

Transformations of trigonometric functions

In this presentation we discuss transformations of trigonometric functions. In particular we will look at the most general forms of four trigonometric functions:

$$f(x) = a \sin(b(x - c)) + d$$

$$f(x) = a \cos(b(x - c)) + d$$

$$f(x) = a \tan(b(x - c)) + d$$

$$f(x) = a \cot(b(x - c)) + d$$

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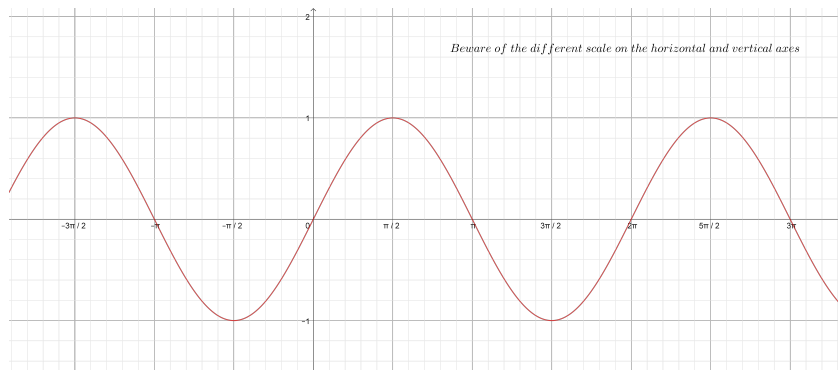
$$f(x) = a \cot(b(x - c)) + d$$

Before you start you need to be familiar with graphs of $\sin x$, $\cos x$, $\tan x$, $\cot x$ and transformations of function - in particular: translations, dilations and reflections.

We will start with a brief review of the graphs of the four trig functions.

Sine function

Graph:



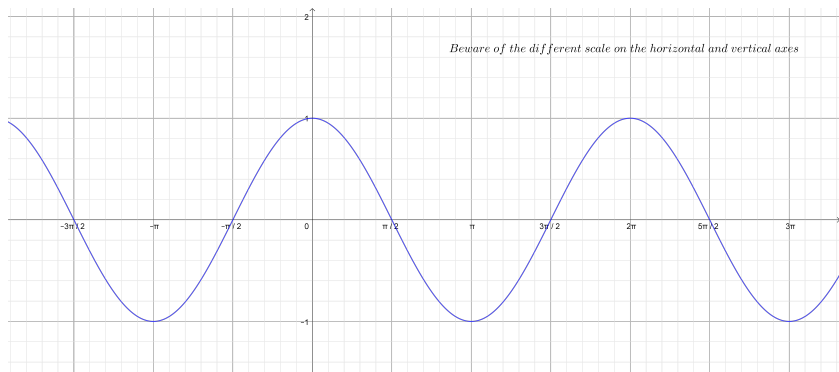
Properties:

Domain: $x \in \mathbb{R}$. Range $y \in [-1, 1]$. Period: 2π .

Zeros: $x = k\pi$, where $k \in \mathbb{Z}$.

Cosine function

Graph:



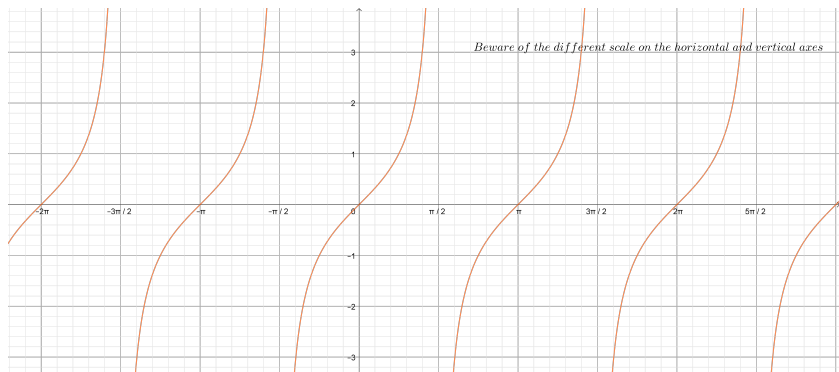
Properties:

Domain: $x \in \mathbb{R}$. Range $y \in [-1, 1]$. Period: 2π .

Zeroes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

Tangent function

Graph:



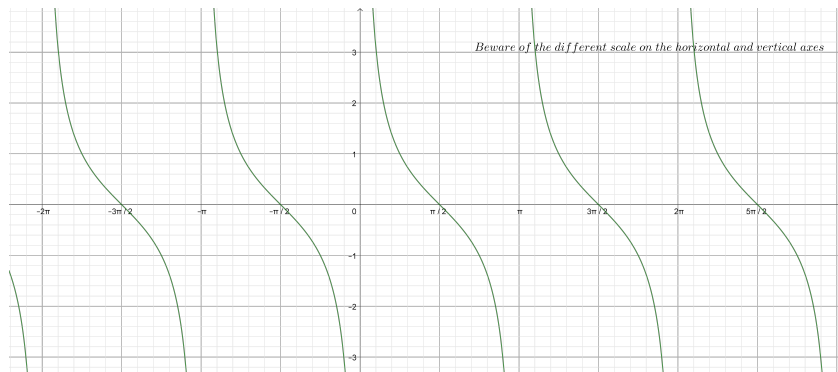
Properties:

Domain: $x \in \mathbb{R} - \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. Range $y \in \mathbb{R}$. Period: π .

Zeroes: $x = k\pi$, where $k \in \mathbb{Z}$. Asymptotes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

Cotangent function

Graph:



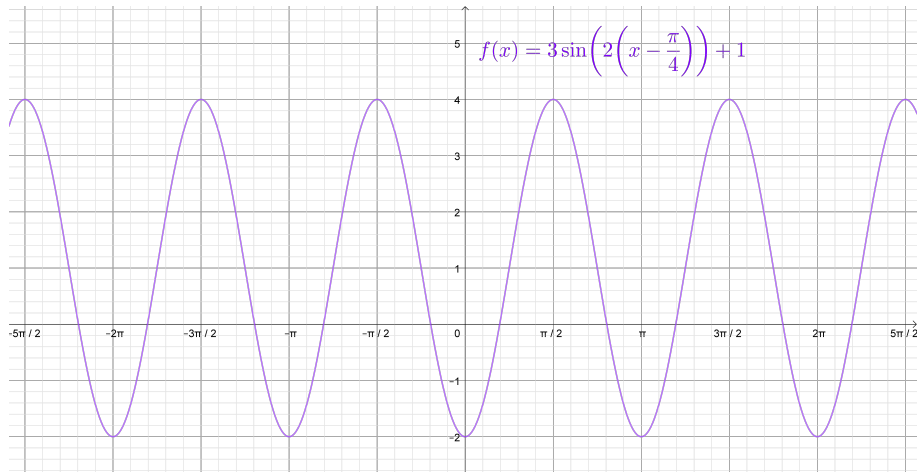
Properties:

Domain: $x \in \mathbb{R} - \{k\pi : k \in \mathbb{Z}\}$. Range $y \in \mathbb{R}$. Period: π .

Zeroes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$. Asymptotes: $x = k\pi$, where $k \in \mathbb{Z}$.

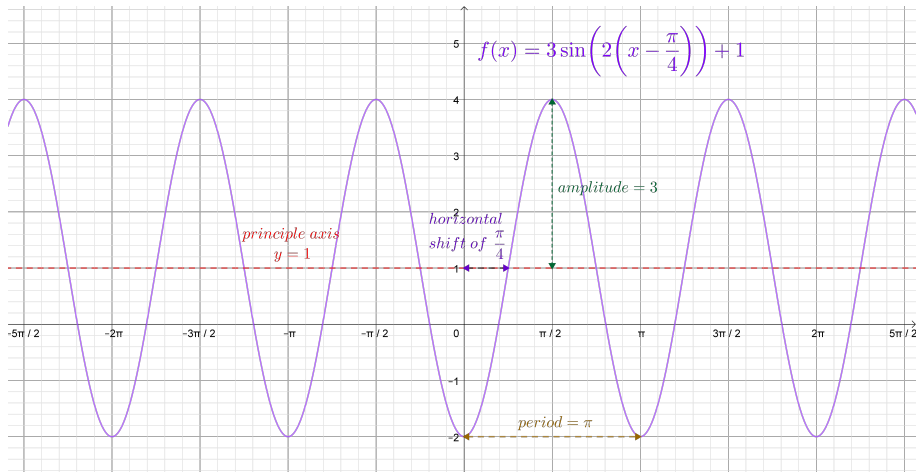
Intro to transformations of sine and cosine

Consider the following function and its graph:



Intro to transformations of sine and cosine

Now I will add some useful features of the graph on the diagram:



Intro to transformations of sine and cosine

The above feature help us deduce the equation from the graph.

If we have a function of the form $f(x) = a \sin(b(x - c)) + d$

- a corresponds to the amplitude, we need to be careful however. We have $|a| = \textit{amplitude}$ i.e. we need to check if the graph has been reflected.

Intro to transformations of sine and cosine

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- a corresponds to the amplitude, we need to be careful however. We have $|a| = \textit{amplitude}$ i.e. we need to check if the graph has been reflected.
- b corresponds to the period. Note that if $b = 2$, then the graph has been stretched (squeezed) horizontally by a factor of $\frac{1}{2}$. It is useful to use the following formula $\frac{\textit{old period}}{b} = \textit{new period}$, where the old period is the period of the original function, so in case of sine and cosine we have $\frac{2\pi}{b} = \textit{new period}$.

Intro to transformations of sine and cosine

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- c corresponds to the horizontal shift.

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- d corresponds to the principle axis (or "middle line").

Intro to transformations of sine and cosine

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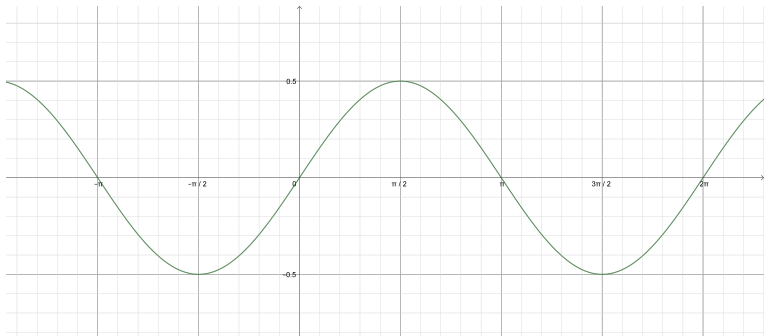
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- a corresponds to the amplitude, we need to be careful however. We have $|a| = \textit{amplitude}$ i.e. we need to check if the graph has been reflected.
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- c corresponds to the horizontal shift.
- d corresponds to the principle axis (or "middle line"). An easy way to find the middle line is to find the average of *max* and *min*.

We will now practice deducing the equation from the graph.

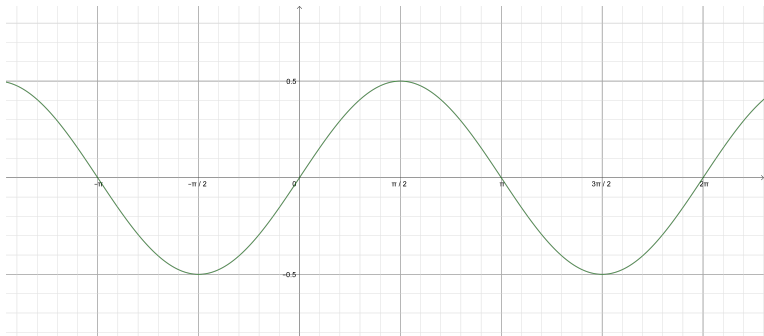
Example 1

The graph of the function $f(x) = a \sin x$ is shown below. Find the value of a .



Example 1

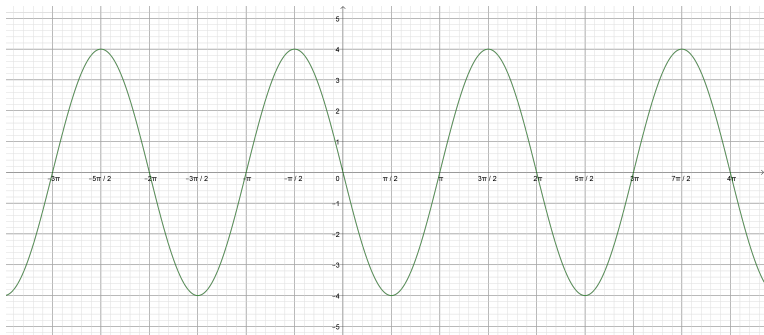
The graph of the function $f(x) = a \sin x$ is shown below. Find the value of a .



The amplitude of the graph is $\frac{1}{2}$, the graph has not been reflected in the x -axis, so $a = \frac{1}{2}$.

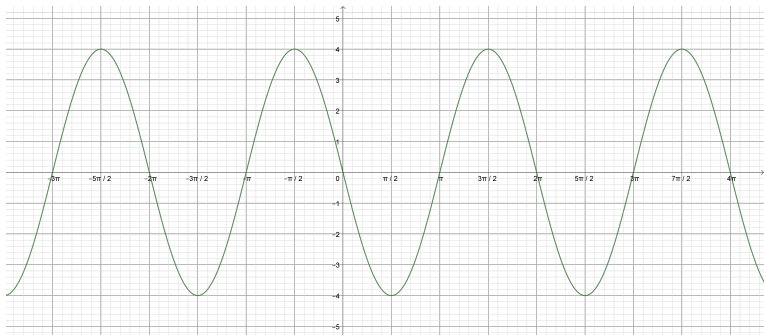
Example 2

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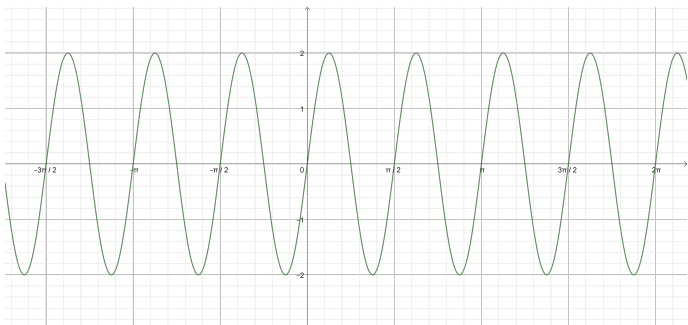
The graph of the function $f(x) = a \sin x$ is shown below. Find the value of a .



The amplitude of the graph is 4, this time the graph has been reflected in the x -axis, so $a = -4$. (To see this note that if we start at the origin and move right, the graph of sine goes up, here the graph goes down.)

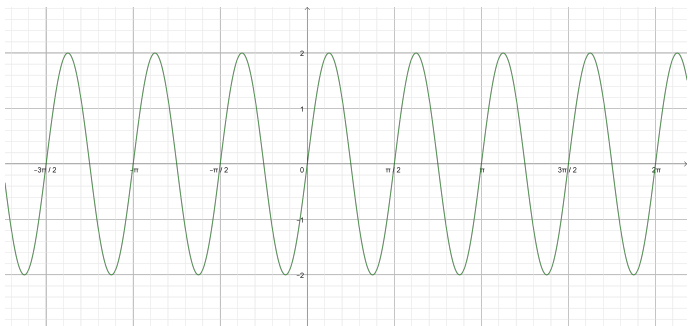
Example 3

The graph of the function $f(x) = a \sin bx$ is shown below. Find the values of a and b .



Example 3

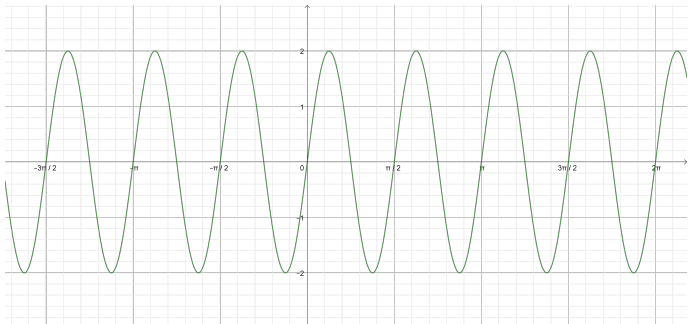
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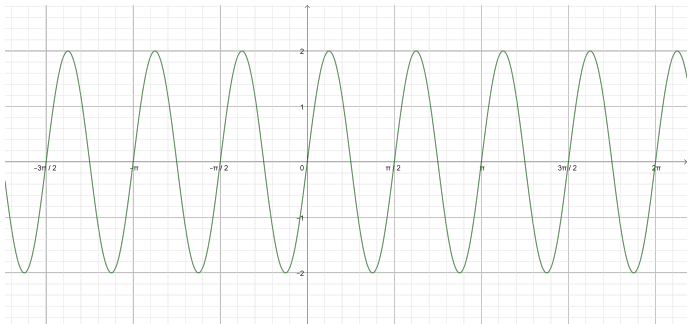
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The amplitude of the graph is 2, the graph has not been reflected in the x -axis, so $a = 2$. The period of the new function is $\frac{\pi}{2}$. You can see this by looking at the zeroes of the function. This means that the graph of sine has been stretched by a factor of $\frac{1}{4}$, so $b = 4$.

Example 3

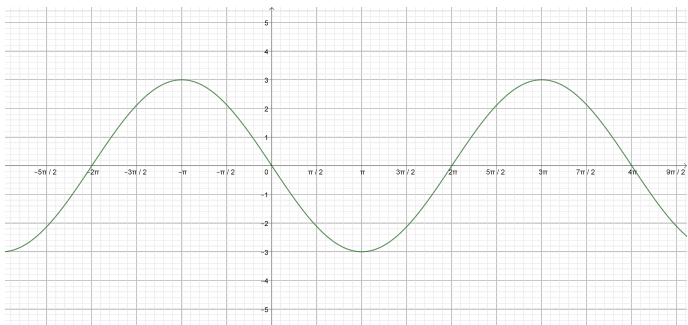
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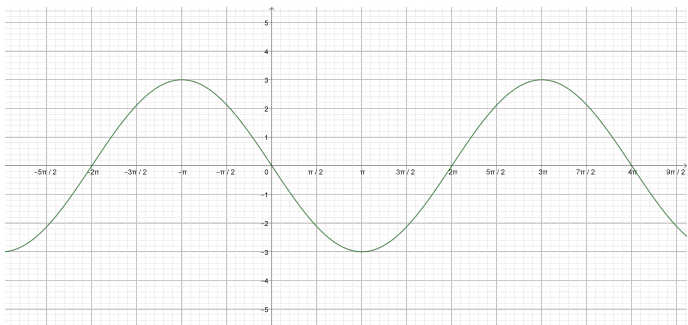
Example 4

The graph of the function $f(x) = a \sin bx$ is shown below. Find the values of a and b .



Example 4

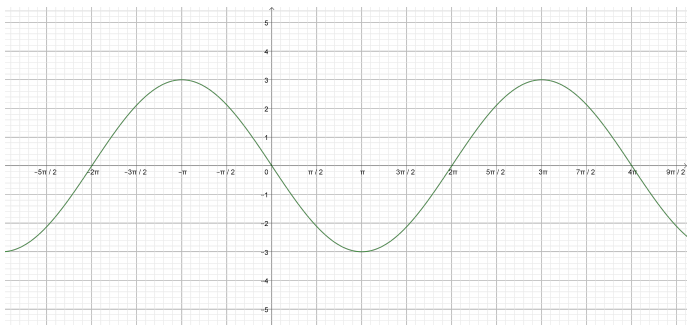
The graph of the function $f(x) = a \sin bx$ is shown below. Find the values of a and b .



The amplitude of the graph is 3, the graph is reflected in the x -axis, so $a = -3$.

Example 4

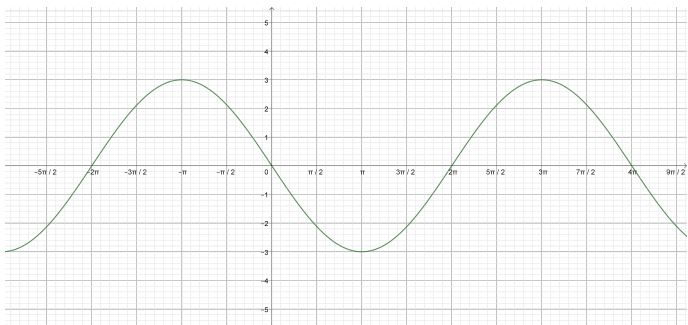
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The amplitude of the graph is 3, the graph is reflected in the x -axis, so $a = -3$. The period of the new function is 4π . You can see this by looking at the zeroes or the peaks of the function. This means that the graph of sine has been stretched by a factor of 2, so $b = \frac{1}{2}$.

Example 4

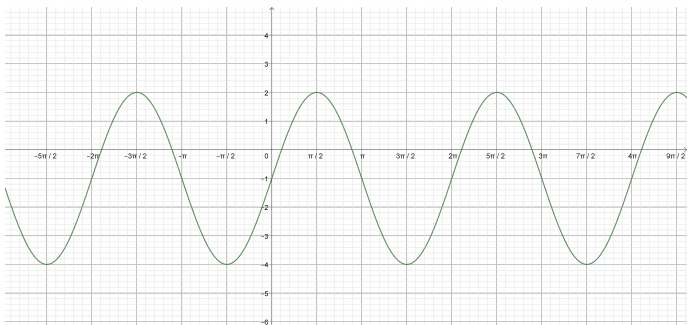
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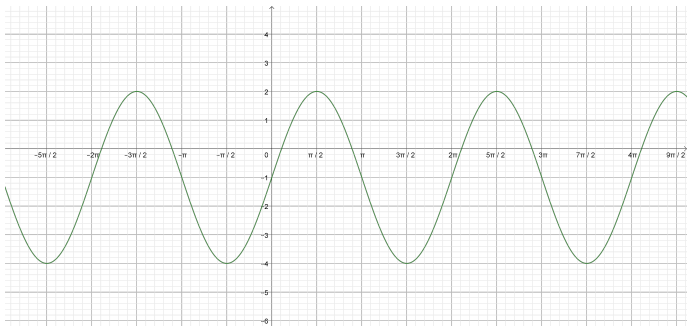
Example 5

The graph of the function $f(x) = a \sin x + d$ is shown below. Find the values of a and d .



Example 5

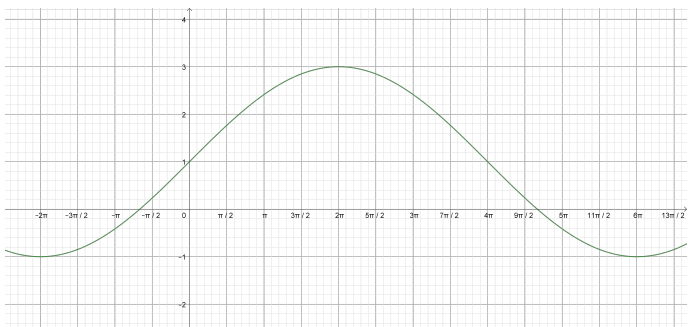
The graph of the function $f(x) = a \sin x + d$ is shown below. Find the values of a and d .



The principle axis is $y = -1$, so $d = -1$. The amplitude is 3 and the graph has not been reflected so $a = 3$.

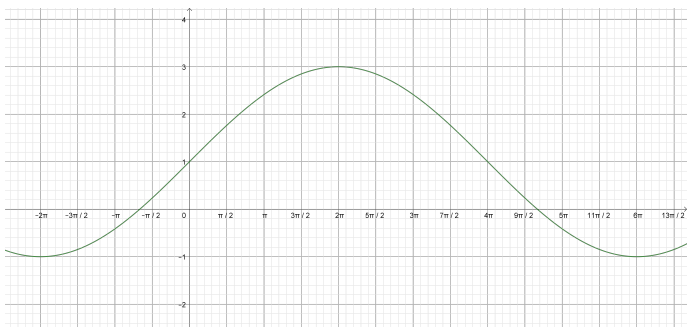
Example 6

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a , b and d .



Example 6

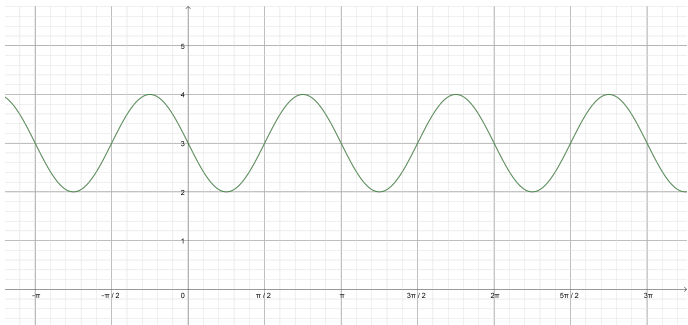
The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a , b and d .



The principle axis is $y = 1$, so $d = 1$. The amplitude is 2 and the graph has not been reflected so $a = 2$. The period is 8π (we can see clearly that half of the period is 4π). This means that the graph has been stretched by a factor of 4, so $b = \frac{1}{4}$.

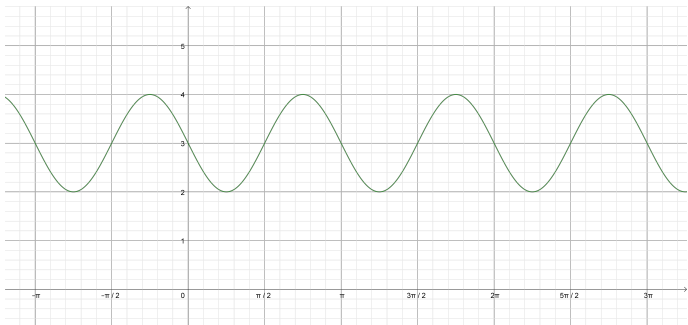
Example 7

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a , b and d .



Example 7

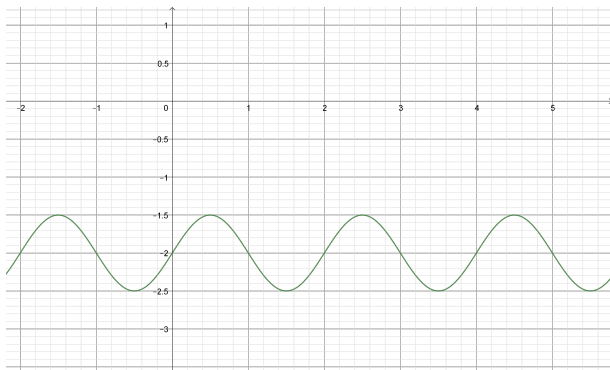
The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a , b and d .



The principle axis is $y = 3$, so $d = 3$. The amplitude is 1 but the graph has been reflected so $a = -1$. The period is π so the graph has been stretched by a factor of $\frac{1}{2}$, so $b = 2$.

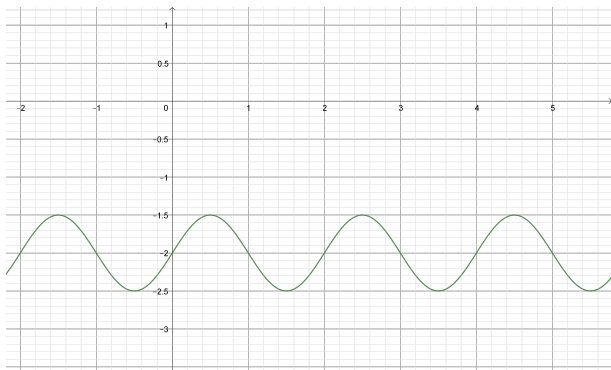
Example 8

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a , b and d .



Example 8

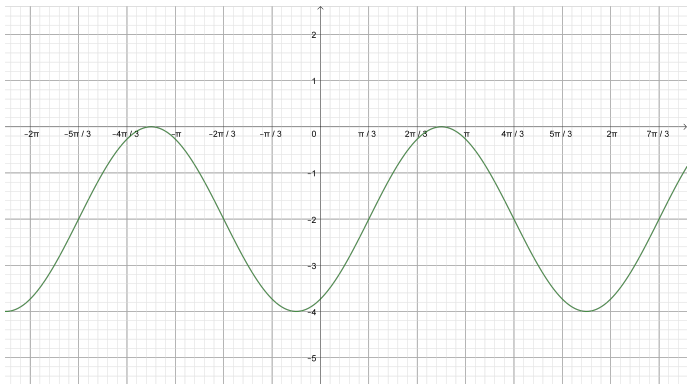
The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a , b and d .



The principle axis is $y = -2$, so $d = -2$. The amplitude is $\frac{1}{2}$, the graph is not reflected so $a = \frac{1}{2}$. The period is 2, it maybe less obvious how the graph was stretched, so let's go straight to the formula $\frac{2\pi}{b} = 2$ and we get that $b = \pi$.

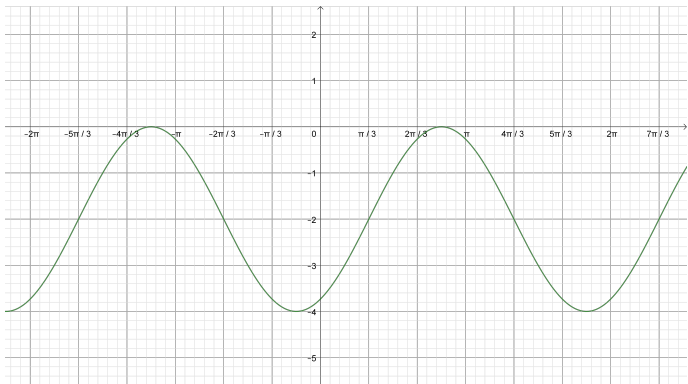
Example 9

The graph of the function $f(x) = a \sin(x - c) + d$ is shown below. Find the values of a , c and d .



Example 9

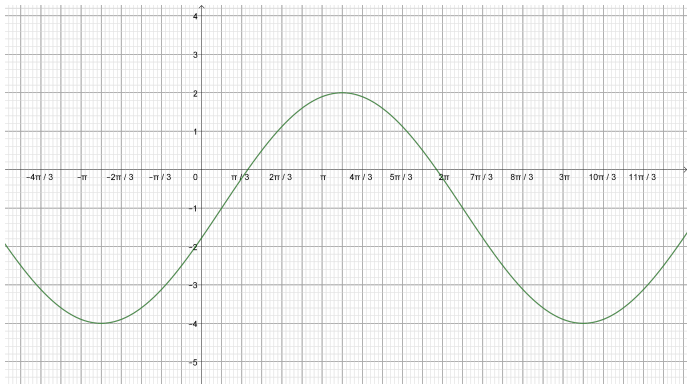
The graph of the function $f(x) = a \sin(x - c) + d$ is shown below. Find the values of a , c and d .



The principle axis is $y = -2$, so $d = -2$. The amplitude is 2, the graph is not reflected so $a = 2$. The graph has been shifted to the right by $\frac{\pi}{3}$, so $c = \frac{\pi}{3}$ (note the negative sign in front of c in the equation).

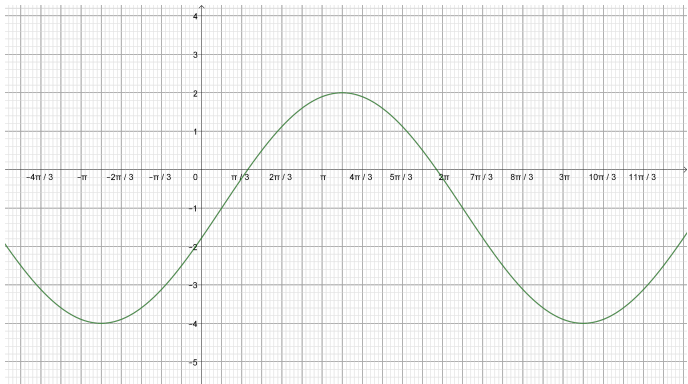
Example 10

The graph of the function $f(x) = a \sin(b(x - c)) + d$ is shown below. Find the values of a , b , c and d .



Example 10

The graph of the function $f(x) = a \sin(b(x - c)) + d$ is shown below. Find the values of a , b , c and d .



The principle axis $y = -1$, so $d = -1$. Amplitude is 3, not reflected so $a = 3$. Period is 4π so $b = \frac{1}{2}$. Horizontal shift is $\frac{\pi}{6}$ units to the right so $c = \frac{\pi}{6}$ (again because we're subtracting c from x).

You may have noticed that the previous example had more possible answers.

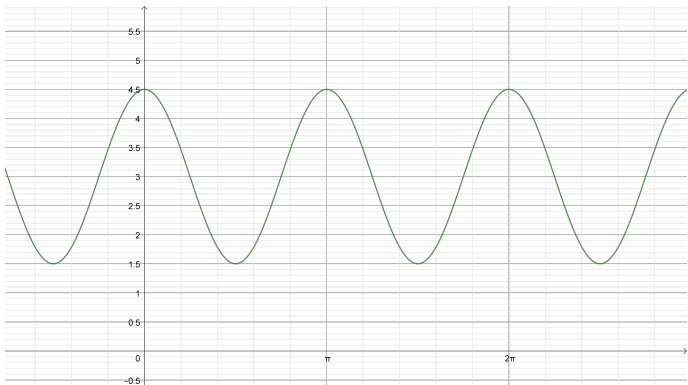
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Now we move on to a few examples on cosine functions, but these are very similar in nature.

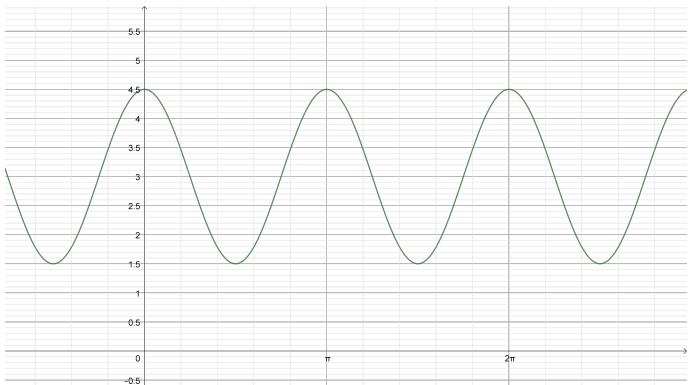
Example 11

The graph of the function $f(x) = a \cos bx + d$ is shown below. Find the values of a , b and d .



Example 11

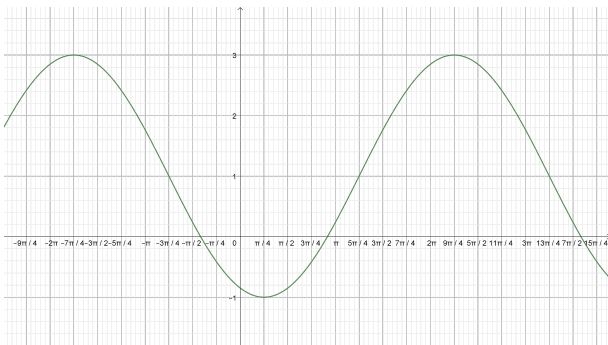
The graph of the function $f(x) = a \cos bx + d$ is shown below. Find the values of a , b and d .



The principle axis is $y = 3$, so $d = 3$. The amplitude is $\frac{3}{2}$, the graph has not been reflected (if we go right from the y -axis, the cosine function starts at 1 and goes down, our function also goes down) so $a = \frac{3}{2}$. The period is π , so $b = 2$.

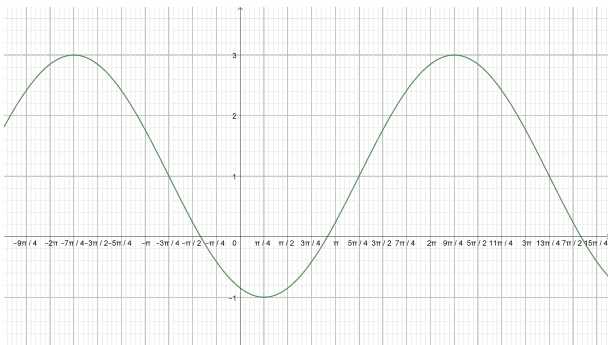
Example 12

The graph of the function $f(x) = a \cos(b(x - c)) + d$ is shown below. Find the values of a , b , c and d .



Example 12

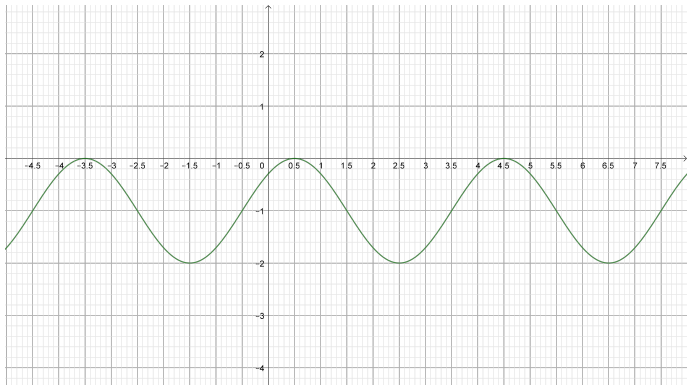
The graph of the function $f(x) = a \cos(b(x - c)) + d$ is shown below. Find the values of a , b , c and d .



The principle axis is $y = 1$, so $d = 1$. The amplitude is 2, but the graph has been reflected so $a = -2$. The period is 4π (we can see that half the period is 2π), so $b = \frac{1}{2}$. The graph has been shifted $\frac{\pi}{4}$ units to the right, so $c = \frac{\pi}{4}$. You can look at the bottom peak to see this.

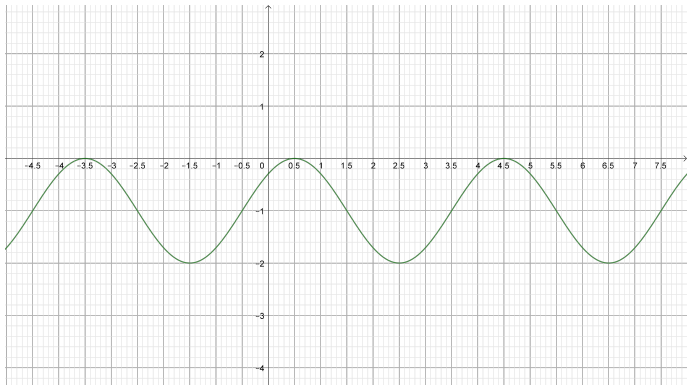
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The principle axis is $y = -1$, so $d = -1$. The amplitude is 1, the graph has not been reflected so $a = 1$. The period is 4, we solve $\frac{2\pi}{b} = 4$ to get $b = \frac{\pi}{2}$. Finally the graph has been shifted 0.5 units to the right, so $c = \frac{1}{2}$.

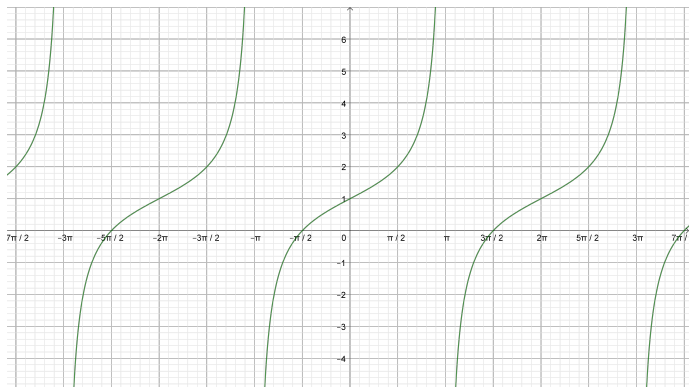
Again notice that the last two examples had multiple solutions. It's a good practice to come up with other solutions, but make sure that you check your answers (by drawing appropriate graph on for instance desmos.com)

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We now turn to tangent and cotangent functions.

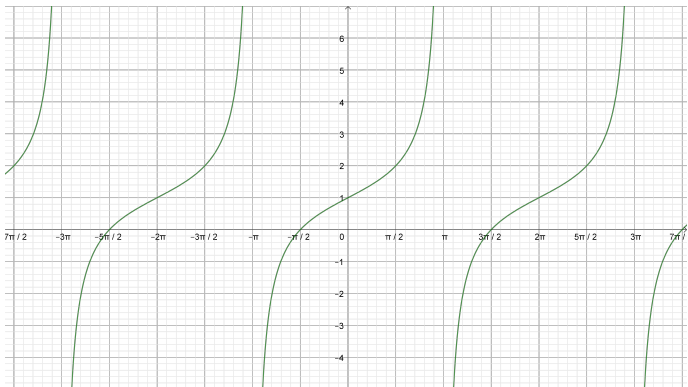
Example 14

The graph of the function $f(x) = \tan(bx) + d$ is shown below. Find the values of b and d .



Example 14

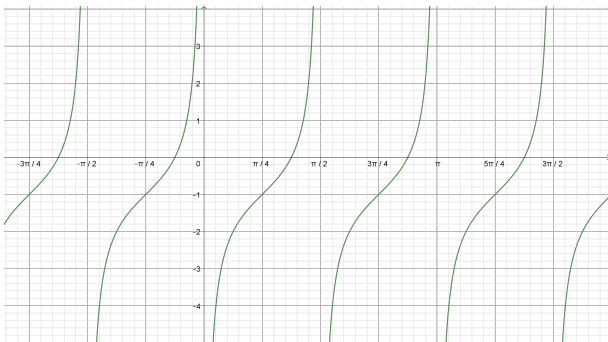
The graph of the function $f(x) = \tan(bx) + d$ is shown below. Find the values of b and d .



This may be less obvious, but still we can identify the vertical translation one unit upwards, so $d = 1$. The period is 2π , the period of $\tan x$ is π , so the it has been stretched by a factor of 2, which means that $b = \frac{1}{2}$.

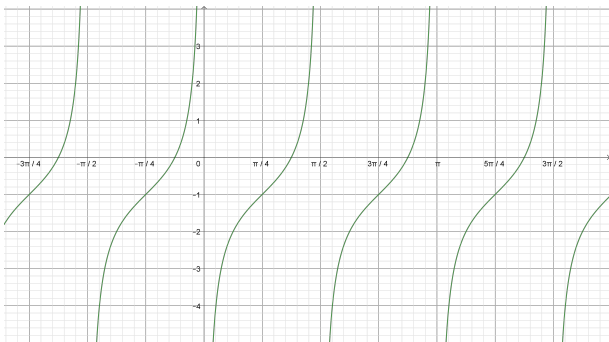
Example 15

The graph of the function $f(x) = \tan(b(x - c)) + d$ is shown below. Find the values of b , c and d .



Example 15

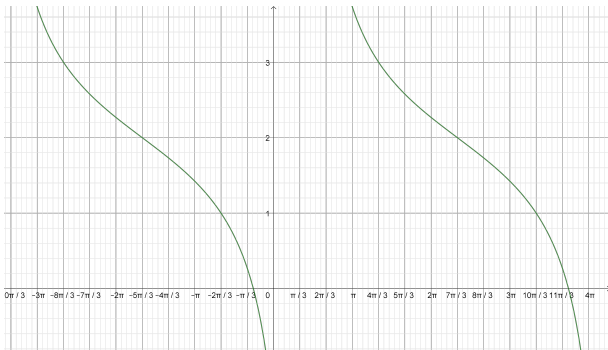
The graph of the function $f(x) = \tan(b(x - c)) + d$ is shown below. Find the values of b , c and d .



The period is $\frac{\pi}{2}$, so the graph of $\tan x$ has been stretched by a factor of $\frac{1}{2}$, which means that $b = 2$. Now we can see the translation by a vector $\begin{pmatrix} \frac{\pi}{4} \\ -1 \end{pmatrix}$, so $c = \frac{\pi}{4}$ and $d = -1$.

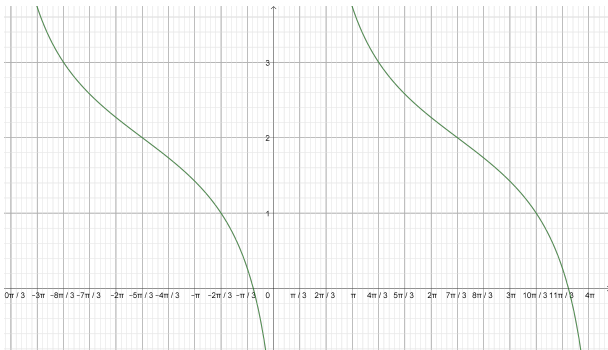
Example 16

The graph of the function $f(x) = \cot(b(x - c)) + d$ is shown below. Find the values of b , c and d .



Example 16

The graph of the function $f(x) = \cot(b(x - c)) + d$ is shown below. Find the values of b , c and d .



The period is 4π , so the graph of $\cot x$ has been stretched by a factor of 4, which means that $b = \frac{1}{4}$. By looking at the image of the point

$(\frac{\pi}{2}, 0) \rightarrow (\frac{7\pi}{3}, 2)$, we can identify the translation by a vector $\begin{pmatrix} \frac{\pi}{3} \\ 2 \end{pmatrix}$, so

$c = \frac{\pi}{3}$ and $d = 2$.

In case of any questions you can email me at T.J.Lechowski@gmail.com.