$, $AC = 10$ and $B\hat{A}C = 150°$. Find the exact length of BC. *[6 marks]*

7. The perimeter of the sector shown in the diagram is 34 cm and its area is 52 cm^2. Find the radius of the sector. *[6 marks]*

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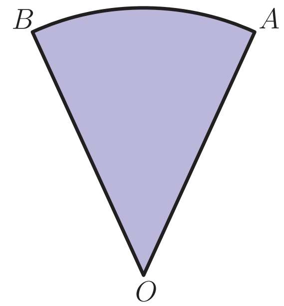
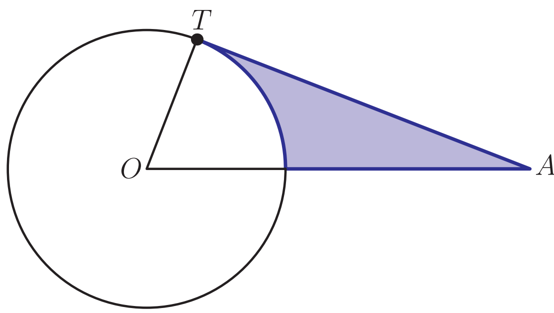


8. In the diagram, O is the centre of the circle and AT is the tangent to the circle at T.



Properties of circles and basic trigonometry are covered in the Prior learning Section W of the CD-ROM.

If $OA = 12$ cm, and the circle has a radius of 6 cm, find the area of the shaded region. *[4 marks]*

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9. The diagram shows a triangle and two arcs of circles.

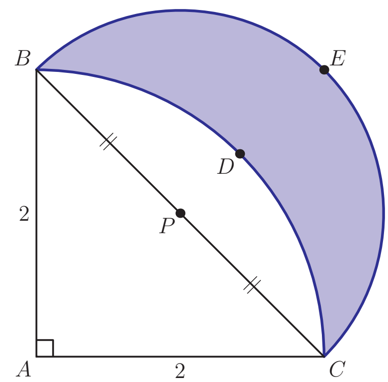
The triangle ABC is a right-angled isosceles triangle, with $AB = AC = 2$. The point P is the midpoint of BC.

The arc BDC is part of a circle with centre A.

The arc BEC is part of a circle with centre P.

(a) Calculate the area of the segment $BDCP$.

(b) Calculate the area of the shaded region $BECD$.



[6 marks]

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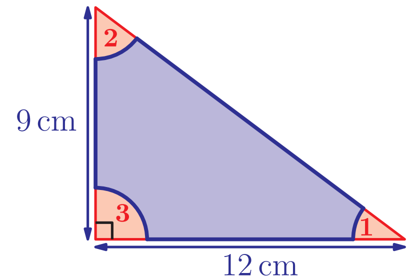
10. A right-angled triangle has sides 12 cm and 9 cm. At each vertex, a sector of radius 2 cm is cut out, as shown in the diagram. The angle at sector 1 is θ.



(a) Write down an expression for the area of sector 2 in terms of θ.

(b) Find the remaining area. *[6 marks]*

11. In the obtuse-angled triangle KLM, $LM = 6.1$ cm, $KM = 4.2$ cm and $K\hat{L}M = 42°$.

Find the area of the triangle. *[6 marks]*

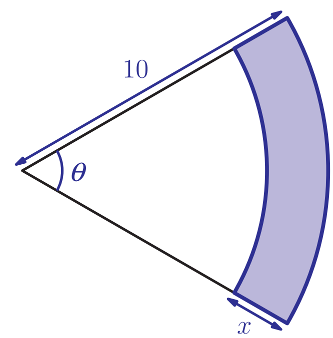
12. In triangle ABC, $AB = 10$ cm, $BC = 8$ cm and $CA = 7$ cm.

(a) Find the exact value of $\cos(A\hat{B}C)$.

(b) Find the exact value of $\sin(A\hat{B}C)$. *[8 marks]*

(c) Find the exact value of the area of the triangle.

13. The diagram shows two circular sectors with angle θ at the centre. The radius of the larger sector is 10 cm, the radius of the smaller sector is x cm smaller.



(a) Show that the area of the shaded region is given by $\frac{x(20-x)\theta}{2}$.

(b) If $\theta = 1.2$ find the value of x such that the area of the shaded region is equal to $54.6\ \text{cm}^2$. *[8 marks]*

Long questions

1. In triangle ABC, $AB = 5$, $AC = x$ and $\hat{A} = \theta$. M is the midpoint of the side AC.

(a) Use the cosine rule to find an expression for MB^2 in terms of x and θ.

(b) Given that $BC = MB$, show that $\cos, = \frac{3x}{20}$.

(c) If $x = 5$, find the value of the angle θ such that $MB = BC$. *[9 marks]*

2. Two circles have equal radius r and intersect at points S and T. The centres of the circles are A and B, and $A\hat{S}B = 90°$.

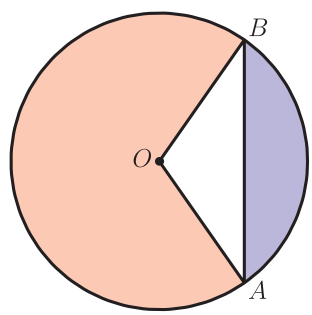
   (a) Explain why $S\hat{A}T$ is also 90°.

   (b) Find the length AB in terms of r.

   (c) Find the area of the sector AST.

   (d) Find the area of the overlap of the two circles. *[10 marks]*

3. The diagram shows a circle with centre O and radius r. Chord AB subtends an angle at the centre of size θ radians. The minor segment and the major sector are shaded.



   (a) Show that the area of the minor segment is $\frac{1}{2}r^2(\theta - \sin\theta)$.

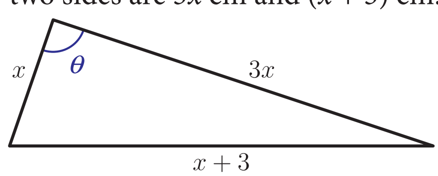
   (b) Find the area of the major sector.

   (c) Given that the ratio of the areas of the blue: pink regions is 1 : 2, show that:

$$\sin , = \frac{3}{2} , - \pi$$

   (d) Find the value of θ. *[10 marks]*

4. The area of the triangle shown is 2.21 cm². The length of the shortest side is x cm and the other two sides are $3x$ cm and $(x + 3)$ cm.



   (a) Using the formula for the area of the triangle, write down an expression for $\sin\theta$ in terms of x.

   (b) Using the cosine rule, write down and simplify an expression for $\cos\theta$ in terms of x.

   (c) (i) Using your answers to parts (a) and (b), show that:

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)$$

   (ii) Hence find the possible values of x and the corresponding values of θ, in radians, using your answer to part (b) above. *[10 marks]*

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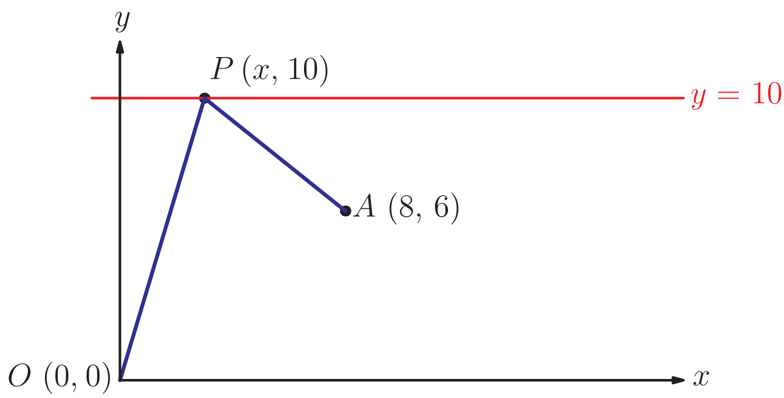
5. In triangle ABC, $AB = 10$, $BC = 5$, $CA = x$ and $B\hat{A}C = \theta°$.

(a) Show that $x^2 - 20x\cos\theta + 75 = 0$.

(b) Find the range of values of $\cos\theta$ for which the above equation has real roots.

(c) Hence find the set of values of θ for which it is possible to construct triangle ABC with the given measurements. *[8 marks]*

6. In the diagram below, the points O (0, 0) and A (8, 6) are fixed. The angle $O\hat{P}A$ varies as the point P $(x, 10)$ moves along the horizontal line $y = 10$.



(a) (i) Show that $AP = \sqrt{x^2 - 16x + 80}$.

(ii) Write down a similar expression for OP in terms of x.

(b) Hence, show that:

$$\cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}$$

(c) Find, in degrees, the angle $O\hat{P}A$ when $x = 8$.

(d) Find the positive value of x such that $O\hat{P}A = 60°$.

Let the function f be defined by

$$f(x) = \cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}, \quad 0 \le x \le 15.$$

(e) Consider the equation $f(x) = 1$.

(i) Explain, in terms of the position of the points O, A and P, why this equation has a solution.

(ii) Find the exact solution to the equation.

[17 marks]

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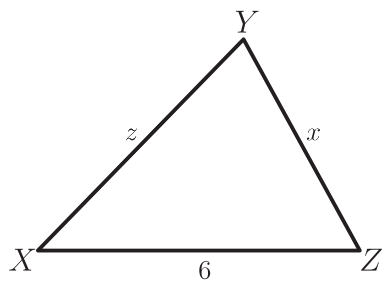
7. (a) Let $y = -16x^2 + 160x - 256$. Given that y has a maximum value, find:

(i) the value of x giving the maximum value of y

(ii) this maximum value of y.

The triangle XYZ has $XZ = 6$, $YZ = x$, $XY = z$ as shown. The perimeter of triangle XYZ is 16.



(b) (i) Express z in terms of x.

(ii) Using the cosine rule, express z^2 in terms of x and $\cos Z$.

(iii) Hence, show that $\cos Z = \frac{5x - 16}{3x}$.

Let the area of triangle XYZ be A.

(c) Show that $A^2 = 9x^2\sin^2 Z$.

(d) Hence, show that $A^2 = -16x^2 + 160x - 256$.

(e) (i) Hence, write down the maximum area for triangle XYZ.

(ii) What type of triangle is the triangle with maximum area?

[15 marks]

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8. Two circular cogs are connected by a chain as shown in diagram 1. The radii of the cogs are 3 cm and 8 cm and the distance between their centres is 25 cm.

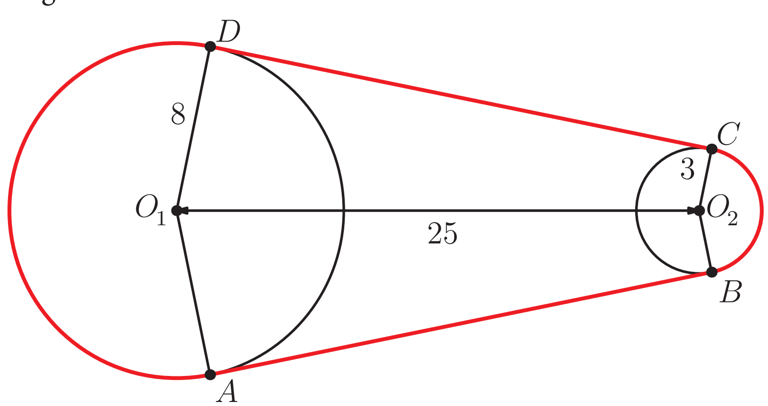


diagram 1

Diagram 2 shows the quadrilateral $O_1\hat{A}BO_2$. Line O_2P is drawn parallel to AB.

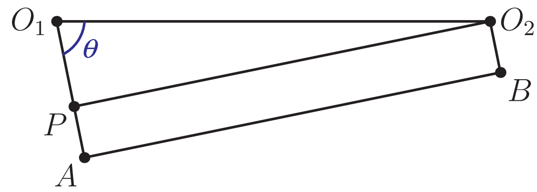


diagram 2

(a) Write down the size of $O_1\hat{A}B$ in radians, giving a reason for your answer.

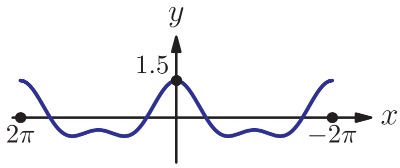
(b) Explain why $PO_2 = AB$.

(c) Hence find the length AB.

(d) Find the size of the angle marked θ, giving your answer in radians correct to 4 significant figures.

(e) Calculate the length of the chain $ABCD$. *[12 marks]*

2. (a)



(b) 2π
(c) $0, \pm\pi, \pm 2\pi$
(d) 1.2
(e) (ii) $2\pi - x_0$

3. (a) $k = \pm 4$
(c) (i) 1
(ii) $\pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$
(iii) $k = 5$
(iv) 7

Chapter 11

Exercise 11B

1. (a) (i) 6.04 (ii) 14.4
(b) (i) 10.6 cm (ii) 23.3 cm
2. (a) (i) 49.7° (ii) 59.2°
(b) (i) 74.6° or 105° (ii) 62.0° or 118°
(c) (i) 50.9° (ii) 54.4°
3. 21.0°, 29.0°, 8.09 cm
4. 10.4 cm, 49.9°, 95.1°; 269 cm, 130°, 15°
5. 9.94 cm

Exercise 11C

1. (a) (i) 5.37 (ii) 3.44
(b) (i) 8.00 (ii) 20.5
2. (a) (i) 60.6° (ii) 120°
(b) (i) 81.5° (ii) 100°
3. (i) 106° (ii) 36.2°
4. 6.12 km
5. 7.95
6. 4.4
7. $2\sqrt{2} + \sqrt{41}$

Exercise 11D

1. (a) (i) 10.7 cm^2 (ii) 24.3 cm^2
(b) (i) 27.6 cm^2 (ii) 26.2 cm^2
2. (a) 81.7 (b) 60.9
3. 17.7 cm, 29.7 cm^2
4. $4\sqrt{3}$ cm^2

Exercise 11E

1. (a) $\sqrt{134}$ cm (b) $4\sqrt{6}$ cm
2. A = 59.7°, B = 47.5°, C = 72.3°, Area = 85.6 cm^2
3. 62.5°
4. $4\sqrt{11} = 13.3$ cm
5. (a) 12.0 cm (b) 17.0 cm
6. (a) 18.8 cm (b) 23.1 m
7. (a) $RA = \frac{h}{\tan\alpha}$ $RB = \frac{h}{\tan\beta}$
(b) 13 m

Exercise 11F

1. (a) 7.8 cm (b) 1.8 cm
2. (a) 82.2 cm (b) 6.84 cm
3. 25 cm
4. (a) 0.938 (b) 53.7°
5. 2.53
6. 7.5 cm
7. 6.69 cm
8. 15.7 cm
9. 31.6 cm
10. $\left(\frac{25\pi}{6} + 10\right)$ cm
11. 5 cm
12. $\frac{6\pi}{5}$

Exercise 11G

1. (a) 16.25 cm^2 (b) 0.072 cm^2
2. (a) 463 cm^2 (b) 4.79 cm^2
3. 0.8
4. 167°
5. 9.49 cm

6. 11.3 cm

7. 5.14 cm^2

8. 48.4 cm^2

9. 2 cm or 1.5 cm

10. 2.54

Exercise 11H

1. (a) (i) 0.935 cm (ii) 3.39 cm
   (b) (i) 21.7 cm (ii) 15.8 cm

2. (a) (i) 1.89 cm (ii) 6.99 cm
   (b) (i) 52.5 cm (ii) 37.1 cm

3. (a) (i) 0.0595 cm^2 (ii) 1.21 cm^2
   (b) (i) 149 cm^2 (ii) 70.1 cm^2

4. (a) $12.5(\theta - \sin\theta)$ (b) 2.08

5. (b) 70.1° 3.67 cm^2

Mixed examination practice 11

Short questions

1. (a) $\frac{\pi}{3}$ (b) 28.9 cm^2
   (c) 23.8 cm

2. 80 cm^2

3. 58.7°, 121°

4. (a) 8.09 m (b) 6.58 m

5. (a) 10.2 cm (b) 18.8 cm

6. $2\sqrt{43}$

7. 4 cm or 13 cm

8. 12.3 cm^2

9. (a) 1.14 cm^2 (b) 2.00 cm^2

10. (a) $\pi - 2\theta$
    (b) $54 - 2\pi$ cm^2

11. 7.23 cm^2

12. (a) $\cos B = \frac{23}{32}$
    (b) $\sin B = \frac{3\sqrt{55}}{32}$
    (c) $\frac{15\sqrt{55}}{4}$ cm^2

13. (b) 7

Long questions

1. (a) $MB^2 = \left(\frac{x}{2}\right)^2 - 5x\cos\theta + 25$
   (c) 41.4°

2. (b) $\sqrt{2}r$
   (c) $\frac{\pi r^2}{4}$
   (d) $\left(\frac{\pi}{2} - 1\right)r^2$

3. (b) $\frac{1}{2}r^2(2\pi - \theta)$
   (d) 2.50

4. (a) $\frac{4.42}{3x^2}$
   (b) $\frac{3x^2 - 2x - 3}{2x^2}$
   (c) (ii) $x = 1.24, \theta = 1.86$
   $x = 2.94, \theta = 0.171$

5. (b) $-1 \le \cos\theta < \frac{-\sqrt{3}}{2}$ or $\frac{\sqrt{3}}{2} < \cos\theta \le 1$
   (c) $0 < \theta < \frac{\pi}{6}$ or $\frac{5\pi}{6} < \theta < \pi$

6. (a) (ii) $\sqrt{x^2 + 100}$
   (c) 38.7°
   (d) 5.63
   (e) (ii) $\frac{40}{3}$

7. (a) (i) 5 (ii) 144
   (b) (i) $z = 10 - x$
   (ii) $z^2 = x^2 + 36 - 12x\cos Z$
   (e) (i) 12
   (ii) Isosceles

8. (a) $\frac{\pi}{2}$, right angle between a tangent and a radius
   (b) ABO_2P is a rectangle, because there are right angles at A and B, and AB is parallel to PO_2.
   (c) 24.5 cm
   (d) 1.369
   (e) 85.6 cm

Chapter 12

Exercise 12A

1. (a) (i) $-\frac{7}{8}$ (ii) $\frac{1}{9}$