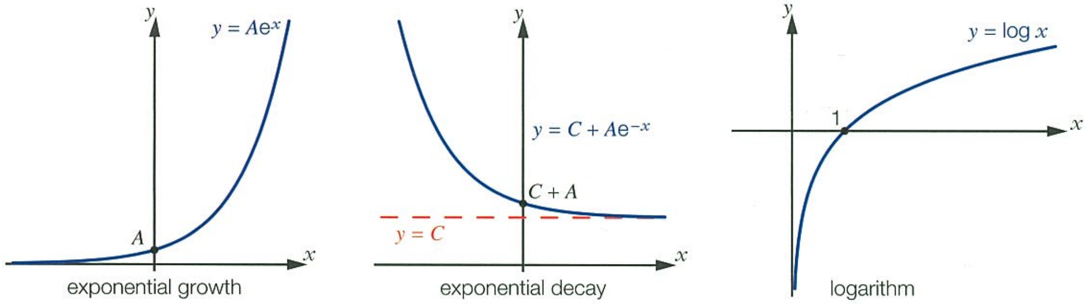
2 EXPONENTS AND LOGARITHMS

WHAT YOU NEED TO KNOW

- The rules of exponents:
  - $a^m \times a^n = a^{m+n}$
  - $\frac{a^m}{a^n} = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
  - $a^{-n} = \frac{1}{a^n}$
  - $a^n \times b^n = (ab)^n$
  - $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$
- The relationship between exponents and logarithms:
  - $a^x = b \Leftrightarrow x = \log_a b$ where a is called the base of the logarithm
  - $\log_a a^x = x$
  - $a^{\log_a x} = x$
- The rules of logarithms:
  - $\log_a x + \log_a y = \log_a xy$
  - $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
  - $k \log_a x = \log_a x^k$
  - $\log_a \left(\frac{1}{x}\right) = -\log_a x$
  - $\log_a 1 = 0$
- The change of base rule: $\log_b a = \frac{\log_c a}{\log_c b}$
- There are two common abbreviations for logarithms to particular bases:
  - $\log_{10} x$ is often written as $\log x$
  - $\log_e x$ is often written as $\ln x$

- The graphs of exponential and logarithmic functions:



EXAM TIPS AND COMMON ERRORS

- You must know what you *cannot* do with logarithms:
  - $\log(x + y)$ cannot be simplified; it is **not** $\log x + \log y$
  - $\log(e^x + e^y)$ cannot be simplified; it is **not** $x + y$
  - $(\log x)^2$ is **not** $2\log x$, whereas $\log x^2 = 2\log x$
  - $e^{2+\log x} = e^2 e^{\log x} = e^2 x$ **not** $e^2 + x$

2.1 SOLVING EXPONENTIAL EQUATIONS

WORKED EXAMPLE 2.1

Solve the equation $4 \times 5^{x+1} = 3^x$, giving your answer in the form $\dfrac{\log a}{\log b}$.

$$\log(4 \times 5^{x+1}) = \log(3^x)$$
$$\Leftrightarrow \log 4 + \log(5^{x+1}) = \log(3^x)$$

Since the unknown is in the power, we take logarithms of each side.

We then use the rules of logarithms to simplify the expression. First use

$\log(ab) = \log a + \log b$

A common mistake is to say that $\log(4 \times 5^{x+1}) = \log 4 \times \log(5^{x+1})$.

$$\Leftrightarrow \log 4 + (x+1)\log 5 = x\log 3$$

We can now use $\log a^k = k\log a$ to get rid of the powers.

$$\Leftrightarrow \log 4 + x\log 5 + \log 5 = x\log 3$$
$$\Leftrightarrow x\log 3 - x\log 5 = \log 4 + \log 5$$
$$\Leftrightarrow x(\log 3 - \log 5) = \log 4 + \log 5$$

Expand the brackets and collect the terms containing x on one side.

$$\Leftrightarrow x = \frac{\log 4 + \log 5}{\log 3 - \log 5}$$
$$\Leftrightarrow x = \frac{\log 20}{\log\left(\frac{3}{5}\right)}$$

Use the rules of logarithms to write the solution in the correct form:

$\log a + \log b = \log ab$

$\log a - \log b = \log\left(\dfrac{a}{b}\right)$

Practice questions 2.1

1. Solve the equation $5^{3x+1} = 15$, giving your answer in the form $\dfrac{\log a}{\log b}$ where a and b are integers.

2. Solve the equation $3^{2x+1} = 4^{x-2}$, giving your answer in the form $\dfrac{\log p}{\log q}$ where p and q are rational numbers.

3. Solve the equation $3 \times 2^{x-3} = \dfrac{1}{5^{2x}}$, giving your answer in the form $\dfrac{\log p}{\log q}$ where p and q are rational numbers.

2.2 SOLVING DISGUISED QUADRATIC EQUATIONS

WORKED EXAMPLE 2.2

Find the exact solution of the equation $3^{2x+1} - 11 \times 3^x = 4$.

$$3^{2x+1} - 11 \times 3^x = 4$$
$$\Leftrightarrow 3 \times 3^{2x} - 11 \times 3^x = 4$$
$$\Leftrightarrow 3 \times (3^x)^2 - 11 \times 3^x = 4$$

We need to find a substitution to turn this into a quadratic equation.
First, express 3^{2x+1} in terms of 3^x:
$3^{2x+1} = 3^{2x} \times 3^1 = 3 \times (3^x)^2$

Look out for an a^{2x} term, which can be rewritten as $(a^x)^2$.

Let $y = 3^x$. Then

$$3y^2 - 11y - 4 = 0$$
$$\Leftrightarrow (3y+1)(y-4)$$
$$\Leftrightarrow y = -\frac{1}{3} \text{ or } y = 4$$

After substituting y for 3^x, this becomes a standard quadratic equation, which can be factorised and solved.

Disguised quadratic equations may also be encountered when solving trigonometric equations, which is covered in Chapter 6.

$$\therefore 3^x = -\frac{1}{3} \text{ or } 3^x = 4$$

Remember that $y = 3^x$.

$3^x = -\frac{1}{3}$ is impossible since $3^x > 0$ for all x.

With disguised quadratic equations, often one of the solutions is impossible.

$$3^x = 4$$
$$\Leftrightarrow \log 3^x = \log 4$$
$$\Leftrightarrow x \log 3 = \log 4$$
$$\Leftrightarrow x = \frac{\log 4}{\log 3}$$

Since x is in the power, we take logarithms of both sides. We can then use $\log a^k = k \log a$ to get rid of the power.

Practice questions 2.2

4. Solve the equation $2^{2x} - 5 \times 2^x + 4 = 0$.

5. Find the exact solution of the equation $e^x - 6e^{-x} = 5$.

6. Solve the simultaneous equations $e^{x+y} = 6$ and $e^x + e^y = 5$.

2.3 LAWS OF LOGARITHMS

WORKED EXAMPLE 2.3

If $x = \log a$, $y = \log b$ and $z = \log c$, write $2x + y - 0.5z + 2$ as a single logarithm.

$$2\log a + \log b - 0.5\log c + 2$$
$$= \log a^2 + \log b - \log c^{0.5} + 2$$

We need to rewrite the expression as a single logarithm. In order to apply the rules for combining logarithms, each log must have no coefficient in front of it. So we first need to use $k\log x = \log x^k$.

$$= \log a^2 b - \log c^{0.5} + 2$$
$$= \log\left(\frac{a^2 b}{\sqrt{c}}\right) + 2$$

We can now use $\log x + \log y = \log(xy)$ and $\log x - \log y = \log\left(\frac{x}{y}\right)$.

$$= \log\left(\frac{a^2 b}{\sqrt{c}}\right) + \log 100$$
$$= \log\left(\frac{100a^2 b}{\sqrt{c}}\right)$$

We also need to write 2 as a logarithm so that it can then be combined with the first term. Since $10^2 = 100$, we can write 2 as $\log 100$.

Remember that log on its own is taken to mean $\log_{10}$.

Practice questions 2.3

7. Given $x = \log a$, $y = \log b$ and $z = \log c$, write $3x - 2y + z$ as a single logarithm.

8. Given $a = \log x$, $b = \log y$ and $c = \log z$, find an expression in terms of a, b and c for $\log\left(\frac{10xy^2}{\sqrt{z}}\right)$

9. Given that $\log a + 1 = \log b^2$, express a in terms of b.

10. Given that $\ln y = 2 + 4\ln x$, express y in terms of x.

11. Consider the simultaneous equations
$e^{2x} + e^{y} = 800$
$3\ln x + \ln y = 5$
(a) For each equation, express y in terms of x.
(b) Hence solve the simultaneous equations.

2.4 SOLVING EQUATIONS INVOLVING LOGARITHMS

WORKED EXAMPLE 2.4

Solve the equation $4\log_4 x = 9\log_x 4$.

$$\log_x 4 = \frac{\log_4 4}{\log_4 x} = \frac{1}{\log_4 x}$$

Therefore

$$4\log_4 x = 9\log_x 4$$

$$\Leftrightarrow 4\log_4 x = 9 \times \frac{1}{\log_4 x}$$

We want to have logarithms involving just one base so that we can apply the rules of logarithms.

Here we use the change of base rule to turn logs with base x into logs with base 4. (Alternatively, we could have turned them all into base x instead.)

$$\Leftrightarrow 4(\log_4 x)^2 = 9$$

$$\Leftrightarrow (\log_4 x)^2 = \frac{9}{4}$$

Multiply through by $\log_4 x$ to get the log terms together.

Make sure you use brackets to indicate that the whole of $\log_4 x$ is being squared, not just x; $(\log_4 x)^2$ is **not** $2\log_4 x$, but $\log_4 x^2$ would be.

$$\log_4 x = \frac{3}{2} \quad \text{or} \quad \log_4 x = -\frac{3}{2}$$

Take the square root of both sides; don't forget the negative square root.

$$\text{So } x = 4^{\frac{3}{2}} \quad \text{or} \quad x = 4^{-\frac{3}{2}}$$

$$= 8 \qquad\qquad = \frac{1}{8}$$

Use $\log_a b = x \Leftrightarrow b = a^x$ to 'undo' the logs.

Practice questions 2.4

12. Solve the equation $\log_4 x + \log_4 (x-6) = 2$.

13. Solve the equation $2\log_2 x - \log_2 (x+1) = 3$, giving your answers in simplified surd form.

Make sure you check your answers by substituting them into the original equation.

14. Solve the equation $25 \log_2 x = \log_x 2$.

15. Solve the equation $\log_4 (4-x) = \log_{16} (9x^2 - 10x + 1)$.

2.5 PROBLEMS INVOLVING EXPONENTIAL FUNCTIONS

WORKED EXAMPLE 2.5

When a cup of tea is made, its temperature is 85°C. After 3 minutes the tea has cooled to 60°C. Given that the temperature T (°C) of the cup of tea decays exponentially according to the function $T = A + Ce^{-0.2t}$, where t is the time measured in minutes, find:

(a) the values of A and C (correct to three significant figures)

(b) the time it takes for the tea to cool to 40°C.

(a) When $t=0$: $85 = A + C \quad \cdots (1)$

When $t=3$: $60 = A + Ce^{-0.6} \quad \cdots (2)$

Substitute the given values for T (temperature) and t (time) into $T = A + Ce^{-0.2t}$, remembering that $e^0 = 1$.

$(1)-(2)$ gives $25 = (1-e^{-0.6})$

$$\text{So } C = \frac{25}{1-e^{-0.6}} = 55.4\ (3\text{SF})$$

Then, from (1),

$A = 85 - C = 85 - 55.4 = 29.6\ (3\text{SF})$

Note that A is the long-term limit of the temperature, which can be interpreted as the temperature of the room.

(b) When $T = 40$:

Now we can substitute for T, A and C.

$29.6 + 55.4e^{-0.2t} = 40$

$$\Rightarrow e^{-0.2t} = \frac{40-29.6}{55.4}$$

$$\Rightarrow \ln(e^{-0.2t}) = \ln\left(\frac{40-29.6}{55.4}\right)$$

$$\Rightarrow -0.2t = \ln\left(\frac{40-29.6}{55.4}\right)$$

$\Rightarrow t = 8.36$ minutes

Since the unknown t is in the power, we take logarithms of both sides and then 'cancel' e and ln using $\log_a(a^x) = x$.

Remember that ln means $\log_e$.

Practice questions 2.5

16. The amount of reactant, V (grams), in a chemical reaction decays exponentially according to the function $V = M + Ce^{-0.32t}$, where t is the time in seconds since the start of the reaction. Initially there was 4.5 g of reactant, and this had decayed to 2.6 g after 7 seconds.

(a) Find the value of C.

(b) Find the value that the amount of reactant approaches in the long term.

17. A population of bacteria grows according to the model $P = Ae^{kt}$, where P is the size of the population after t minutes. Given that after 2 minutes there are 200 bacteria and after 5 minutes there are 1500 bacteria, find the size of the population after 10 minutes.

Mixed practice 2

1. Solve the equation $3 \times 9^x - 10 \times 3^x + 3 = 0$.

2. Find the exact solution of the equation $2^{3x+1} = 5^{5-x}$.

3. Solve the simultaneous equations

$$\ln x^2 + \ln y = 15$$

$$\ln x + \ln y^3 = 10$$

4. Given that $y = \ln x - \ln(x+2) + \ln(x^2-4)$, express x in terms of y.

5. The graph with equation $y = 4\ln(x-a)$ passes through the point $(5, \ln 16)$. Find the value of a.

6. (a) An economic model predicts that the demand, D, for a new product will grow according to the equation $D = A - Ce^{-0.2t}$, where t is the number of days since the product launch. After 10 days the demand is 15 000 and it is increasing at a rate of 325 per day.

   (i) Find the value of C.

   (ii) Find the initial demand for the product.

   (iii) Find the long-term demand predicted by this model.

   (b) An alternative model is proposed, in which the demand grows according to the formula $D = B\ln\left(\frac{t+10}{5}\right)$. The initial demand is the same as that for the first model.

   (i) Find the value of B.

   (ii) What is the long-term prediction of this model?

   (c) After how many days will the demand predicted by the second model become larger than the demand predicted by the first model?

Going for the top 2

1. Find the exact solution of the equation $2^{3x-4} \times 3^{2x-5} = 36^{x-2}$, giving your answer in the form $\frac{\ln p}{\ln q}$ where p and q are integers.

2. Given that $\log_a b^2 = c^2$ and $\log_b a = c + 1$, express a in terms of b.

3. In a physics experiment, Maya measured how the force, F, exerted by a spring depends on its extension, x. She then plotted the values of $a = \ln F$ and $b = \ln x$ on a graph, with b on the horizontal axis and a on the vertical axis. The graph was a straight line, passing through the points (2, 4.5) and (4, 7.2). Find an expression for F in terms of x.