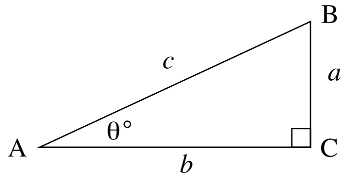
9.1 TRIGONOMETRIC RATIOS

9.1.1 REVIEW OF TRIGONOMETRIC FUNCTIONS FOR RIGHT-ANGLED TRIANGLES

The trigonometric functions are defined as **ratio functions** in a right-angled triangle. As such they are often referred to as the **trigonometric ratios**.

The trigonometric ratios are based on the right-angled triangle shown alongside. Such right-angled triangles are defined in reference to a nominated angle. In the right-angled triangle ABC the longest side [AB] (opposite the right-angle) is the **hypotenuse**. Relative to the angle $\angle BAC$ of size $\theta°$, the side BC is called the **opposite** side while the side AC is called the **adjacent** side.



The trigonometric ratios are defined as

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}, \quad \cos\theta° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}, \quad \tan\theta° = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$

Note then, that $\tan\theta° = \frac{\sin\theta°}{\cos\theta°}, \cos\theta \neq 0$.

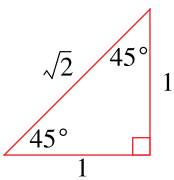
There also exists another important relation between the side lengths of a right-angled triangle.

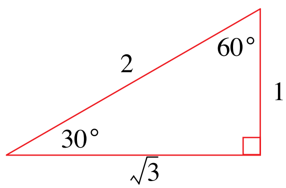
This relationship, using **Pythagoras's Theorem** is $a^2 + b^2 = c^2$

Do not forget to adjust the mode of your calculator to degree mode when necessary. On the TI–83, this is done by pressing **MODE** and then selecting the **Degree** mode. As angles can be quoted in degrees '°', minutes '′' and seconds '″' we make use of the **DMS** option under the **ANGLE** menu (accessed by pressing **2nd APPS**) to convert an angle quoted as a decimal into one quoted in degrees, minutes and seconds.

9.1.2 EXACT VALUES

There are a number of special right-angled triangles for which exact values of the trigonometric ratios exist. Two such triangles are shown below:





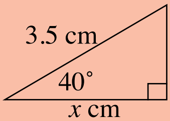
From these triangles we can tabulate the trigonometric ratios as follows:

θ	$\sin\theta°$	$\cos\theta°$	$\tan\theta°$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

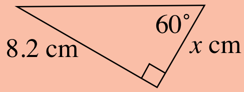
EXAMPLE 9.1

Find x in each of the following triangles (correct to 4 d.p)

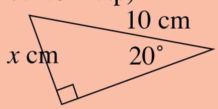
(a)



(b)

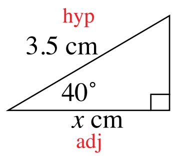


(c)



Solution

(a) We label the sides relative to the given angle:



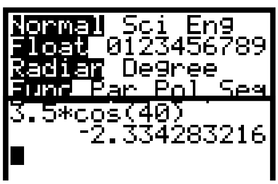
As the sides involved are the adjacent (adj) and the hypotenuse (hyp). The appropriate ratio is the cosine ratio, i.e., $\cos\theta = \frac{\text{adj}}{\text{hyp}}$. Then, substituting the information into the expression we can solve for x:

$$\cos 40° = \frac{x}{3.5} \Leftrightarrow 3.5 \times \cos 40° = x$$

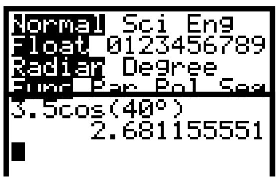
$$\therefore x = 2.6812 \text{ (to 4 d.p)}$$

A quick word about using the TI–83. Below we show that depending on the mode setting we obtain different values. In particular, note that in Case B, even though the mode setting was in radians, we were able to over ride this by using the degree measure, '°', under the **ANGLE** menu.

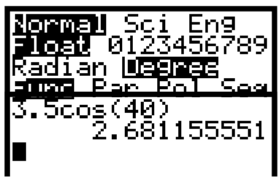
Case A:



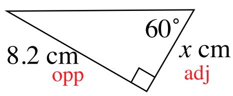
Case B:



Case C:



(b) We label the sides relative to the given angle.

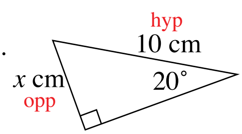


The sides involved are the adjacent (adj) and the opposite (opp) the appropriate ratio is the tangent ratio, i.e., $\tan\theta = \frac{\text{opp}}{\text{adj}}$. Then, substituting the information into the expression we can solve for x:

$$\tan 60° = \frac{8.2}{x} \Leftrightarrow x\tan 60° = 8.2 \Leftrightarrow x = \frac{8.2}{\tan 60°}$$

$$\therefore x = 4.7343 \text{ (to 4 d.p)}$$

(c) We label the sides relative to the given angle.



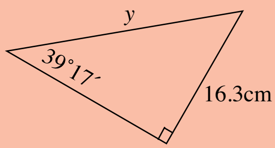
The sides involved are the opposite (opp) and the hypotenuse (hyp). The appropriate ratio is the sine ratio, i.e., $\sin\theta = \frac{\text{opp}}{\text{hyp}}$. Then, substituting the information into the expression we can solve for x:

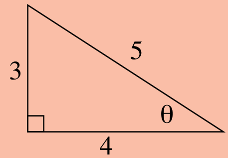
$$\sin 20° = \frac{x}{10} \Leftrightarrow 10 \times \sin 20° = x$$

$$\therefore x = 3.4202 \text{ (to 4 d.p)}$$

EXAMPLE 9.2

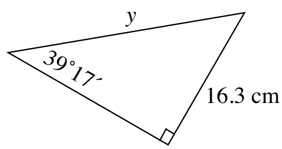
Find x and θ in the following triangles

(a) 

(b) 

Solution

(a) The important sides are the opposite and hypotenuse. So,



$$\sin 39°17' = \frac{16.3}{y} \Leftrightarrow y \times \sin 39°17' = 16.3$$

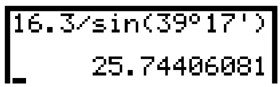
$$\Leftrightarrow y = \frac{16.3}{\sin 39°17'}$$

$$\therefore y = 25.7 \text{ cm}$$

The TI–82/83 calculators accept angle inputs using the **2nd ANGLE** menu. **Option 1** allows entry of angles in degrees irrespective of the **MODE** setting of the calculator. **Option 2** allows the entry of degrees, minutes, seconds.

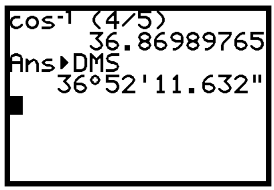
The problem would be solved using the keying sequence

16.3÷sin 39 **2nd ANGLE 1** 17 **2nd ANGLE 2** ENTER.



(b) When using a calculator to find an angle, **option 4** of the **2nd ANGLE** menu will allow you to display an answer in degree, minute, second format.

Any of the three trigonometric ratios will do, but when finding angles, it is generally best to use the cosine ratio. The reason for this should become apparent as this chapter progresses.



$$\cos\theta = \frac{4}{5} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{5}\right) \therefore \theta \approx 36°52'12''$$

rounded to the nearest second.

EXAMPLE 9.3

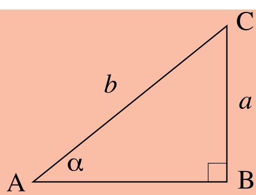
Using the triangle shown, find

(a) i. AB

ii. $\cos\alpha$

iii. $\tan\alpha$

(b) If $\cos\alpha = 0.2$ find $\sin(90° - \alpha)$.



Solution

(a) Using Pythagoras's Theorem we have $AC^2 = AB^2 + BC^2 \therefore b^2 = AB^2 + a^2$

$$\Leftrightarrow AB^2 = b^2 - a^2$$

$$\Rightarrow AB = \sqrt{b^2 - a^2}$$

(b) $\cos\alpha = \dfrac{AB}{AC} = \dfrac{\sqrt{b^2 - a^2}}{b}$

(c) $\tan\alpha = \dfrac{CB}{AB} = \dfrac{a}{\sqrt{b^2 - a^2}}$

(d) $\sin(90° - \alpha) = \dfrac{AB}{AC}$, but $\cos\alpha = \dfrac{AB}{AC} \therefore \sin(90° - \alpha) = \cos\alpha$.

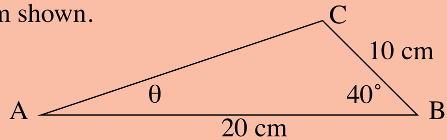
i.e., $\sin(90° - \alpha) = 0.2$

Often we are dealing with non right-angled triangles. However, these can be 'broken up' into at least two right-angled triangles, which then involves solving simultaneous equations. This is illustrated in the next example.

EXAMPLE 9.4

Find the angle θ in the diagram shown.

Note that $\angle ACB \neq 90°$.

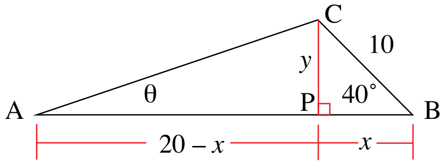


Solution

We start by 'breaking-up' the triangle into two right-angled triangles as follows:

Using ΔACP:

$$\tan\theta = \frac{PC}{AP} = \frac{y}{20 - x} \quad -(1)$$



We now need to determine x and y.

Using ΔBPC:

$$\sin 40° = \frac{PC}{BC} = \frac{y}{10}$$

$$\Leftrightarrow y = 10\sin 40° \quad -(2)$$

and $$\cos 40° = \frac{BP}{BC} = \frac{x}{10}$$

$$\Leftrightarrow x = 10\cos 40° \quad -(3)$$

Therefore, substituting (3) and (2) into (1) we have:

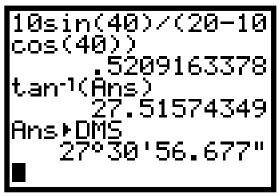
$$\tan\theta = \frac{10\sin 40°}{20 - 10\cos 40°}$$

$$= 0.5209$$

$$\therefore \theta = \tan^{-1}(0.5209)$$

$$= 27.5157$$

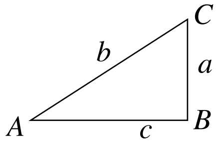
$$= 27°31'$$



Note that we have not rounded down our answer until the very last step.

EXERCISES 9.1

1. The parts of this question refer to the triangle shown. Complete the blank spaces in this table, giving lengths correct to three significant figures and angles correct to the nearest degree.

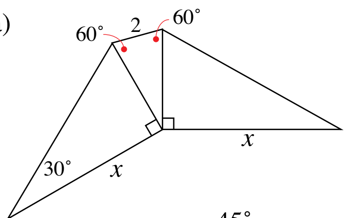


	a cm	*b* cm	*c* cm	*A*	*B*	*C*
1.			1.6		90°	23°
2.		98.3			90°	34°
3.			33.9		90°	46°
4.		30.7			90°	87°
5.	2.3				90°	33°
6.		77			90°	51°
7.	44.4		68.4		90°	57°
8.			12.7	13°	90°	
9.		94.4		52°	90°	

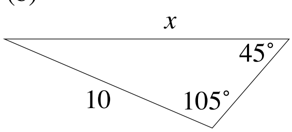
	a cm	*b* cm	*c* cm	*A*	*B*	*C*
10.	71.8		64.6	48°	90°	
11.		34.1		43°	90°	
12.			2.3	87°	90°	
13.	71.5			63°	90°	
14.	33.5		6.5		90°	
15.	6.1	7.2			90°	
16.		30	7.3		90°	
17.	29.0		2.0		90°	
18.	34.5	88.2			90°	
19.	24.0	29.7			90°	
20.		46.2			90°	27°
21.	59.6		41.8		90°	35°
22.		6.8			90°	37°
23.			14.9	41°	90°	49°
24.			16.1	41°	90°	49°
25.			33.3	68°	90°	22°

2. Find the exact value of x in each of the following

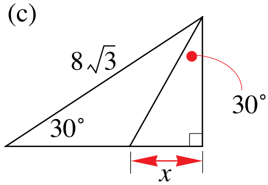
(a)



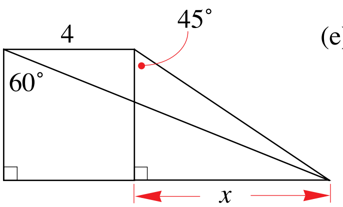
(b)



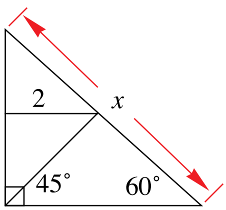
(c)



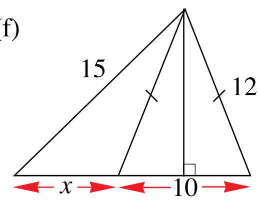
(d)



(e)



(f)

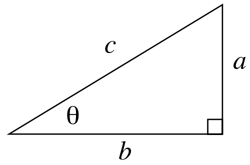


3. Using the triangle on the right, show that

(a) $\sin(90° - \theta) = \cos\theta$

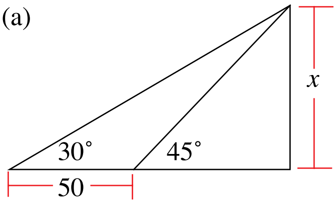
(b) $\cos(90° - \theta) = \sin\theta$

(c) $\tan(90° - \theta) = \dfrac{1}{\tan\theta}$

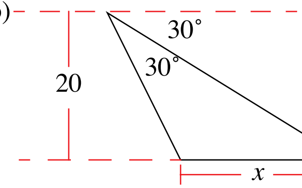


4. Find the exact value of x in each of the following.

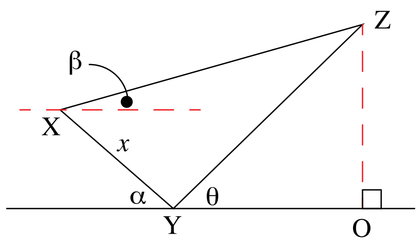
(a)



(b)



5. Show that $OZ = \dfrac{x\tan\theta(\sin\alpha + \cos\alpha\tan\beta)}{\tan\theta - \tan\beta}$

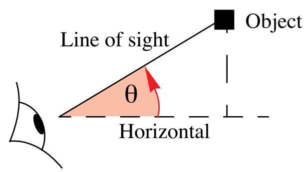


9.2 APPLICATIONS

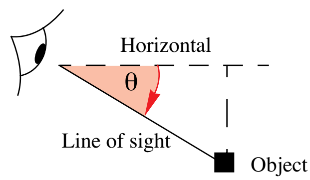
Applications that require the use of trigonometric ratios and right-angled triangles are many and varied. In this section we consider a number of standard applications to highlight this.

9.2.1 ANGLE OF ELEVATION AND DEPRESSION

The **angle of elevation** is the angle of the line of sight **above** the horizontal of an object seen above the horizontal



The **angle of depression** is the angle of the line of sight **below** the horizontal of an object seen below the horizontal



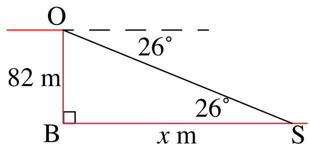
Note that the angle of depression and elevation for the same line of sight are **alternate angles**.

EXAMPLE 9.5

An observer standing on the edge of a cliff 82 m above sea level sees a ship at an angle of depression of 26°. How far from the base of the cliff is the ship situated?

Solution

We first draw a diagram to represent this situation: Let the ship be at point S, x metres from the base of the cliff, B, and let O be where the observer is standing.



Using the right-angled triangle OBS we have:

$$\tan 26° = \frac{82}{x} \Leftrightarrow x\tan 26° = 82$$

$$\Leftrightarrow x = \frac{82}{\tan 26°}$$

$$= 168.1249\ldots$$

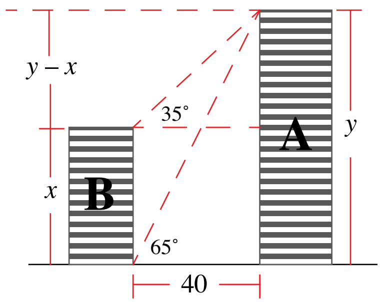
Therefore, the ship is 168 m from the base of the cliff.

EXAMPLE 9.6

The angle of depression from the roof of building A to the foot of a second building, B, across the same street and 40 metres away is 65°. The angle of elevation of the roof of building B to the roof of building A is 35°. How tall is building B?

Solution

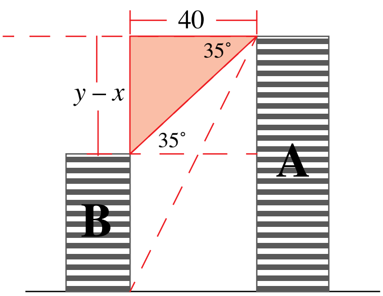
Let the height of building B be x m and that of building A be y m.



Note that we are using the fact that for the same line of sight, the angle of depression and elevation is equal.

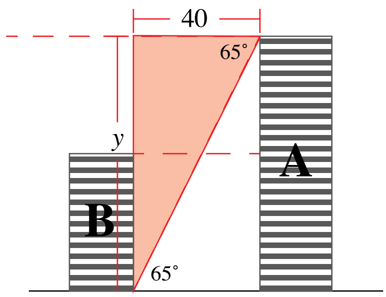
The height difference between the two buildings must then be $(y - x)$ m.

We now have two right-angled triangles to work with:



$$\tan 35° = \frac{y-x}{40}$$

$$\Leftrightarrow y - x = 40\tan 35° \quad -(1)$$



$$\tan 65° = \frac{y}{40}$$

$$\Leftrightarrow y = 40\tan 65° \quad -(2)$$

Substituting (2) into (1) we have:

$$40\tan 65° - x = 40\tan 35°$$

$$\Leftrightarrow x = 40\tan 65° - 40\tan 35°$$

$$\therefore x = 57.7719\ldots$$

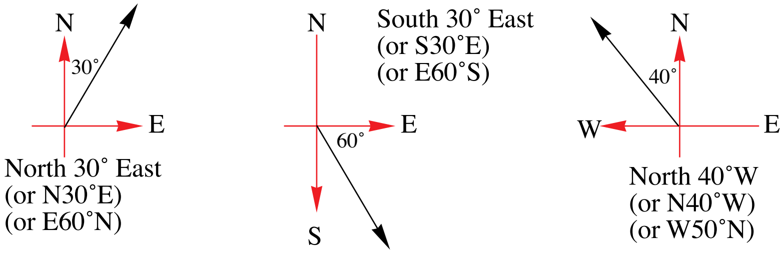
That is, building B is 57.77 m

9.2.2 BEARINGS

In the sport of orienteering, participants need to be skilled in handling bearings and reading a compass. Bearings can be quoted by making reference to the North, South, East and West directions or using true bearings. We look at each of these.

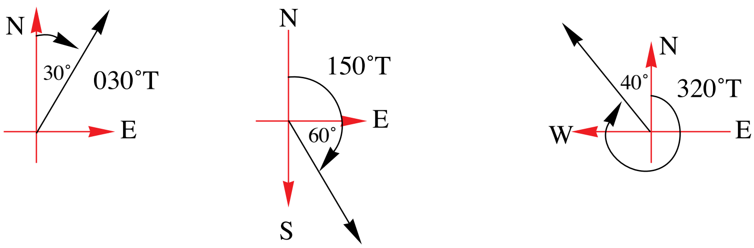
Compass bearings

These are quoted in terms of an angle measured East, West, North or South, or somewhere in between. For example, North 30˚ East, expressed as N30˚E, informs us that from the North direction we rotate 30˚ towards the East and then follow that line of direction. The following diagrams display this for a number of bearings.



True bearings

These are quoted in terms of an angle measured in a clockwise direction from North (and sometimes a capital T is attached to the angle to highlight this fact). So, for example, a bearing of 030˚T would represent a bearing of 30˚ in a clockwise direction from the North – this corresponds to a compass bearing of N30˚E. Using the above compass bearings we quote the equivalent true bearings:



EXAMPLE 9.7

Janette walks for 8 km North-East and then 11 km South-East. Find the distance and bearing from her starting point.

Solution

First step is to draw a diagram.

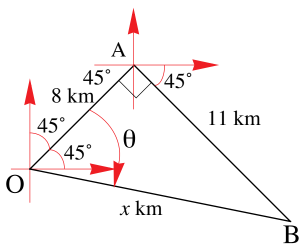
As $\angle OAB = 90°$ we can make use of Pythagoras's Theorem:

$$x^2 = 8^2 + 11^2$$

$$\therefore x = 13.60 \text{ [taking +ve square root]}$$

Let $\theta = \angle AOB$ so that $\tan\theta = \frac{11}{8}$. $\therefore \theta = \tan^{-1}\left(\frac{11}{8}\right)$

$$= 53.97°$$



Therefore, bearing is $45° + \theta = 45° + 53.97° = 98.97°$

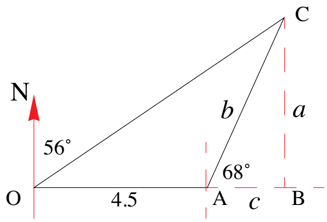
That is, B has a bearing of 98.97˚T from O and is (approx) 13.6 km away.

EXAMPLE 9.8

The lookout, on a ship sailing due East, observes a light on a bearing of 056˚T. After the ship has travelled 4.5 km, the lookout now observes the light to be on a bearing of 022˚T. How far is the light source from the ship at its second sighting?

Solution

As always, we start with a diagram.



Using ΔOBC we have, $\tan 34° = \frac{BC}{OB} = \frac{a}{4.5 + c}$

$\therefore a = (4.5 + c)\tan 34°$ – (1)

Using ΔABC we have, $\tan 68° = \frac{BC}{AB} = \frac{a}{c}$

$\therefore a = c\tan 68°$ – (2)

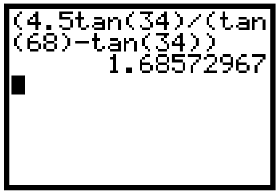
Equating (1) and (2) we have, $c\tan 68° = (4.5 + c)\tan 34°$

$$c\tan 68° = 4.5\tan 34° + c\tan 34°$$

$$\Leftrightarrow c(\tan 68° - \tan 34°) = 4.5\tan 34°$$

$$\Leftrightarrow c = \frac{4.5\tan 34°}{(\tan 68° - \tan 34°)}$$

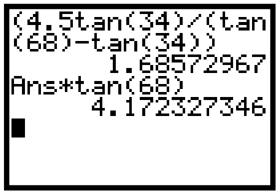
$$\therefore c = 1.6857$$



Substituting this result into (2) we have,

$$a = \frac{4.5\tan 34°}{(\tan 68° - \tan 34°)} \times \tan 68°$$

$$\therefore a = 4.1723$$



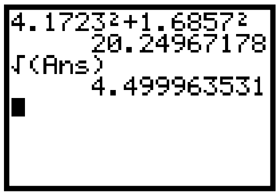
Then, using ΔABC and Pythagoras's Theorem, we have

$$b^2 = a^2 + c^2$$

$$= 4.1723^2 + 1.6857^2$$

$$\therefore b = \sqrt{20.2496}$$

$$= 4.4999$$



That is, the light is 4.5 km from the ship (at the second sighting).
Can you see a much quicker solution? Hint – think isosceles triangle.

EXERCISES 9.2

1. (a) Change the following compass bearings into true bearings

i. N30˚E ii. N30˚W iii. S15˚W iv. W70˚S

(b) Change the following true bearings into compass bearings

i. 025˚T ii. 180˚T iii. 220˚T iv. 350˚T

2. The angle of depression from the top of a building 60 m high to a swing in the local playground is 58˚. How far is the swing from the foot of the building?

3. From a point A on the ground, the angle of elevation to the top of a tree is 52˚. If the tree is 14.8 m away from point A, find the height of the tree.

4. Find the angle of elevation from a bench to the top of an 80 m high building if the bench is 105 m from the foot of the building.

5. Patrick runs in a direction N60˚E and after 45 minutes finds himself 3900 m North of his starting point. What is Patrick's average speed in ms^{-1}.

6. A ship leaves Oldport and heads NW. After covering a distance of 16 km it heads in a direction of N68˚22'W travelling a distance of 22 km where it drops anchor. Find the ship's distance and bearing from Oldport after dropping anchor.

7. From two positions 400 m apart on a straight road, running in a northerly direction, the bearings of a tree are N36˚40'E and E33˚22'S. What is the shortest distance from the tree to the road?

8. A lamp post leaning away from the sun and at 6˚ from the vertical, casts a shadow 12 m long when the sun's angle of elevation is 44˚. Assuming that the level of the ground where the pole is situated is horizontal, find its length.

9. From a window, 29.6 m above the ground, the angle of elevation of the top of a building is 42˚, while the angle of depression to the foot of the building is 32˚. Find the height of the building.

10. Two towns P and Q are 50 km apart, with P due west of Q. The bearing of a station from town P is 040˚T while the bearing of the station from town Q is 300˚T. How far is the station from town P?

11. When the sun is 74˚ above the horizon, a vertical flagpole casts a shadow 8.5 m onto a horizontal ground. Find the shadow cast by the sun when it falls to 62˚ above the horizontal.

12. A hiker walks for 5km on a bearing of 053˚ true (North 53˚ East). She then turns and walks for another 3km on a bearing of 107˚ true (East 17˚ South).

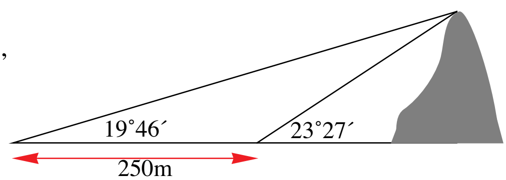
(a) Find the distance that the hiker travels North/South and the distance that she travels East/West on the first part of her hike.

(b) Find the distance that the hiker travels North/South and the distance that she travels East/West on the second part of her hike.

(c) Hence find the total distance that the hiker travels North/South and the distance that she travels East/West on her hike.

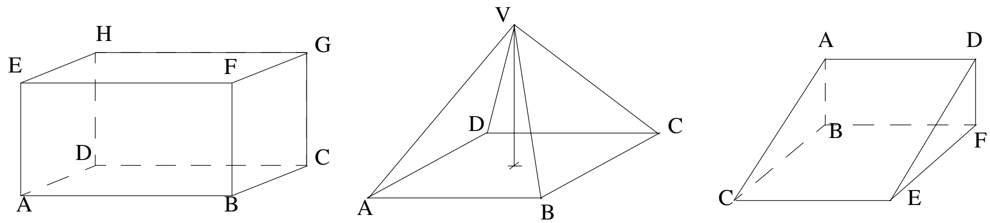
(d) If the hiker intends to return directly to the point at which she started her hike, on what bearing should she walk and how far will she have to walk?

13. A surveying team are trying to find the height of a hill. They take a 'sight' on the top of the hill and find that the angle of elevation is 23˚27´. They move a distance of 250 metres on level ground directly away from the hill and take a second 'sight'. From this point, the angle of elevation is 19˚46´. Find the height of the hill, correct to the nearest metre.



9.3 RIGHT ANGLES IN 3–DIMENSIONS

When dealing with problems in three dimensions, we draw the figures in perspective, so that a model can be more accurately visualised. This does not mean that you must be an artist, simply that you take a little time (and a lot of practice) when drawing such diagrams. The key to many 3–D problems is locating the relevant right–angled triangles within the diagram. Once this is done, all of the trigonometric work that has been covered in the previous two sections can be applied. As such, we will not be learning new theory, but rather developing new drawing and modelling skills. Some typical examples of solids that may be encountered are:

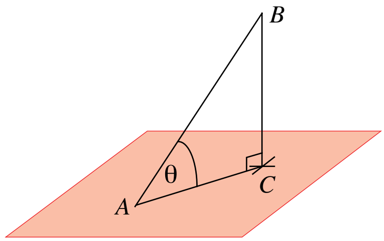


Cuboid ABCD, EFGH | Right Pyramid V, ABCD | Wedge ABC, DEF

We look at two basic concepts and drawing techniques to help us.

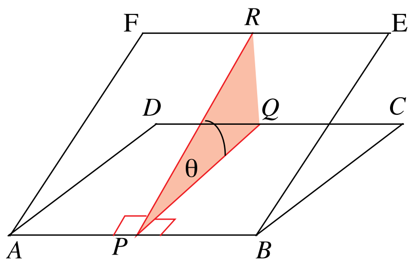
1. A line and a plane:

A line will always cut a plane at some point (unless the line is parallel to the plane). To find the angle between a line and a plane construct a perpendicular from the line to the plane and complete a right–angled triangle. In our diagram, we have that the segment $[AB]$ is projected onto the plane. A perpendicular, $[BC]$ is drawn, so that a right–angled triangle, ABC is completed. The angle that the line then makes with the plane is given by θ (which can be found by using the trig–ratios).



2. A plane and a plane:

To find the angle between two planes ABCD and ABEF (assuming that they intersect), take any point P on the intersecting segment $[AB]$ and draw $[PQ]$ and $[PR]$ on each plane in such a way that they are perpendicular to $[AB]$. Then, the angle between $[PQ]$ and $[PR]$ (θ) is the angle between the two planes.



EXAMPLE 9.9

A cube ABCD, EFGH has a side length measuring 6 cm.

(a) Find the length of the segment $[AC]$.

(b) The length of the diagonal $[AG]$.

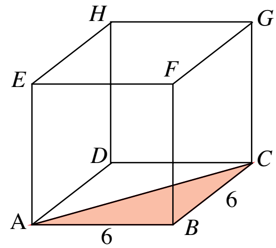
(c) The angle that the diagonal $[AG]$ makes with the base.

Solution

First we need to draw a cube:

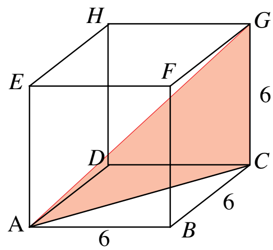
(a) Now the base of the cube is a square, so that $\angle ABC = 90°$, i.e., we have a right–angled triangle, so we can use Pythagoras's Theorem:

$$AC^2 = AB^2 + BC^2$$
$$= 6^2 + 6^2$$
$$= 72$$
$$\therefore AC = \sqrt{72} \approx 8.49$$



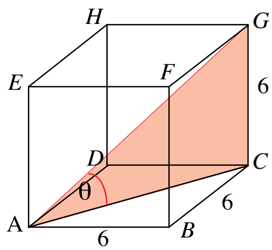
(b) This time we have that $\angle ACG = 90°$, therefore,

$$AG^2 = AC^2 + CG^2$$
$$= (\sqrt{72})^2 + 6^2$$
$$= 108$$
$$\therefore AC = \sqrt{108} \approx 10.39$$



(c) Using triangle ACG, $\tan\theta = \frac{CG}{AC} \therefore \tan\theta = \frac{6}{\sqrt{72}}$

$$\theta = \tan^{-1}\left(\frac{6}{\sqrt{72}}\right)$$
$$= 35.26°$$
$$= 35°16'$$



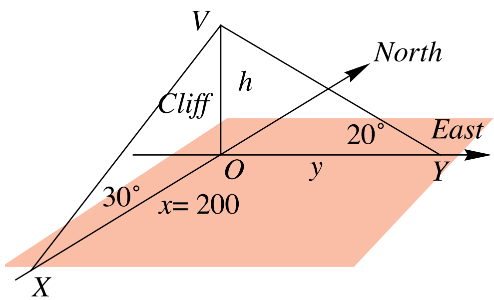
EXAMPLE 9.10

From a point *X*, 200 m due South of a cliff, the angle of elevation of the top of the cliff is 30°. From a point *Y*, due East of the cliff, the angle of elevation of the top of the cliff is 20°. How far apart are the points *X* and *Y*?

Solution

We start by illustrating this information on a 3–D diagram (Note that North–South and West–East are drawn on a plane. It is necessary to do this otherwise the diagram will not make sense).

Let the cliff be h metres high. The distance from X to the base of the cliff be x metres and the distance from Y to the base of the cliff be y metres.



As $\angle XOY = 90°$, then $XY^2 = x^2 + y^2$
$$= 200^2 + y^2$$

But, $\tan 20° = \frac{h}{y}$, of which we know neither h or y.

However, using triangle XOV, we have that $\tan 30° = \frac{h}{200} \Rightarrow h = 200 \times \tan 30°$.

Therefore, we have that $\tan 20° = \dfrac{200 \times \tan 30°}{y} \Leftrightarrow y = \dfrac{200 \times \tan 30°}{\tan 20°}$

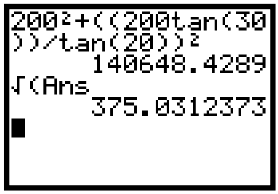
That is, $y = 317.25$

Therefore, $XY^2 = x^2 + y^2 = 200^2 + \left(\dfrac{200 \times \tan 30°}{\tan 20°}\right)^2$

$= 140648.4289$

$XY = 375.0312$.

That is, X and Y are approximately 375 m apart.



EXERCISES 9.3

1. For the diagram shown, determine the angle of inclination between the plane

(a) ABCD and the base, EABH (Figure 1).

(b) ABC and the base EBFA (Figure 2).

Figure 1.

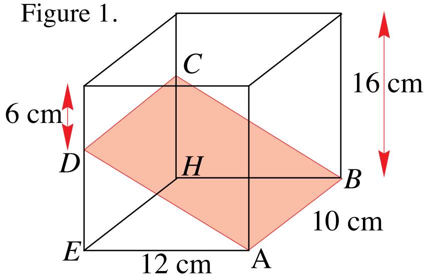
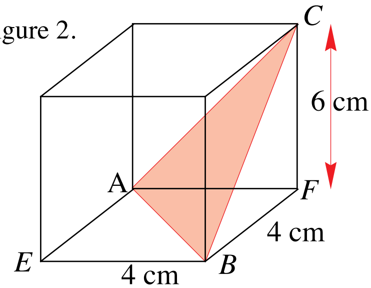


Figure 2.



2. A right pyramid with a rectangular base and a vertical height of 60 cm is shown in the diagram alongside. The points X and Y are the midpoints of the sides [AB] *and* [BC] *respectively*

Find

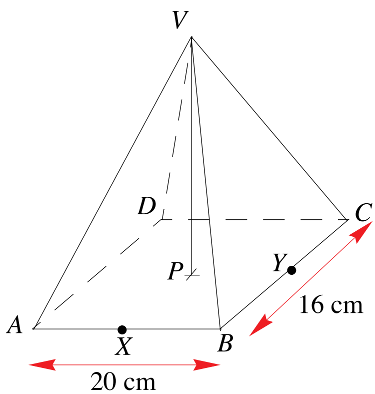
(a) the length, AP.

(b) the length of the edge [AV].

(c) the angle that the edge AV makes with the base ABCD.

(d) the length, YV .

(e) The angle that the plane BCV makes with the base.

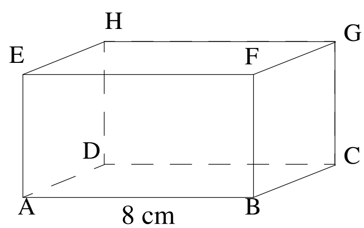


3. The diagram alongside shows a rectangular box with side lengths AB = 8 cm, BC = 6 cm and CG = 4 cm. Find the angle between

(a) the line [BH] and the plane $ABCD$.

(b) the lines [BH and [BA].

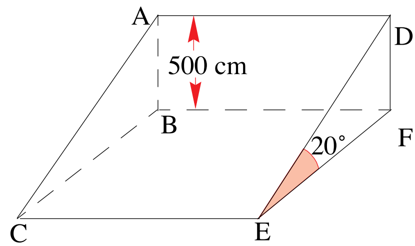
(c) the planes $ADGF$ and $ABCD$.



4. For the wedge shown alongside, given that the angle between the lines EA and ED is 50°find

(a) the length of [AE].

(b) the $\angle AEB$.



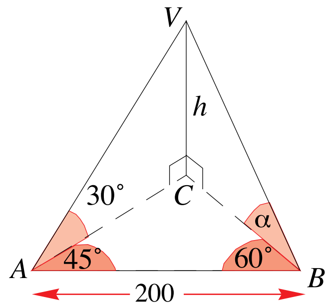
5. From a point A, 100 m due South of a tower, the angle of elevation of the top of the tower is 40°. From a point B, due East of the tower, the angle of elevation of the top of the tower is 20°. How far apart are the points A and B?

6. For the triangular prism shown alongside find

(a) the value of h

(b) the value of α

(c) the angle that the plane ABV makes with the base ABC.



7. The angle of depression from the top of a tower to a point X South of the tower, on the ground and 120 m from the foot of the tower is 24°. From point Y due West of X the angle of elevation to the top of the tower is 19°.

(a) Illustrate this information on a diagram.

(b) Find the height of the tower.

(c) How far is Y from the foot of the tower?

(d) How far apart are the points X and Y?

8. A mast is held in a vertical position by four ropes of length 60 metres. All four ropes are attached at the same point at the top of the mast so that their other ends form the vertices of a square when pegged into the (level) ground. Each piece of rope makes an angle of 54° with the ground.

(a) Illustrate this information on a diagram.

(b) How tall is the mast?

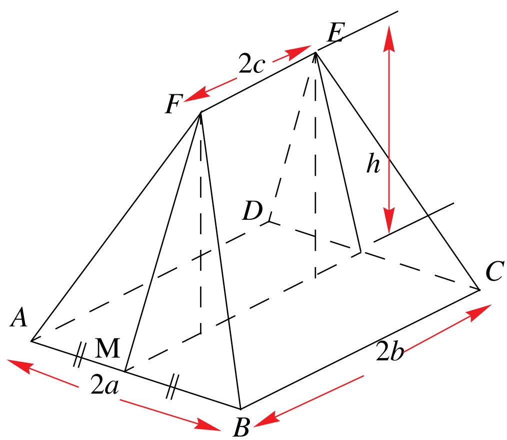
9. A symmetrical sloping roof has dimensions as shown in the diagram. Find

(a) the length of [FM].

(b) the angle between the plane BCEF and the ground.

(c) the angle between the plane ABF and the ground

(d) the total surface area of the roof.



10. The angle of elevation of the top of a tower from a point A due South of it is $68°$. From a point B, due East of A, the angle of elevation of the top is $54°$. If A is 50 m from B, find the height of the tower.

11. A tower has been constructed on the bank of a long straight river. From a bench on the opposite bank and 50 m downstream from the tower, the angle of elevation of the top of the tower is $30°$. From a second bench on the same side of the tower and 100 m upstream from the tower, the angle of elevation of the top of the tower is $20°$. Find
(a) the height of the tower.
(b) the width of the river.

12. A right pyramid of height 10 m stands on a square base of side lengths 5 m. Find
(a) the length of the slant edge.
(b) the angle betwen a sloping face and the base.
(c) the angle between two sloping faces.

13. A camera sits on a tripod with legs 1.5 m long. The feet rest on a horizontal flat surface and form an equilateral triangle of side lengths 0.75 m. Find
(a) the height of the camera above the ground.
(b) the angles made by the legs with the ground.
(c) the angle between the sloping faces formed by the tripod legs.

14. From a point A due South of a mountain, the angle of elevation of the mountaintop is α. When viewed from a point B, x m due East of A, the angle of elevation of the mountaintop is β. Show that the height, h m, of the mountain is given by $h = \dfrac{x\sin\alpha\sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$.

9.4 AREA OF A TRIANGLE

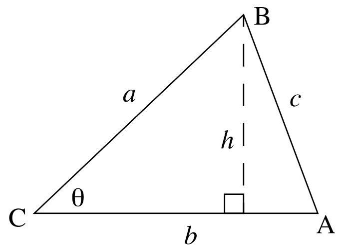
Given **any** triangle with sides a and b, and the included angle θ, the area, A, is given by

$$A = \frac{1}{2}bh$$

However, $\sin\theta = \frac{h}{a} \Leftrightarrow h = a \times \sin\theta$ and so, we have that

$$A = \frac{1}{2}b \times a \times \sin\theta$$

where θ is the angle between sides a and b.



Note that the triangle need not be a right-angled triangle.

Because of the standard labelling system for triangles, the term $\sin\theta$ is often replaced by $\sin C$, given the expresion Area $= \frac{1}{2}ab\sin C$.

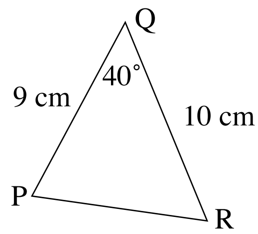
A similar argument can be used to generate the formulas: Area $= \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$

EXAMPLE 9.11

Find the area of the triangle PQR given that PQ = 9 cm, QR = 10 cm and $\angle PQR = 40°$.

Solution

Based on the given information we can contruct the following triangle:



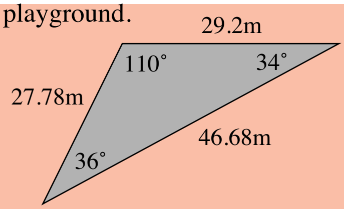
Meaning that, the required area, A, is given by

$$A = \frac{1}{2}ab\sin\theta = \frac{1}{2} \times 9 \times 10 \times \sin 40°$$
$$= 28.9$$

That is, the area is 28.9 cm^2.

EXAMPLE 9.12

The diagram shows a triangular children's playground. Find the area of the playground



Solution

Since all the measurements of the triangle are known, any one of the three formulas could be used. Many people remember the formula as 'Area equals half the product of the lengths of two sides times the sine of the angle between them'.

$$\text{Area} = \frac{1}{2} \times 27.78 \times 46.68 \times \sin 36° = 381\text{m}^2$$

$$\text{Area} = \frac{1}{2} \times 27.78 \times 29.2 \times \sin 110° = 381\text{m}^2$$

$$\text{Area} = \frac{1}{2} \times 29.2 \times 46.68 \times \sin 34° = 381\text{m}^2$$

EXERCISES 9.4

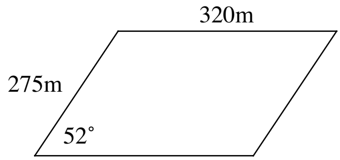
1. Find the areas of these triangles that are labelled using standard notation.

	***a* cm**	***b* cm**	***c* cm**	***A***	***B***	***C***
(a)	35.94	128.46	149.70	12°	48°	120°
(b)	35.21	54.55	81.12	20°	32°	128°
(c)	46.35	170.71	186.68	14°	63°	103°
(d)	33.91	159.53	163.10	12°	78°	90°
(e)	42.98	25.07	48.61	62°	31°	87°
(f)	39.88	24.69	34.01	84°	38°	58°
(g)	43.30	30.26	64.94	34°	23°	123°
(h)	12.44	2.33	13.12	68°	10°	102°

	a cm	*b* cm	*c* cm	*A*	*B*	*C*
(i)	43.17	46.44	24.15	67°	82°	31°
(j)	23.16	32.71	24.34	45°	87°	48°
(k)	50.00	52.91	38.64	64°	72°	44°
(l)	44.31	17.52	48.77	65°	21°	94°
(m)	12.68	23.49	22.34	32°	79°	69°
(n)	42.37	42.37	68.56	36°	36°	108°
(o)	40.70	15.65	41.26	77°	22°	81°

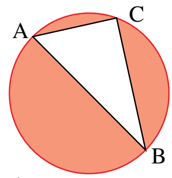
2. A car park is in the shape of a parallelogram. The lengths of the sides of the car park are given in metres.

What is the area of the car park?

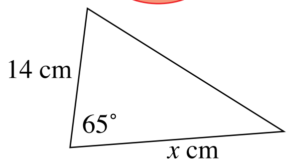


3. The diagram shows a circle of radius 10 cm. AB is a diameter of the circle. AC = 6 cm.

Find the area of the shaded region, giving an exact answer.

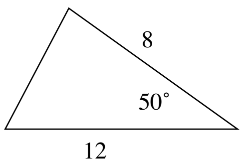


4. The triangle shown has an area of 110 cm^2. Find x.

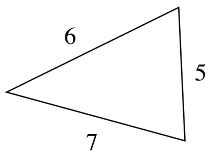


5. Find the area of the following

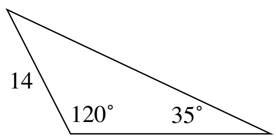
(a)



(b)



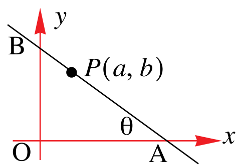
(c)



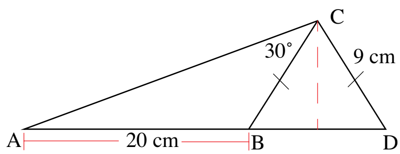
6. A napkin is in the shape of a quadrilateral with diagonals 9 cm and 12 cm long. The angle between the diagonals is 75°. What area does the napkin cover when laid out flat?

7. A triangle of area 50 cm^2 has side lengths 10 cm and 22 cm. What is the magnitude of the included angle?

8. A variable triangle OAB is formed by a straight line passing through the point $P(a, b)$ on the Cartesian plane and cutting the x–axis and y–axis at A and B respectively.
If $\angle OAB = \theta$, find the area of ΔOAB in terms of a, b and θ.



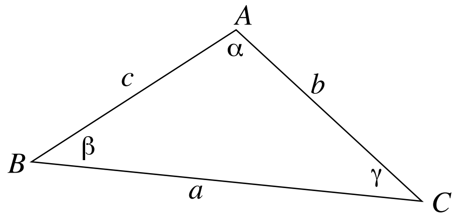
9. Find the area of ΔABC for the given diagram.



9.5 NON RIGHT-ANGLED TRIANGLES

9.5.1 THE SINE RULE

Previous sections dealt with the trigonometry of right-angled triangles. The trigonometric ratios can be used to solve non right-angled triangles. There are two main methods for solving non right-angled triangles, the **sine rule** and the **cosine rule** (which we look at later in this chapter). Both are usually stated using a standard labelling of the triangle. This uses capital letters to label the vertices and the corresponding small letters to label the sides opposite these vertices.



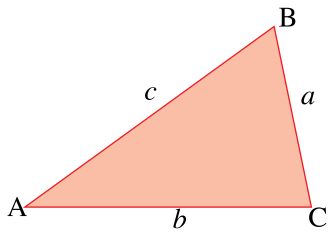
$$\sin A = \sin\alpha$$
$$\sin B = \sin\beta$$
$$\sin C = \sin\gamma$$

Using this labelling of a triangle, the sine rule can be stated as:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Note: the sine rule can only be used in a triangle in which an angle and the side **opposite** that angle are known.
So, why does this work?

Using the results of the last section and labelling a triangle in the standard manner we have:



$$Area = \frac{1}{2}bc\sin A,$$

$$Area = \frac{1}{2}ac\sin B$$

and $$Area = \frac{1}{2}ab\sin C$$

However, each of these are equal, meaning that

$$\frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A \Leftrightarrow a\sin B = b\sin A \Leftrightarrow \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly,

$$\frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C \Leftrightarrow c\sin B = b\sin C \Leftrightarrow \frac{c}{\sin C} = \frac{b}{\sin B}$$

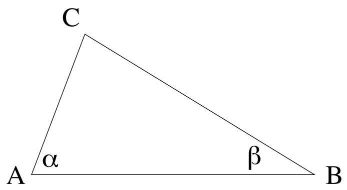
Combining these results we have that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

So, when should/can we make use of the sine rule?

Although the sine rule can be used for right-angled triangles, it is more often used for situations when we do not have a right-angled triangle, and when the given triangle has either of the following pieces of information:

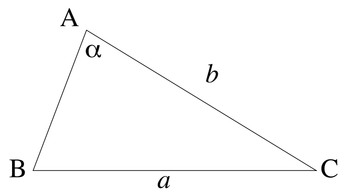
(a) Two angles and one side



Either the length CB, AB or AC can be given and the triangle can be 'solved'. i.e., we can find the length of every side and every angle.

In this case, if we are give the length AB, we need $\angle ACB$, which can be found using $\angle ACB = 180° - \alpha - \beta$

(b) Two sides and a non-included angle



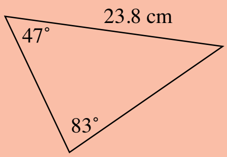
In this case, we first need to determine angle B using $\frac{\sin\alpha}{a} = \frac{\sin B}{b}$.

Once we have angle B, we can the find angle C and then the length AB.

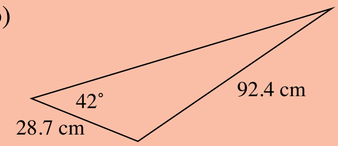
EXAMPLE 9.13

Solve the following triangles giving the lengths of the sides in centimetres, correct to one decimal place and angles correct to the nearest degree.

(a)

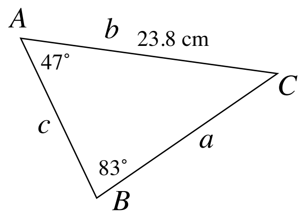


(b)



Solution

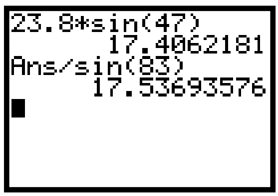
(a) Firstly, label the triangle using the standard method of lettering. 'Solve the triangle' means find all the angles and the lengths of all the sides. Since two of the angles are known, the third is; $C = 180° - 47° - 83° = 50°$.
The lengths of the remaining sides can be found using the known pairing of side and angle, b and B.



$$\frac{a}{\sin A} = \frac{b}{\sin B} \Leftrightarrow \frac{a}{\sin 47°} = \frac{23.8}{\sin 83°}$$

$$a = \frac{23.8 \times \sin 47°}{\sin 83°}$$

$$= 17.5369\ldots$$



That is, BC is 17.5 cm (correct to one d.p).

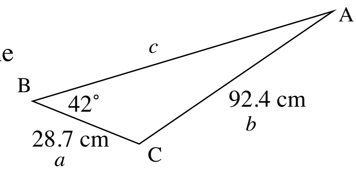
Similarly, the remaining side can be calculated: $\frac{c}{\sin C} = \frac{b}{\sin B} \Leftrightarrow \frac{c}{\sin 50°} = \frac{23.8}{\sin 83°}$

$$\therefore c = \frac{23.8 \times \sin 50°}{\sin 83°}$$

$$= 18.3687\ldots$$

That is, AB is 18.4 cm (correct to one d.p).

(b) This triangle is different from the previous example in that only one angle is known. It remains the case that a pair of angles and an opposite side are known and that the sine rule can be used. The angle A must be found first.



$$\frac{\sin A}{a} = \frac{\sin B}{b} \Leftrightarrow \frac{\sin A}{28.7} = \frac{\sin 42°}{92.4}$$

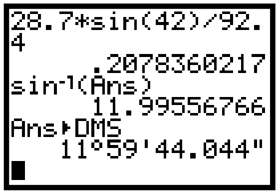
$$\Leftrightarrow \sin A = \frac{28.7 \times \sin 42°}{92.4}$$

$$= 0.207836$$

$$\therefore A = \sin^{-1} 0.207836$$

$$= 11.9956°$$

$$= 11°59'44''$$



The answer to the first part of the question is 12° correct to the nearest degree. It is important, however, to carry a much more accurate version of this angle through to subsequent parts of the calculation. This is best done using the calculator memory.

The third angle can be found because the sum of the three angles is 180°.
So, $C = 180° - 12° - 42° = 126°$

An accurate version of this angle must also be carried to the next part of the calculation. Graphics calculators have multiple memories labelled A, B, C etc. and students are advised to use these in such calculations.

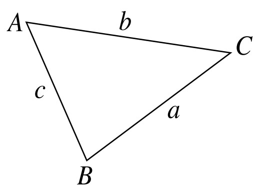
The remaining side is: $\frac{c}{\sin 126°} = \frac{28.7}{\sin 12°} \Leftrightarrow c = \frac{28.7 \sin 126°}{\sin 12°}$

$$\therefore c = 111.6762\ldots$$

That is, AB is 111.7 cm (correct to one d.p)

EXERCISES 9.5.1

Use the sine rule to complete the following table, which refers to the standard labelling of a triangle.

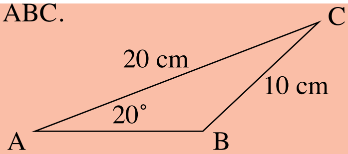


	***a* cm**	***b* cm**	***c* cm**	***A***	***B***	***C***
1.			48.2		29°	141°
2.		1.2		74°	25°	
3.			11.3	60°		117°
4.			51.7	38°		93°
5.	18.5	11.4		68°		
6.	14.6	15.0			84°	
7.		7.3			16°	85°
8.			28.5	39°		124°
9.	0.8		0.8	82°		
10.			33.3	36°		135°
11.	16.4			52°	84°	

	a cm	b cm	c cm	A	B	C
12.			64.3		24°	145°
13.	30.9	27.7		75°		
14.			59.1	29°		102°
15.		9.8	7.9		67°	
16.			54.2	16°		136°
17.	14.8		27.2			67°
18.			10.9		3°	125°
19.			17.0		15°	140°
20.			40.1	30°		129°

EXAMPLE 9.14

For the triangle shown, find the angle ABC.



Solution

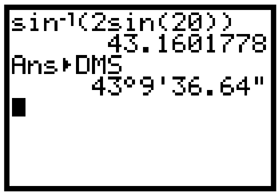
Making use of the sine rule we have:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Leftrightarrow \frac{\sin 20°}{10} = \frac{\sin B}{20}$$

$$\Leftrightarrow \sin B = \frac{20\sin 20°}{10}$$

$$\therefore B = \sin^{-1}(2\sin 20°)$$

$$= 43.1601\ldots$$



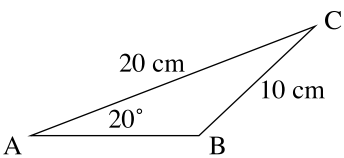
That is, $B = 43°10'$

However, from our diagram, the angle ABC should have been greater than 90°! That is, we should have obtained an **obtuse angle** (90° < B < 180°) rather than an **acute angle** (0° < B < 90°).

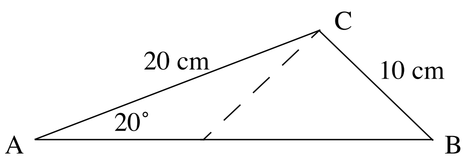
So, what went wrong?

This example is a classic case of what is known as the **ambiguous case**, in that, from the given information it is possible to draw two different diagrams, both having the same data. we show both these triangles:

Case 1



Case 2



Notice that the side BC can be pivoted about the point C and therefore two different triangles can be formed with BC = 10. This is why there are two possible triangles based on the same information.

In the solution above, $B = 43°10'$ – representing Case 2. However, our diagram is represented by Case 1! Therefore, the correct answer is $180 - 43°10' = 136°50'$.

9.5.2 THE AMBIGUOUS CASE

From Example 9.14, it can be seen that an ambiguous case can arise when using the sine rule. In the given situation we see that the side CB can be pivoted about its vertex, forming two posible triangles.

We consider another such triangle.

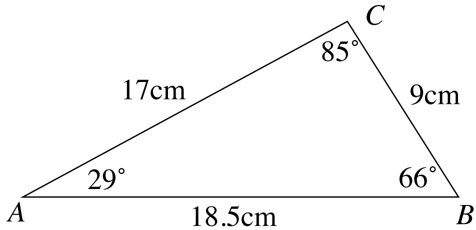
EXAMPLE 9.15

Draw diagrams showing the triangles in which AC = 17 cm, BC = 9 cm and A = 29˚ and solve these triangles.

Solution

Applying the sine rule to the triangle gives:

$$\frac{\sin B}{17} = \frac{\sin 29°}{9} \Leftrightarrow \sin B = \frac{17 \times \sin 29°}{9}$$
$$= 0.91575$$
$$\therefore B = 66°$$

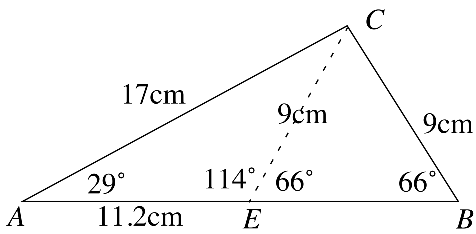


Next, we have,

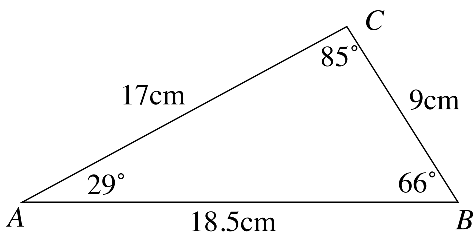
$$C = 180° - 29° - 66° = 85°$$

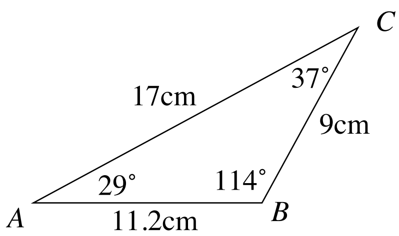
$$\frac{c}{\sin 85°} = \frac{9}{\sin 29°} \Leftrightarrow c = 18.5$$

There is, however, a second solution that results from drawing an isosceles triangle BCE. This creates the triangle AEC which also fits the data. The third angle of this triangle is 37˚ and the third side is: $\frac{\text{AE}}{\sin 37°} = \frac{9}{\sin 29°} \Leftrightarrow \text{AE} = 11.2$



The original data is ambiguous in the sense that there are two triangles that are consistent with it.





You should also notice that the two angles in the solution are 66˚ and 114˚ and that sin66˚ = sin114˚. (That is, sin 66˚ = sin(180˚ – 66˚) = sin 114˚. This will be developed further in Chapter 10).

In fact, we can go one step further and make the following statement:

If we are given two sides of a triangle and the magnitude of an angle opposite one of the sides, there may exist one, two or no possible solutions for the given information.

We summarise our findings:

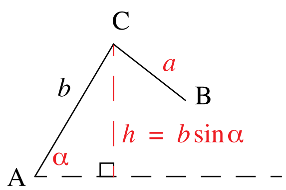
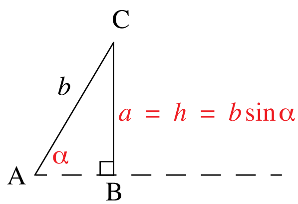
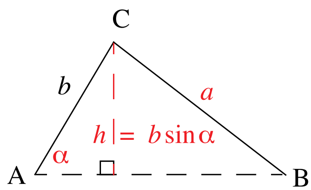
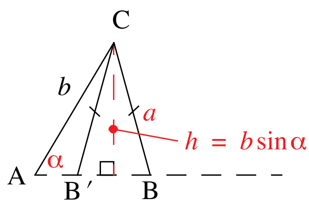
Type 1

Given the **acute angle** α and the side lengths a and b, there are four possible outcomes:

Notice that the number of solutions **depends on** the length a relative to the perpendicular height, h, of the triangle as well as the length b. Where the height h is based on the right-angled triangle formed in each case, i.e., $\sin\alpha = \frac{h}{b} \Leftrightarrow h = b\sin\alpha$.

So that

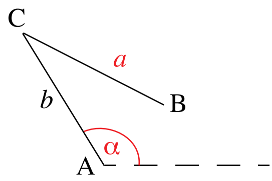
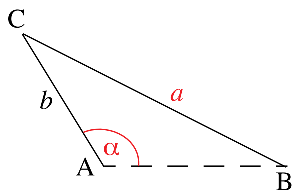
- if $a < h$, then the triangle cannot be completed.
- if $a = h$, then we have a right-angled triangle.
- if $a > b$, then we have a triangle that is consistent with the given information.
- if $h < a < b$, then the side BC can be pivoted about the vertex C, forming two triangles.

Number of Δ s	Necessary condition	Type of triangle that can be formed	
None	$a < h$		In this case, the triangle cannot be constructed.
One	$a = h$		In this case we have formed a right-angled triangle.
One	$a > b$		In this case there can be only one triangle that is consistent with the given information.
Two	$h < a < b$		In this case there are two possible triangles, ΔABC and $\Delta AB'C$. This is because BC can be pivoted about C and still be consistent with the given information.

The table above reflects the case where α is acute. What if α is obtuse?

Type 2

Given the **obtuse angle** α and the side lengths a and b, there are two possible outcomes:

Number of Δ s	Necessary condition	Type of triangle that can be formed	
None	$a \le b$		In this case, the triangle cannot be constructed.
One	$a > b$		In this case there can be only one triangle that is consistent with the given information.

EXAMPLE 9.16

Find $\angle ABC$ for the triangle ABC given that $a = 50$, $b = 80$ and $A = 35°$.

Solution

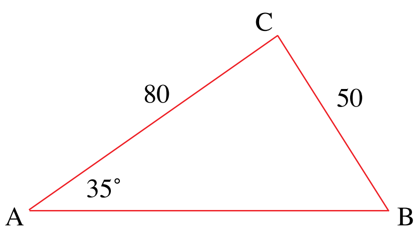
We first determine the value of $b\sin\alpha$ and compare it to the value a :

Now, $b\sin\alpha = 80\sin 35° = 45.89$

Therefore we have that $b\sin\alpha$ $(= 45.89) < a$ $(= 50) < b$ $(= 80)$

meaning that we have an ambiguous case.

Case 1:

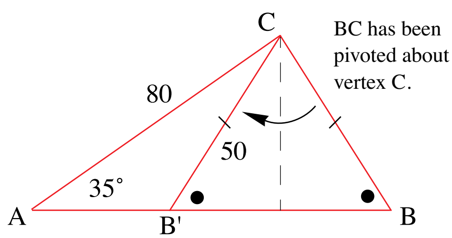


Using the sine rule, $\frac{\sin A}{a} = \frac{\sin B}{b}$, we have

$$\frac{\sin 35°}{50} = \frac{\sin B}{80} \Leftrightarrow \sin B = \frac{80\sin 35°}{50}$$

$$\therefore B = 66°35'$$

Case 2:



From case 1, the obtuse angle B' is given by $180° - 66°35' = 113°25'$.

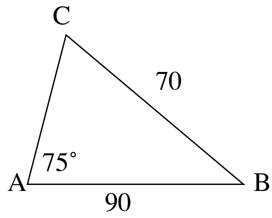
This is because $\Delta B'CB$ is an **isosceles** triangle, so that $\angle AB'C = 180° - \angle CB'B$

EXAMPLE 9.16

Find $\angle ACB$ for the triangle ABC given that $a = 70$, $c = 90$ and $A = 75^\circ$.

Solution

We start by drawing the triangle with the given information:



Using the sine rule we have

$$\frac{\sin C}{90} = \frac{\sin 75}{70} \Leftrightarrow \sin C = \frac{90\sin 75}{70}$$

$$\therefore \sin C = 1.241\ldots$$

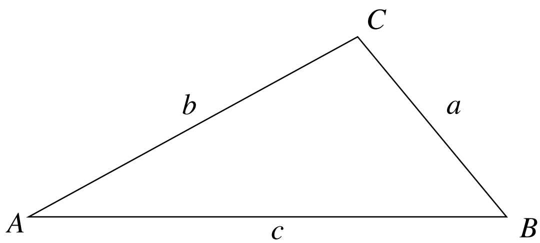
Which is imposible to solve for as the sine of an angle can never be greater than one.

Therefore no such triangle exists.

EXERCISES 9.5.2

Find the two solutions to these triangles which are defined using the standard labelling:

	***a* cm**	***b* cm**	***A***
1.	7.4	18.1	20°
2.	13.3	19.5	14°
3.	13.5	17	28°
4.	10.2	17	15°
5.	7.4	15.2	20°
6.	10.7	14.1	26°
7.	11.5	12.6	17°
8.	8.3	13.7	24°
9.	13.7	17.8	14°
10.	13.4	17.8	28°
11.	12.1	16.8	23°
12.	12	14.5	21°
13.	12.1	19.2	16°
14.	7.2	13.1	15°
15.	12.2	17.7	30°
16.	9.2	20.9	14°
17.	10.5	13.3	20°
18.	9.2	19.2	15°
19.	7.2	13.3	19°
20.	13.5	20.4	31°



21. Solve the following triangles

(a) $\alpha = 75°, a = 35, c = 45$

(b) $\alpha = 35°, a = 30, b = 80$

(c) $\beta = 40°, a = 22, b = 8$

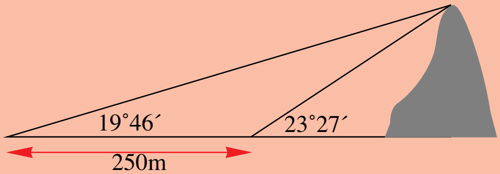
(d) $\gamma = 50°, a = 112, c = 80$

9.5.3 APPLICATIONS OF THE SINE RULE

Just as in the case of right-angled triangles, the sine rule becomes very useful. In particular, it means that previous problems that required the partitioning of a non right-angled triangle into two (or more) right-angled triangles can be solved using the sine rule.

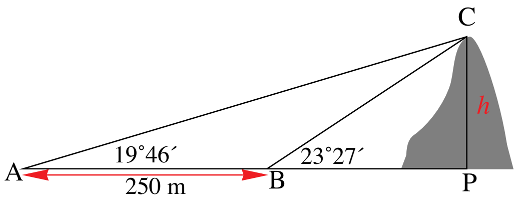
We start by considering Q.13., Exercise 9.2:

EXAMPLE 9.17 A surveying team are trying to find the height of a hill. They take a 'sight' on the top of the hill and find that the angle of elevation is 23˚27´. They move a distance of 250 metres on level ground directly away from the hill and take a second 'sight'. From this point, the angle of elevation is 19˚46´. Find the height of the hill, correct to the nearest metre.



Solution

Labelling the given diagram using the standard notation we have:



With $\beta = 180 - 23°27' = 156°33'$

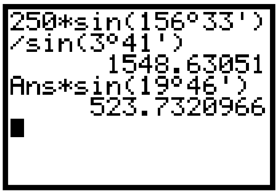
and $\gamma = 180 - 19°46' - 156°33' = 3°41'$

Then, using the sine rule,

$$\frac{b}{\sin 156°33'} = \frac{250}{\sin 3°41'}$$

$$\Leftrightarrow b = \frac{250 \sin 156°33'}{\sin 3°41'}$$

$$= 1548.63\ldots$$


250*sin(156°33')
/sin(3°41')
1548.63051
Ans*sin(19°46')
523.7320966

Then, using ΔACP we have,

$$\sin 19°46' = \frac{h}{b} \Leftrightarrow h = b\sin 19°46'$$

$$= 523.73$$

So, the hill is 524 m high (to nearest metre).
A much neater solution (as opposed to solving simultaneous equations – as was required previously).

EXERCISES 9.5.3

1. A short course biathlon meet requires the competitors to run in the direction S60˚W to their bikes and then ride S40˚E to the finish line, situated 20 km due South of the starting point. What is the distance of this course?

2. A pole is slanting towards the sun and is making an angle of 10˚ to the vertical. It casts a shadow 7 metres long along the horizontal ground. The angle of elevation of the top of the pole to the tip of its shadow is 30˚. Find the length of the pole, giving your answer to 2 d.p.

3. A statue A, is observed from two other statues B and C which are 330 m apart. The angle between the lines of sight AB and BC is 63° and the angle between the lines of sight AC and CB is 75°. How far is statue A from statue B?

4. Town A is 12 km from town B and its bearing is 132°T from B. Town C is 17 km from A and its bearing is 063°T from B. Find the bearing of A from C.

5. The angle of elevation of the top of a building from a park bench on level ground is 18°. The angle of elevation from a second park bench, 300 m closer to the base of the building is 30°. Assuming that the two benches and the building all lie on the same vertical plane, find the height of the building.

6. (a) A man standing 6 metres away from a lamp post casts a shadow10 metres long on a horizontal ground. The angle of elevation from the tip of the shadow to the lamp light is 12°. How high is the lamp light?

(b) If the shadow is cast onto a road sloping at 30° upwards, how long would the shadow be if the man is standing at the foot of the sloping road and 6 metres from the lamp post?

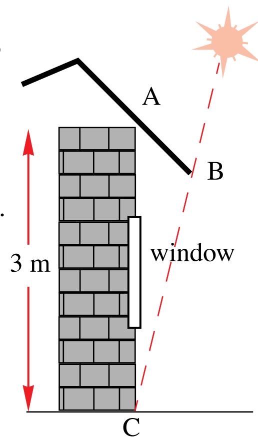
7. At noon the angle of elevation of the sun is 72° and is such that a three metre wall AC, facing the sun, is just in the shadow due to the overhang AB.

The angle that the overhang makes with the vertical wall is 50°.

(a) Copy and illustrate this information on the diagram shown.
(b) Find the length of the overhang.

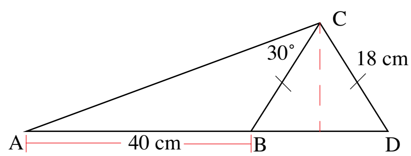
At 4 p.m. the angle of elevation of the sun is 40° and the shadow due to the overhang just reaches the base of the window.

(c) How far from the ground is the window?



8. The lookout on a ship sailing due East at 25 km/h observes a reef N62°E at a distance of 30 km.
(a) How long will it be before the ship is 15 km from the reef, assuming that it continues on its easterly course.
(b) How long is it before it is again 15 km from the reef?
(c) What is the closest that the ship will get to the reef?

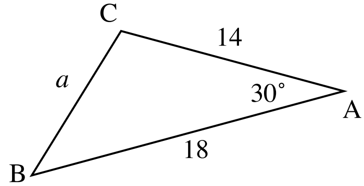
9. The framework for an experimental design for a kite is shown. Material for the kite costs $12 per square cm. How much will it cost for the material if it is to cover the framework of the kite.



10. A boy walking along a straight road notices the top of a tower at a bearing of 284°T. After walking a further 1.5 km he notices that the top of the tower is at a bearing of 293°T. How far from the road is the tower?

9.5.4 THE COSINE RULE

Sometimes the sine rule is not enough to help us solve for a non right-angled triangle. For example, in the triangle shown, we do not have enough information to use the sine rule. That is, the sine rule only provided the following:

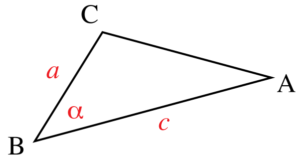


$$\frac{a}{\sin 30°} = \frac{14}{\sin B} = \frac{18}{\sin C}$$

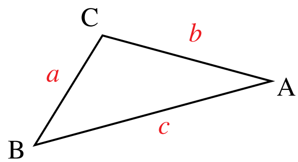
where there are too many unknowns.

For this reason we derive another useful result, known as the cosine rule. The cosine rule may be used when

1. **two sides and an included angle are given**:
This means that the third side can be determined and then we can make use of the sine rule (or the cosine rule again).



2. **three sides are given**:
This means we could then determine any of the angles.



The cosine rule, with the standard labelling of the triangle has three versions:

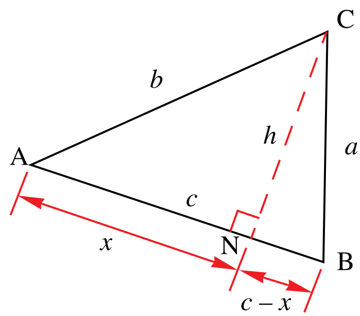
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

The cosine rule can be remembered as a version of Pythagoras's Theorem with a correction factor. We now show why this works.

Consider the case where there is an acute angle at A. Draw a perpendicular from C to N as shown in the diagram.

In ΔANC we have $b^2 = h^2 + x^2$

$$\Leftrightarrow h^2 = b^2 - x^2 \quad -(1)$$



In ΔBNC we have $a^2 = h^2 + (c-x)^2$

$$\Leftrightarrow h^2 = a^2 - (c-x)^2 \quad -(2)$$

Equating (1) and (2) we have,

$$\begin{aligned} a^2 - (c-x)^2 &= b^2 - x^2 \\ \Leftrightarrow a^2 - c^2 + 2cx - x^2 &= b^2 - x^2 \\ \Leftrightarrow a^2 &= b^2 + c^2 - 2cx \end{aligned}$$

However, from ΔANC we have that $\cos A = \frac{x}{b} \Leftrightarrow x = b\cos A$

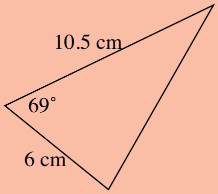
Substituting this result for x, we have

$$a^2 = b^2 + c^2 - 2cb\cos A$$

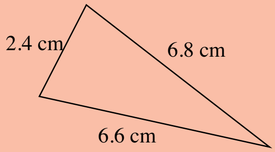
Although we have shown the result for an acute angle at A, the same rule applies if A is obtuse.

EXAMPLE 9.18 Solve the following triangles giving the lengths of the sides in centimetres, correct to one decimal place and angles correct to the nearest degree.

(a)

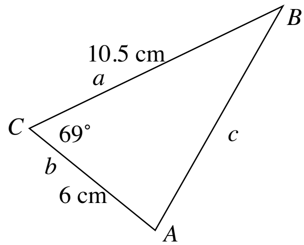


(b)



Solution

(a) The data does not include an angle and the opposite side so the sine rule cannot be used. The first step, as with the sine rule, is to label the sides of the triangle. Once the triangle has been labelled, the correct version of the cosine rule must be chosen. In this case, the solution is:



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 10.5^2 + 6^2 - 2 \times 10.5 \times 6 \times \cos 69°$$

$$= 101.0956$$

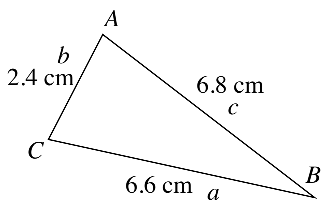
$$a = 10.1$$

The remaining angles can be calculated using the sine rule. Again, it is important to carry a high accuracy for the value of c to the remaining problem:

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Leftrightarrow \sin B = \frac{6 \times \sin 69°}{10.0546} \quad \therefore B = 34°$$

Finally, $A = 180° - 34° - 69° = 77°$

(b) In this case, there are no angles given. The cosine rule can be used to solve this problem as follows:



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$6.6^2 = 2.4^2 + 6.8^2 - 2 \times 2.4 \times 6.8 \times \cos A$$

$$2 \times 2.4 \times 6.8 \times \cos A = 2.4^2 + 6.8^2 - 6.6^2$$

$$\cos A = \frac{2.4^2 + 6.8^2 - 6.6^2}{2 \times 2.4 \times 6.8}$$

$$= 0.25858$$

$$A = 75.014°$$

$$= 75°1'$$

Next, use the sine rule: $\frac{\sin B}{b} = \frac{\sin A}{a} \Leftrightarrow \sin B = \frac{2.4 \times \sin 75}{6.6} \quad \therefore B = 20°34'$

So that $C = 180° - 75° - 21° = 84°$

The three angles, correct to the nearest degree are $A = 75°$, $B = 21°$ & $C = 84°$.

EXERCISES 9.5.4

Solve the following triangles

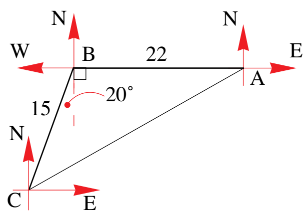
	a cm	*b* cm	*c* cm	*A*	*B*	*C*
1.	13.5		16.7		36˚	
2.	8.9	10.8				101˚
3.	22.8		12.8		87˚	
4.	21.1	4.4				83˚
5.		10.6	15.1	74˚		
6.		13.6	20.3	20˚		
7.	9.2		13.2		46˚	
8.	23.4	62.5				69˚
9.		9.6	15.7	41˚		
10.	21.7	36.0	36.2			
11.	7.6	3.4	9.4			
12.	7.2	15.2	14.3			
13.	9.1		15.8		52˚	
14.	14.9	11.2	16.2	63˚	42˚	75˚
15.	2.0	0.7	2.5			
16.	7.6	3.7	9.0			
17.	18.5	9.8	24.1			
18.	20.7	16.3	13.6			
19.		22.4	29.9	28˚		
20.	7.0		9.9		42˚	
21.	21.8	20.8	23.8			
22.	1.1		1.3		89˚	
23.		1.2	0.4	85˚		
24.	23.7	27.2				71˚
25.	3.4	4.6	5.2			

9.5.5 APPLICATIONS OF THE COSINE RULE

EXAMPLE 9.19 A cyclist rode her bike for 22 km on a straight road heading in a westerly direction towards a junction. Upon reaching the junction, she headed down another straight road bearing 200˚T for a distance of 15 km. How far is the cyclist from her starting position?

Solution

We start with a diagram:



Note that $\angle ABC = 90° + 20° = 110°$

Using the cosine rule we have,

$$AC^2 = 15^2 + 22^2 - 2 \times 15 \times 22 \cos 110°$$

$$\Rightarrow AC = \sqrt{225 + 484 - 660 \times (-0.3420\ldots)}$$

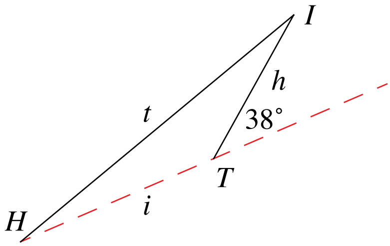
$$\therefore AC = 30.5734\ldots$$

That is, she is (approximately) 30.57 km from her starting point.

EXAMPLE 9.20 A yacht starts from a harbour and sails for a distance of 11 km in a straight line. The yacht then makes a turn to port (left) of 38° and sails for 7 km in a straight line in this new direction until it arrives at a small island. Draw a diagram that shows the path taken by the yacht and calculate the distance from the harbour to the island.

Solution

The question does not give the bearing of the first leg of the trip so the diagram can show this in any direction. H is the harbour, I the island and T the point where the yacht makes its turn.



The angle in the triangle at T is 180° – 38° = 142°. The problem does not contain an angle and the opposite side and so must be solved using the cosine rule.

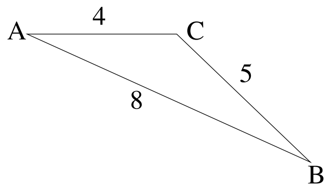
$$\begin{aligned} t^2 &= h^2 + i^2 - 2hi\cos T \\ &= 7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 142° \\ &= 291.354 \\ \therefore t &= 17.1 \end{aligned}$$

That is, distance from the harbour to the port is 17.1 km (to one d.p)

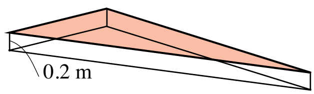
EXAMPLE 9.21 A triangular sandpit having side lengths 5 m, 4 m and 8 m is to be constructed to a depth of 20 cm. Find the volume of sand required to fill this sandpit.

Solution

We will need to find an angle. In this case we determine the largest angle, which will be the angle opposite the longest side.
From our diagram we have



$$\begin{aligned} 8^2 &= 4^2 + 5^2 - 2 \times 4 \times 5\cos C \\ \therefore 64 &= 16 + 25 - 40\cos C \\ \Leftrightarrow \cos C &= \frac{16 + 25 - 64}{40} \\ &= -\frac{23}{40} \\ \therefore C &= 125°6' \end{aligned}$$



To find the volume of sand we first need to find the surface area of the sandpit.

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2} \times 4 \times 5 \times \sin(125°6') = 8.1815 \text{ m}^2.$$

Therefore, volume of sand required is $0.2 \times 8.1815 = 1.64 \text{ m}^3$.

EXERCISES 9.5.5

1. Thomas has just walked 5 km in a direction N70˚E when he realises that he needs to walk a further 8 km in a direction E60˚S.
(a) How far from the starting point will Thomas have travelled?
(b) What is his final bearing from his starting point?

2. Two poles, 8 m apart are facing a rugby player who is 45 m from the left pole and 50 from the right one. Find the angle that the player makes with the goal mouth.

3. The lengths of the adjacent sides of a parallelogram are 4.80 cm and 6.40 cm. If these sides have an inclusive angle of $40°$, find the length of the shorter diagonal.

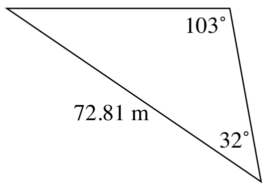
4. During an orienteering venture, Patricia notices two rabbit holes and estimates them to be 50 m and 70 m away from her. She measures the angle between the line of sight of the two holes as 54˚. How far apart are the two rabbit holes?

5. To measure the length of a lake, a surveyor chooses three points. Starting at one end of the lake she walks in a straight line for 223.25 m to some point X, away from the lake. She then heads towards the other end of the lake in a straight line and measures the distance covered to be 254.35 m. If the angle between the paths she takes is 82˚25', find the width of the lake.

6. A light airplane flying N87˚W for a distance of 155 km, suddenly needs to alter its course and heads S 34˚E for 82 km to land on an empty field.
(a) How far from its starting point did the plane land.
(b) What was the plane's final bearing from its starting point?

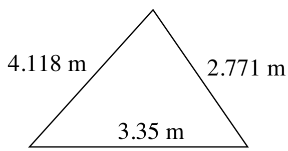
EXERCISES 9.5.6 – MISCELLANEOUS EXERCISES

1. The diagram shows a triangular building plot. The distances are given in metres. Find the length of the two remaining sides of the plot giving your answers correct to the nearest hundredth of a metre.



2. Xiang is standing on level ground. Directly in front of him and 32 metres away is a flagpole. If Xiang turns 61˚ to his right, he sees a post box 26.8 metres in front of him. Find the distance between the flagpole and the post box.

3. A triangular metal brace is part of the structure of a bridge. The lengths of the three parts are shown in metres. Find the angles of the brace.



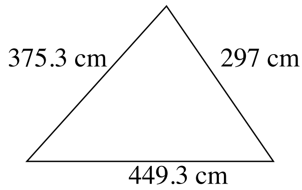
4. Find the smallest angle in the triangle whose sides have length 35.6 cm, 58.43 cm and 52.23 cm.

5. Ayton is directly north of Byford. A third town, Canfield, is 9.93km from Ayton on a bearing of 128˚ true. The distance from Byford to Canfield is 16.49km. Find the bearing of Canfield from Byford.

6. A parallelogram has sides of length 21.90 cm and 95.18 cm. The angle between these sides is 121˚. Find the length of the long diagonal of the parallelogram.

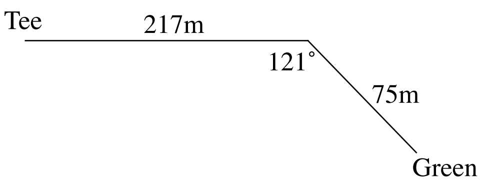
7. A town clock has 'hands' that are of length 62cm and 85cm.
(a) Find the angle between the hands at half past ten.
(b) Find the distance between the tips of the hands at half past ten.

8. A shop sign is to be made in the shape of a triangle. The lengths of the edges are shown. Find the angles at the vertices of the sign.

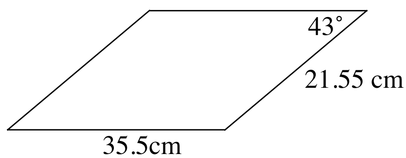


9. An aircraft takes off from an airstrip and then flies for 16.2 km on a bearing of 066˚ true. The pilot then makes a left turn of 88˚ and flies for a further 39.51 km on this course before deciding to return to the airstrip.
(a) Through what angle must the pilot turn to return to the airstrip?
(b) How far will the pilot have to fly to return to the airstrip?

10. A golfer hits two shots from the tee to the green.
How far is the tee from the green?

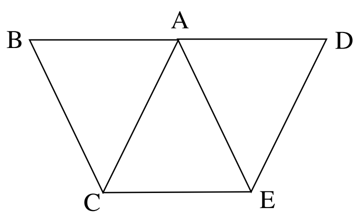


11. The diagram shows a parallelogram. Find the length of the longer of the two diagonals.



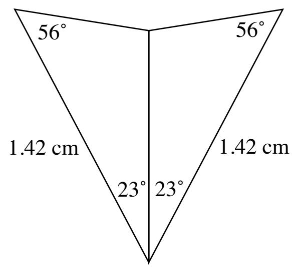
12. A triangle has angles 64˚, 15˚ and 101˚. The shortest side is 49 metres long. What is the length of the longest side?

13. The diagram shows a part of the support structure for a tower. The main parts are two identical triangles, ABC and ADE.
AC = DE = 27.4cm and BC = AE = 23.91cm
The angles ACB and AED are 58˚.
Find the distance BD.



14. The diagram shows a design for the frame of a piece of jewellery. The frame is made of wire.

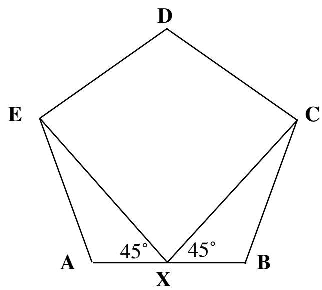
Find the length of wire needed to make the frame.



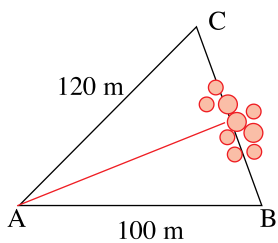
15. A triangular cross-country running track begins with the runners running North for 2050 metres. The runners then turn right and run for 5341 metres on a bearing of 083˚ true. Finally, the runners make a turn to the right and run directly back to the starting point.
(a) Find the length of the final leg of the run.
(b) Find the total distance of the run.
(c) What is the angle through which the runners must turn to start the final leg of the race?
(d) Find the bearing that the runners must take on the final leg of the race.

16. Show that for any standard triangle ABC, $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$.

17. A sandpit in the shape of a pentagon ABCDE is to be built in such a way that each of its sides are of equal length, but its angles are not all equal.
The pentagon is symmetrical about DX, where X is the midpoint of AB.
The angle AXE and BXC are both 45˚ and each side is 2 m long.
(a) Find $\angle XEA$.
(b) Find the length of EX.
(c) How much sand is required if the sandpit is 30 cm deep? Give your answer to three decimal place.



18. A triangular region has been set aside for a housing development which is to be divided into two sections. Two adjacent street frontages, AB and AC measuring 100 m and 120 m respectively. With the 100 m frontage running in an easterly direction, while the 120 m frontage runs in a north-east direction. A plan for this development is shown alongside. Give all answers to the nearest metre.
(a) Find the area covered by the housing development.



During the development stages, an environmental group specified that existing trees were not to be removed from the third frontage. This made it difficult for the surveyors to measure the length of the third frontage.

(b) Calculate the length of the third frontage, BC.

The estate is to be divided into 2 regions, by bisecting the angle at A with a stepping wall running from A to the frontage BC.

(c) How long is this stepping wall?

9.6 MORE APPLICATIONS IN 3–D

EXAMPLE 9.22

A vertical tower PA is due West of a point B. From C, bearing 210˚ and 500 m from B, the bearing of the foot of the tower A is 290˚, and the angle of elevation of the top of the tower P is 1.5˚. The points A, B and C are on level ground. Given that h is the height of the tower, show that $h = \frac{250\sqrt{3}\tan 1.5°}{\sin 20°}$ and find the height to the nearest metre.

Solution

We start by drawing a diagram to depict the situation, producing both a perspective and aerial diagram:

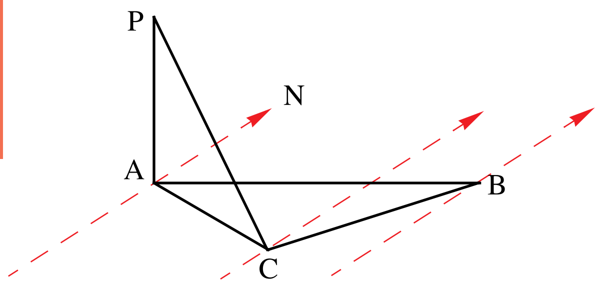


Figure 1

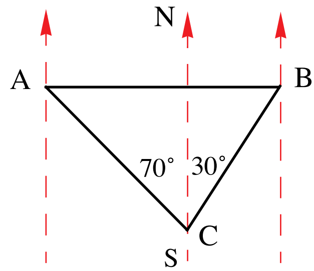


Figure 2

In figure 1, $PA = h$, $\angle PCA = 1.5°$ and thus, $h = AC \times \tan 1.5°$.

In figure 2, $\angle NCA = 70°$ (= 360˚ – bearing of A from C), $\angle NCB = 30°$ (alternate to $\angle CBS$) and $\angle ABC = 60°$ (complementary to $\angle CBS$) and thus, $\angle CAB = 20°$ (using the angle sum of a triangle).

In ΔABC we have (using the sine rule) that $\frac{AC}{\sin 60°} = \frac{500}{\sin 20°} \Leftrightarrow AC = \frac{500}{\sin 20°} \times \sin 60°$

But, $AC = \frac{h}{\tan 1.5°}$ (from above), $\therefore \frac{h}{\tan 1.5°} = \frac{500}{\sin 20°} \times \sin 60°$

$$\Leftrightarrow h = \tan 1.5° \times \frac{500}{\sin 20°} \times \sin 60°$$

$$= \tan 1.5° \times \frac{500}{\sin 20°} \times \frac{\sqrt{3}}{2}$$

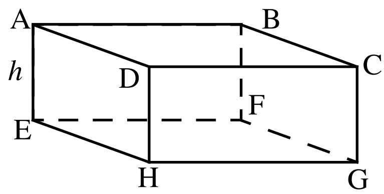
$$= \frac{250\sqrt{3}\tan 1.5°}{\sin 20°}$$

= 33 (to the nearest metre)

That is, the height of the tower (to the nearest metre) is 33 m.

EXERCISES 9.6

1. A rectagular box is constructed as shown, with measurements HG = 10 cm, $\angle FHE = 30°$, $\angle CEG = 15°$. Find the height of the box.



2. From a point A due South of a vertical tower, the angle of elevation of the top of the tower is 15°, and from a point B due West of the tower it is 32°. If the distance from A to B is 50 metres, find the height of the tower.

3. From a point P, the angle of elevation of the top of a radio mast due North of P is 17°. From Q, due West of the radio mast, the angle of elevation is 13°. Given that P and Q are 130 m apart, show that h, the height of the mast, can be given by

$$h = \frac{130}{\sqrt{\tan^2 73 + \tan^2 77}}$$

and find h to the nearest m.

4. From a point due South of a radio tower, an observer measures the angle of elevation to the top of the tower to be 41 °. A second observer is standing on a bearing of 130° from the base of the tower, and on a bearing of 50° from the first observer. If the height of the tower is 45m, find the distance between the two observers, and the angle of elevation of the top of the tower as measured by the second observer.

5. A small plane is flying due east at a constant altitude of 3 km and a constant speed of 120 km/h. It is approaching a small control tower that lies to the South of the plane's path. At time t_0 the plane is on a bearing of 300° from the tower, and elevated at 4.5°. How long does it take for the plane to be due North of the tower, and what is its angle of elevation from the tower at this time?

6. A plane is flying at a constant altitude h with a constant speed of 250km/h. At 10:30 AM it passes directly over a town T heading towards a second town R. A fisherman located next to a river 50km due South of T observes the angle of elevation to the plane to be 4.5°. Town R lies on a bearing of 300° from where the fisherman is standing, and when the plane flies directly over R, the angle of depression to the fisherman is 2.5°. At what time does the plane pass directly over town R?

7. Frank and Stella are walking along a straight road heading North, when they spot the top of a tower in the direction N$\theta°$ E, behind low lying trees. The angle of elevation to the top of the tower is $\alpha°$. After walking d m along the road they notice that the tower is now N$\phi°$ E of the road and that the angle of elevation of the top of the tower is now $\beta°$. Let the height of the tower be h m.

(a) Find the distance of the tower from the
  i. first sighting.
  ii. second sighting.

(b) Find an expression of the height, h m, of the tower.

(c) How much further must Stella and Frank walk before the tower is located in an easterly direction?

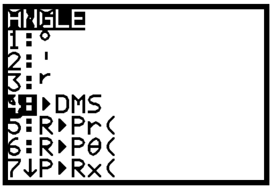
9.7 ARCS, SECTORS AND SEGMENTS

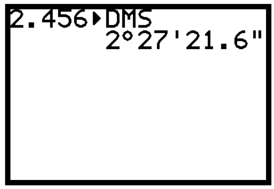
9.7.1 RADIAN MEASURE OF AN ANGLE

So far we have been dealing with angles that have been measured in degrees. However, while this has been very useful, such measurements are not suitable for many topics in mathematics. Instead, we introduce a new measure, called the **radian measure**.

The degree measure of angle is based on dividing the complete circle into 360 equal parts known as degrees. Each degree is divided into sixty smaller parts known as minutes, and each minute is divided into sixty seconds.

Decimal parts of a degree can be converted into degrees, minutes and seconds using the **2nd ANGLE** menu and selecting option 4.





This can be useful as calculators generally produce answers in the decimal format. It should also be noted that the degree, minute, second angle system is the same as the hours minutes seconds system that we use to measure time. The above screen could be interpreted as 2.456° and is equal to 2 degrees 27 minutes and 21.6 seconds or as 2.456 hours which is the same as 2 hours 27 minutes and 21.6 seconds.

The degree system is arbitrary in the sense that the decision was made (in the past and due to astronomical measurements) to divide the complete circle into 360 parts. The radian system is an example of a natural measurement system.

One radian is defined as the size of angle needed to cut off an arc of the same length as the radius.

Two radians is the angle that gives an arc length of twice the radius, etc., giving a natural linear conversion between the measure of a radian, the arc length and the radius of a circle.

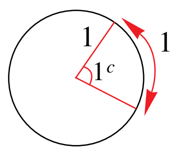
A complete circle has an arc length of $2\pi r$. It follows that a complete circle corresponds to $\frac{2\pi r}{r} = 2\pi$ radians. This leads to the conversion factor between these two systems:

$$360^\circ = 2\pi \text{ radians or } 180^\circ = \pi \text{ radians (often written as } \pi^c\text{)}$$

So, exactly how large is a radian?

Using the conversion above, if $360^\circ = 2\pi^c$, then $1^c = \frac{360}{2\pi} \approx 57.2957^\circ$

That is, the angle which subtends an **arc of length 1 unit** in a **circle of radius 1** unit, is **1 radian**.



More generally we have:

To convert from **degrees to radians**, multiply angle by $\frac{\pi^c}{180}$.

To convert from **radians to degrees**, multiply angle by $\frac{180}{\pi^c}$.

All conversions between the two systems follow this ratio. It is not generally necessary to convert between the systems as problems are usually worked either entirely in the degree system (as in the previous sections) or in radians (as in the functions and calculus chapters). In the case of arc length and sector areas, it is generally better to work in the radian system.

EXAMPLE 9.23

Convert

(a) 70° into radians (b) 2.34^c into degrees (c) $\frac{\pi^c}{6}$ into degrees

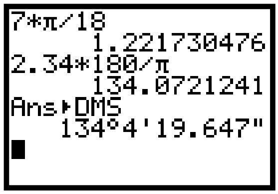
Solution

Using the above conversion factors we have:

(a) $70^\circ = 70 \times \frac{\pi^c}{180} = \frac{7\pi^c}{18}$ or 1.2217^c

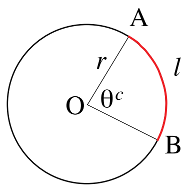
(b) $2.34^c = 2.34 \times \frac{180^\circ}{\pi} = 134.0721^\circ = 134^\circ 4' 20''$

(c) $\frac{\pi^c}{6} = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$



9.7.2 ARC LENGTH

As the arc length AB of a circle is directly proportional to the angle which AB subtends at its centre, then, the arc length AB is a fraction of the circumference of the circle of radius r.



So, if the angle is θ^c, then the arc length is $\frac{\theta}{2\pi}$ of the circumference.

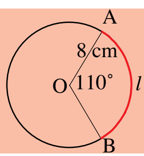
Then, the **(minor) arc length**, AB, denoted by l, is given by $l = \frac{\theta}{2\pi} \times 2\pi r = r\theta$.

i.e., $$l = r\theta^c$$

The longer arc AB, called the **major arc**, has a length of $2\pi r - l$.

EXAMPLE 9.24

Using the circle shown, find the arc length AB.



Solution

First we need to convert 110° into radian measure.

$$110° = 110 \times \frac{\pi^c}{180} = \frac{11\pi^c}{18}$$

Then, the arc length, l, is given by, $l = r\theta = 8 \times \frac{11\pi}{18} = \frac{44\pi}{9}$

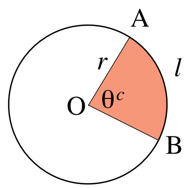
$$= 15.3588\ldots$$

Therefore, the arc length is 15.36 cm.

9.7.3 AREA OF SECTOR

The formula for the area of a sector is derived as follows:

If a sector is cut from a circle of radius r using an angle at the centre of θ radians, the area of the complete circle is πr^2. The fraction of the circle that forms the sector is $\frac{\theta}{2\pi}$ of the complete circle, so the area of the sector is $\pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta$.

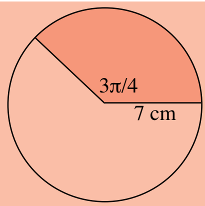


i.e.,

$$A = \frac{1}{2}r^2\theta^c$$

EXAMPLE 9.25

Find the area and perimeter of the sector shown



Solution

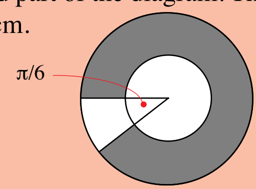
Area of sector $= \frac{1}{2} \times 7^2 \times \frac{3\pi}{4} = \frac{147\pi}{8}$ cm^2

The perimeter is made up from two radii (14 cm) and the arc $l = r\theta = 7 \times \frac{3\pi}{4} = \frac{21\pi}{4}$

The perimeter is $14 + \frac{21\pi}{4}$ cm.

EXAMPLE 9.25

Find the area and perimeter of the shaded part of the diagram. The radius of the inner circle is 4 cm and the radius of the outer circle is 9 cm.



Solution

The angle of the shaded segment = $2\pi - \frac{\pi}{6} = \frac{11\pi^c}{6}$

The shaded area can be found by subtracting the area of the sector in the smaller circle from that in the larger circle.

Shaded area = $\frac{1}{2} \times 9^2 \times \frac{11\pi}{6} - \frac{1}{2} \times 4^2 \times \frac{11\pi}{6} = \frac{11\pi}{12}(9^2 - 4^2) = 59\frac{7}{12}\pi \text{ cm}^2$

The perimeter is made up from two straight lines (each 9 – 4 = 5 cm long) and two arcs.

Perimeter = $10 + 4 \times \frac{11\pi}{6} + 9 \times \frac{11\pi}{6} = 10 + \frac{143\pi}{6}$ cm.

EXERCISES 9.7

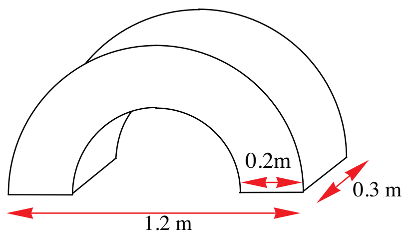
1. Find the areas and perimeters of the following sectors:

	Radius	Angle		Radius	Angle
i.	2.6cm	$\frac{\pi}{3}$	ii.	11.5cm	$\frac{\pi}{4}$
iii.	44cm	$\frac{\pi}{4}$	iv.	6.8m	$\frac{2\pi}{3}$
v.	0.64cm	$\frac{3\pi}{4}$	vi.	7.6cm	$\frac{5\pi}{6}$
vii.	324m	$\frac{\pi}{10}$	viii.	8.6cm	$\frac{7\pi}{6}$
ix.	6.2cm	$\frac{4\pi}{3}$	x.	76m	$\frac{11\pi}{6}$
xi.	12cm	30˚	xii.	14m	60˚
xiii.	2.8cm	120˚	xiv.	24.8cm	270˚
xv.	1.2cm	15˚			

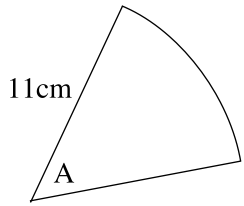
2. A cake has a circumference of 30cm and a uniform height of 7cm. A slice is to be cut from the cake with two straight cuts meeting at the centre. If the slice is to contain 50 cm^3 of cake, find the angle between the two cuts, giving the answer in radians to 2 significant figures and in degrees correct to the nearest degree.

3. The diagram shows a part of a Norman arch. The dimensions are shown in metres.

Find the volume of stone in the arch, giving your answer in cubic metres, correct to three significant figures.

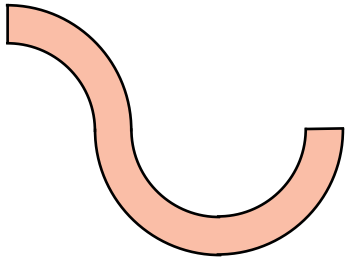


4. In the diagram, find the value of the angle A in radians, correct to three significant figures, if the perimeter is equal to 40cm.



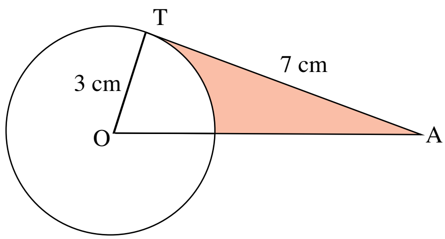
5. The diagram shows a design for a shop sign. The arcs are each one quarter of a complete circle. The radius of the smaller circle is 7 cm and the radius of the larger circle is 9cm.

Find the perimeter of the shape, correct to the nearest centimetre.

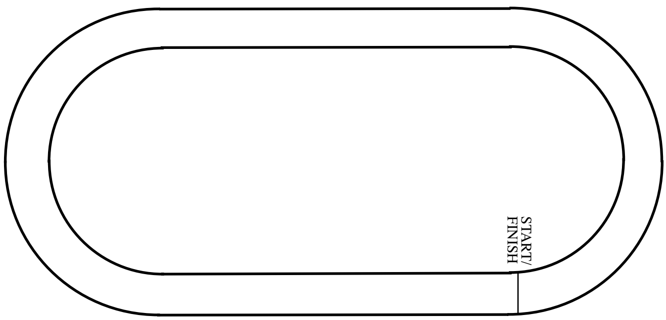


6. Find the shaded area in the diagram. The dimensions are given in centimetres. O is the centre of the circle and AT is a tangent.

Give your answer correct to three significant figures.



7. The diagram shows a running track. The perimeter of the inside line is 400 metres and the length of each straight section is 100 metres.

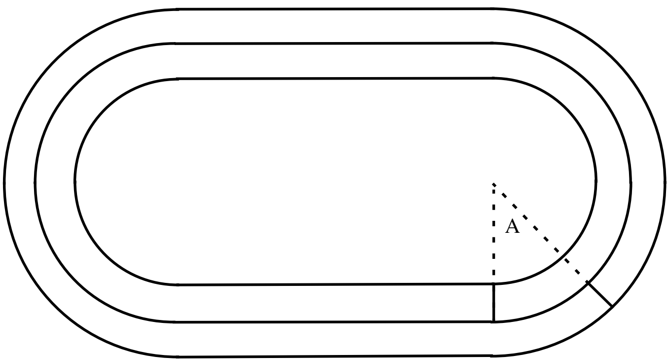


(a) Find the radius of each of the semi-circular parts of the inner track.

(b) If the width of the lane shown is 1 metre, find the perimeter of the outer boundary of the lane.

A second lane is added on the outside of the track. The starting positions of runners who have to run (anti-clockwise) in the two lanes are shown.

(c) Find the value of angleA° (to the nearest degree) if both runners are to run 400 metres.



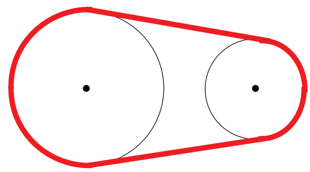
8. Find the angle subtended at the centre of a circle of radius length 12 cm by an arc which forms a sector of area 80 sq. cm

9. Find the angle subtended at the circumference of a circle of radius length 10 cm by an arc which forms a sector of area 75 sq. cm

10. A chord of length 32 cm is drawn in a circle of radius 20 cm.
(a) Find the angle it subtends at the centre.
(b) Find :
i. the minor arc length.
ii. the major arc length.
(c) Find the area of the minor sector.

11. Two circles of radii 6 cm and 8 cm have their centres 10 cm apart. Find the area common to both circles.

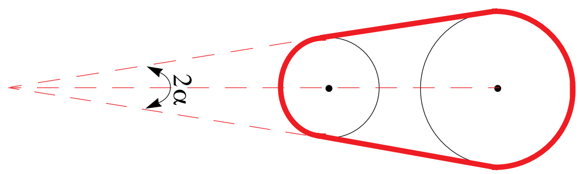
12. Two pulleys of radii 16 cm and 20 cm have their centres 40 cm apart. Find the length of the piece of string that will be required to pass tightly round the circles if the string does not cross over.



13. Two pulleys of radii 7 cm and 11 cm have their centres 24 cm apart. Find the length of the piece of string that will be required to pass tightly round the circles if
(a) the string cannot cross over.
(b) the string crosses over itself.

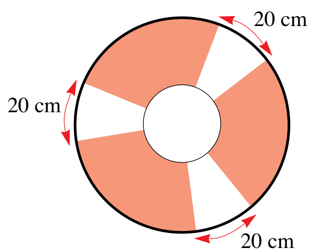
14. A sector of a circle has a radius of 15 cm and an angle of $216°$. The sector is folded in such a way that it forms a cone, so that the two straight edges of the sector do not overlap.
(a) Find the base radius of the cone.
(b) Find the vertical height of the cone.
(c) Find the semi–vertical angle of the cone.

15. A taut belt passes over two discs of radii 12 cm and 4 cm as shown in the diagram.



(a) If the total length of the belt is 88 cm, show that $1 = (5.5 - \pi - \alpha)\tan\alpha$
(b) On the same set of axes, sketch the graphs of
i. $y = \dfrac{1}{\tan\alpha}$ ii. $y = 5.5 - \pi - \alpha$.
(c) Hence find $\{\alpha : 1 = (5.5 - \pi - \alpha)\tan\alpha\}$, giving your answer to two d.p.

16. The diagram shows a disc of radius 40 cm with parts of it painted. The smaller circle (having the same centre as the disc) has a radius of 10 cm. What area of the disc has not been painted in red?



10.1 TRIGONOMETRIC RATIOS

10.1.1 THE UNIT CIRCLE

We saw in Chapter 9 that we were able to find the sine, cosine and tangent of acute angles contained within a right-angled triangle. We extended this to enable us to find the sine and cosine ratio of obtuse angles. To see why this worked, or indeed why it would work for an angle of any magnitude, we need to reconsider how angles are measured. To do this we start by making use of the unit circle and introduce some definitions.

From this point on we define the angle θ as a real number that is measured in either degrees or radians. So that, an expression such as $\sin(180° - \theta)$ will imply that θ is measured in degrees as opposed to the expression $\sin(\pi^c - \theta)$ which would imply that θ is measured in radians. In both cases, it should be clear from the context of the question which one it is.

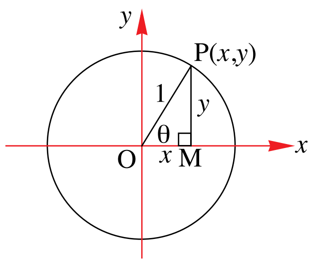
From the work in §9.7 we have the following conversions between degrees and radians and the exact value of their trigonometric ratios:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30° = \frac{\pi^c}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45° = \frac{\pi^c}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60° = \frac{\pi^c}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90° = \frac{\pi^c}{2}$	1	0	–

Note that $\tan 90°$ is undefined. We will shortly see why this is the case.

By convention, an angle θ is measured in terms of the rotation of a ray OP from the positive direction of the x–axis, so that a rotation in the **anticlockwise** direction is described as a **positive** angle, whereas a rotation in the **clockwise** direction is described as a **negative** angle.

Let the point $P(x, y)$ be a point on the circumference of the unit circle, $x^2 + y^2 = 1$, with centre at the origin and radius 1 unit.



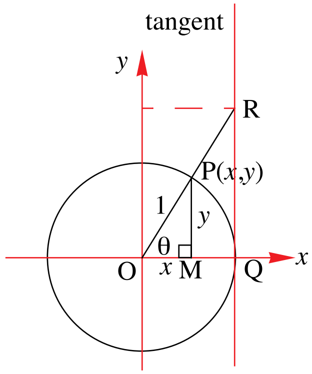
With OP making an angle of θ with the positive direction of the x–axis, we draw the perpendicular from P to meet the x–axis at M. This then provides the following definitions:

$$\sin\theta = \frac{MP}{OP} = \frac{y\text{-coordinate of P}}{OP} = \frac{y}{1} = y$$

$$\cos\theta = \frac{MP}{OP} = \frac{x\text{-coordinate of P}}{OP} = \frac{x}{1} = x$$

$$\tan\theta = \frac{MP}{OM} = \frac{y\text{-coordinate of P}}{x\text{-coordinate of P}} = \frac{y}{x}$$

Note that this means that the y-coordinate corresponds to the sine of the angle θ. That the x-coordinate corresponds to the cosine of the angle θ and that the tangent,. . . , well, for the tangent, let's revisit the unit circle, but this time we will make an addition to the diagram.



Using the existing unit circle, we draw a tangent at the point where the circle cuts the positive x-axis, Q.

Next, we extend the ray OP to meet the tangent at R.

Using similar triangles, we have that $\frac{PM}{OM} = \frac{RQ}{OQ} = \frac{RQ}{1}$.

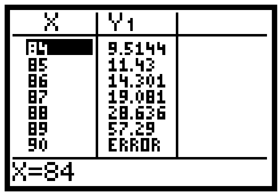
That is, $\tan\theta = RQ$ – which means that the value of the tangent of the angle θ corresponds to the y-coordinates of point R cut off on the tangent at Q by the extended ray OP.

Also, it is worth noting that $\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$ (as long as $\cos\theta \neq 0$).

That is, $\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0$

From our table of exact values, we note that $\tan 90°$ was undefined. This can be observed from the above diagram. If $\theta = 90°$, P lies on the y-axis, meaning that OP would be parallel to QR, and so, P would never cut the tangent, meaning that no y-value corresponding to R could ever be obtained.

Using a table of values for $\tan\theta$ on the TI-83, we see how the tangent ratio increases as θ increases to 90˚ and in particular how it is undefined for $\theta = 90°$.



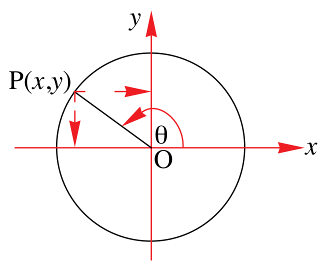
X	Y1
84	9.5144
85	11.43
86	14.301
87	19.081
88	28.636
89	57.29
90	ERROR

X=84

10.1.2 ANGLE OF ANY MAGNITUDE

From the unit circle we have seen how the trigonometric ratios of an acute angle can be obtained – i.e., for the sine ratio we read off the y–axis, for the cosine ratio, we read off the x-axis and for the tangent ratio we read off the tangent. As the point P is located in the first quadrant, then $x \geq 0$, $y \geq 0$ and $\frac{y}{x} \geq 0, x \neq 0$. Meaning that we obtain positive trigonometric ratios.

So, what if P lies in the second quadrant?



We start by drawing a diagram for such a situation:

From our diagram we see that if P lies in the second quadrant, the y-value is still positive, the x-value is negative and therefore the ratio, $\frac{y}{x}$ is negative.

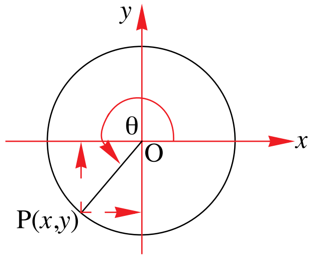
This means that, $\sin\theta > 0$, $\cos\theta < 0$ and $\tan\theta < 0$.

In a similar way, we can conclude that if $180° < \theta < 270°$,
i.e., the point P is in the **third quadrant**, then,

y-value is negative $\Rightarrow \sin\theta < 0$,

x-value is negative $\Rightarrow \cos\theta < 0$

and therefore the ratio $\frac{y}{x}$-value is positive $\Rightarrow \tan\theta > 0$

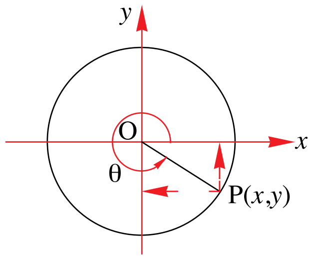


For the **fourth quadrant** we have, $270 < \theta < 360$, so that

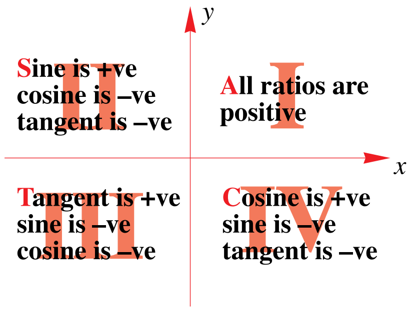
y-value is negative $\Rightarrow \sin\theta < 0$,

x-value is positive $\Rightarrow \cos\theta > 0$

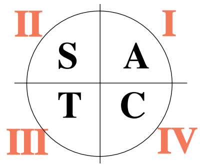
and therefore the ratio $\frac{y}{x}$-value is negative $\Rightarrow \tan\theta < 0$



So far, so good. We now know that depending on which quadrant an angle lies in, the sign of the trigonometric ratio will be either positive or negative. In fact, we can summarise this as follows:



This may be usefully recalled from the following diagram, or by use of a mnemonic sentence

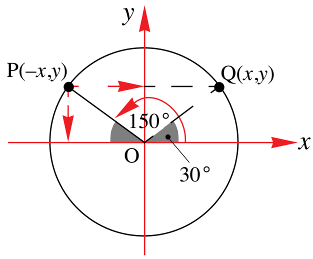


All Stations To City

However, knowing the sign of a trigonometric ratio reflects only half the information. We still need to determine the numerical value. We start by considering a few examples:

Consider the value of $\sin 150°$. Using the unit circle we have:
By symmetry we see that the y-coordinate of Q and the y-coordinate of P are the same and so, $\sin 150° = \sin 30°$.

Therefore, $\sin 150° = \frac{1}{2}$



Note that $150° = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ and $30° = \frac{\pi}{6}$, so that in radian form we have, $\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$.

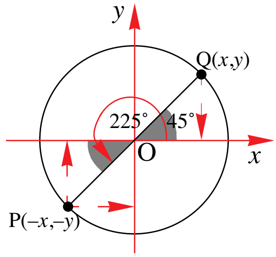
In other words, we were able to express the sine of an angle in the second quadrant in terms of the sine of an angle in the first quadrant. In particular, we have that

$$\text{If } 0° < \theta < 90°, \sin(180 - \theta) = \sin\theta$$

Or,

$$\text{If } 0^c < \theta < \frac{\pi^c}{2}, \sin(\pi^c - \theta) = \sin\theta$$

Next, consider the value of $\cos 225°$. Using the unit circle we have:



By symmetry we see that the x-coordinate of P has the same magnitude as the x-coordinate of Q but is of the opposite sign.
So, we have that $\cos 225° = -\cos 45°$.

Therefore, $\cos 225° = -\frac{1}{\sqrt{2}}$.

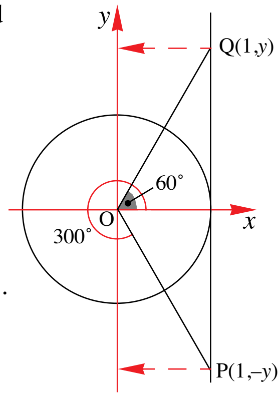
Similarly, as $225° = \frac{3\pi^c}{4}$ and $45° = \frac{\pi^c}{4}$, $\cos\frac{3\pi^c}{4} = -\cos\frac{\pi^c}{4} = -\frac{1}{\sqrt{2}}$.

In other words, we were able to express the cosine of an angle in the third quadrant in terms of the cosine of an angle in the first quadrant. In particular, we have that

If $0° < \theta < 90°$, $\cos(180 + \theta) = -\cos\theta$

Or, If $0^c < \theta < \frac{\pi^c}{2}$, $\cos(\pi^c + \theta) = -\cos\theta$

As a last example we consider the value of $\tan 300°$. This time we need to add a tangent to the unit circle cutting the positive x–axis:



By symmetry we see that the y-coordinate of P has the same magnitude as the y-coordinate of Q but is of the opposite sign.
So, we have that $\tan 300° = -\tan 60°$.

Therefore, $\tan 300° = -\sqrt{3}$.

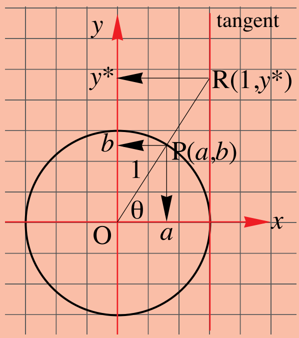
Similarly, as $300° = \frac{5\pi^c}{3}$ and $60° = \frac{\pi^c}{3}$, $\tan\frac{5\pi^c}{3} = -\tan\frac{\pi^c}{3} = -\sqrt{3}$.

In other words, we were able to express the tangent of an angle in the fourth quadrant in terms of the tangent of an angle in the first quadrant. In particular, we have that

If $0° < \theta < 90°$, $\tan(360 - \theta) = -\tan\theta$

Or, If $0^c < \theta < \frac{\pi^c}{2}$, $\tan(2\pi^c - \theta) = -\tan\theta$

In summary we have:



To find the sine of θ, i.e., $\sin\theta$ we read off the y–value of P.
So that, $\sin\theta = b$

To find the cosine of θ, i.e., $\cos\theta$ we read off the x–value of P.
So that, $\cos\theta = a$

To find the tangent of θ, i.e., $\tan\theta$ we read off the y^*–value of R.
So that, $\tan\theta = \frac{b}{a} = y^*$

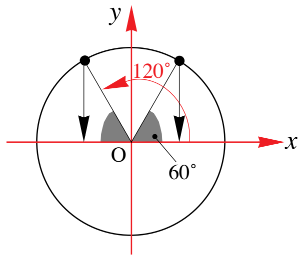
EXAMPLE 10.1

Find the exact values of

(a) $\cos 120°$ (b) $\sin 210°$ (c) $\cos\frac{7\pi}{4}$ (d) $\tan\frac{5\pi}{4}$

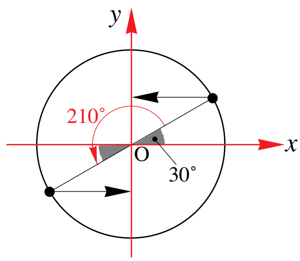
Solution

(a) **Step 1:** Start by drawing the unit circle:
Step 2: Trace out an angle of 120˚
Step 3: Trace out the **reference angle** in the first quadrant. In this case it is 60˚.
Step 4: Use the symmetry between the reference angle and the given angle.
Step 5: State relationship and give answer;



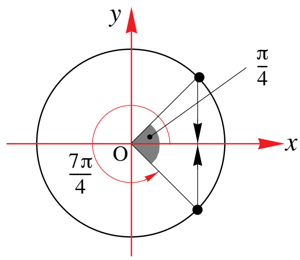
$$\cos 120° = -\cos 60° = -\frac{1}{2}$$

(b) **Step 1:** Start by drawing the unit circle:
Step 2: Trace out an angle of 210˚
Step 3: Trace out the **reference angle** in the first quadrant. In this case it is 30˚.
Step 4: Use the symmetry between the reference angle and the given angle.
Step 5: State relationship and give answer;



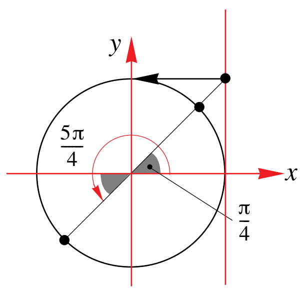
$$\sin 210° = -\sin 30° = -\frac{1}{2}$$

(c) **Step 1:** Start by drawing the unit circle:
Step 2: Trace out an angle of $\frac{7\pi}{4}$ (= 315°)
Step 3: Trace out the **reference angle** in the first quadrant. In this case it is $\frac{\pi}{4}$.
Step 4: Use the symmetry between the reference angle and the given angle.
Step 5: State relationship and give answer;



$$\cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

(c) **Step 1:** Start by drawing the unit circle:
Step 2: Trace out an angle of $\frac{5\pi}{4}$ (= 225°)
Step 3: Trace out the **reference angle** in the first quadrant. In this case it is $\frac{\pi}{4}$.
Step 4: Use the symmetry between the reference angle and the given angle.
Step 5: State relationship and give answer;



$$\tan\frac{5\pi}{4} = \tan\frac{\pi}{4} = 1$$

The results we have obtained, that is, expressing trigonometric ratios of any angle in terms of trigonometric ratios of acute angles in the first quadrant (i.e., reference angles) are known as **trigonometric reduction formulae**. There are too many formulae to commit to memory, and so it is advisible to draw a unit circle and then use symmetry properties as was done in Examples 10.1. We list a number of these formulae in the table below, where $0 < \theta < \frac{\pi}{2}$ (= 90˚).

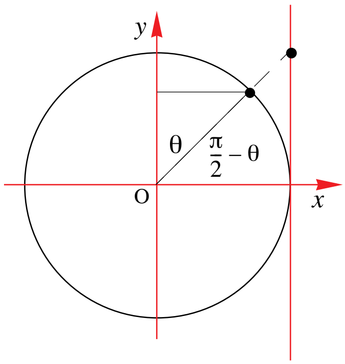
Note: From this point on angles without the degree symbol or radian symbol will mean an angle measured in radian mode.

Quadrant	θ in degrees	θ in radians
2	$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$	$\sin(\pi - \theta) = \sin\theta$ $\cos(\pi - \theta) = -\cos\theta$ $\tan(\pi - \theta) = -\tan\theta$
3	$\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = \tan\theta$	$\sin(\pi + \theta) = -\sin\theta$ $\cos(\pi + \theta) = -\cos\theta$ $\tan(\pi + \theta) = \tan\theta$
4	$\sin(360° - \theta) = -\sin\theta$ $\cos(360° - \theta) = \cos\theta$ $\tan(360° - \theta) = -\tan\theta$	$\sin(2\pi - \theta) = -\sin\theta$ $\cos(2\pi - \theta) = \cos\theta$ $\tan(2\pi - \theta) = -\tan\theta$

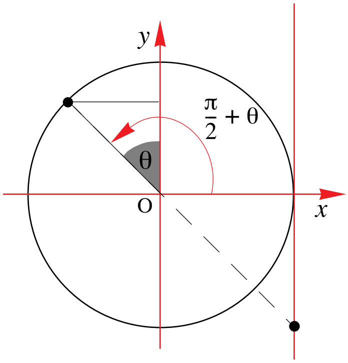
There is another set of results that is suggested by symmetry through the fourth quadrant:

Quadrant	θ in degrees or θ in radians
4	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

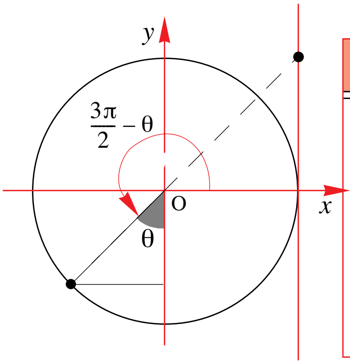
There are other trigonometric reduction formulae, where $0 < \theta < \frac{\pi}{2}$ or $0 < \theta < 90°$. These formulae however, are expressed in terms of their variation from the vertical axis. That is:



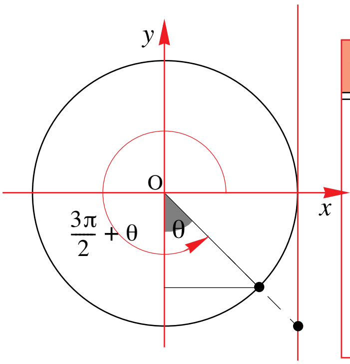
Quadrant	θ in radians or θ in degrees
1	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\sin(90° - \theta) = \cos\theta$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, $\cos(90° - \theta) = \sin\theta$ $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$, $\tan(90° - \theta) = \cot\theta$



Quadrant	θ in radians	or θ in degrees
2	$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$,	$\sin(90°+\theta) = \cos\theta$
	$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$,	$\cos(90°+\theta) = -\sin\theta$
	$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$,	$\tan(90°+\theta) = -\cot\theta$



Quadrant	θ in radians	or θ in degrees
3	$\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta$,	$\sin(270°-\theta) = -\cos\theta$
	$\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$,	$\cos(270°-\theta) = -\sin\theta$
	$\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta$,	$\tan(270°-\theta) = \cot\theta$



Quadrant	θ in radians	or θ in degrees
4	$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$,	$\sin(270°+\theta) = -\cos\theta$
	$\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$,	$\cos(270°+\theta) = \sin\theta$
	$\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$,	$\tan(270°+\theta) = -\cot\theta$

Note the introduction of a new trigonometric ratio, $\cot\theta$. This is one of a set of three other trigonometric ratios known as the **reciprocal trigonometric ratios**, namely **cosecant**, **secant** and **cotangent** ratios. These are defined as

cosecant ratio : $\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$

secant ratio : $\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0$

cotangent ratio : $\cot\theta = \frac{1}{\tan\theta}, \tan\theta \neq 0$

Note then, that $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$

EXAMPLE 10.2

Given that $\sin\theta = 0.3$, where $0 < \theta < \frac{\pi}{2}$ find

(a) $\sin(\pi + \theta)$ (b) $\sin(2\pi - \theta)$ (c) $\cos\left(\frac{\pi}{2} - \theta\right)$

Solution

(a) From the reduction formulae, we have that $\sin(\pi + \theta) = -\sin\theta$.
Therefore, $\sin(\pi + \theta) = -0.3$.

(b) From the reduction formulae, we have that $\sin(2\pi - \theta) = -\sin\theta$.
Therefore, $\sin(\pi + \theta) = -0.3$.

(c) From the reduction formulae, we have that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

Therefore, $\cos\left(\frac{\pi}{2} - \theta\right) = 0.3$.

EXAMPLE 10.3

Given that $\cos\theta = k$ and $0 < \theta < \frac{\pi}{2}$ find

(a) $\cos(\pi + \theta)$ (b) $\cos(2\pi - \theta)$ (c) $\cos\left(\frac{\pi}{2} + \theta\right)$

Solution

(a) $\cos(\pi + \theta) = -\cos\theta \therefore \cos(\pi + \theta) = -k$.

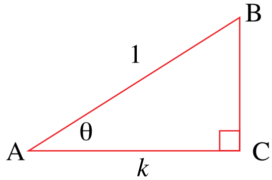
(b) $\cos(2\pi - \theta) = \cos\theta \therefore \cos(2\pi - \theta) = k$.

(c) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$. However, we only have a value for $\cos\theta$.

To determine the value of $\sin\theta$ that corresponds to $\cos\theta = k$ we make use of a right-angled triangle where $\cos\theta = k$.

Construct a right-angled triangle ABC, where $\angle BAC = \theta$

so that $AC = k$ and $AB = 1$ $\left(\text{i.e., } \cos\theta = \frac{AC}{AB} = \frac{k}{1} = k\right)$.



Then, from Pythagoras's theorem, we have

$$1^2 = k^2 + BC^2 \Leftrightarrow BC = \pm\sqrt{1 - k^2}$$

Therefore, as $\sin\theta = \frac{BC}{AB} \Rightarrow \sin\theta = \frac{\pm\sqrt{1-k^2}}{1} = \pm\sqrt{1-k^2}$.

However, as $0 < \theta < \frac{\pi}{2}$, then θ is in the first quadrant and so, $\sin\theta > 0 \therefore \sin\theta = \sqrt{1-k^2}$.

Now that we have the value of $\sin\theta$ we can complete the question:

$$\sin\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \therefore \sin\left(\frac{\pi}{2} + \theta\right) = -\sqrt{1-k^2}$$

Part (c) in Example 10.3 shows a useful approach, i.e., contstructing a right-angled triangle to help in determining the trigonometric ratio of one of the six trig ratios based on any one of the remaining five trig ratios.

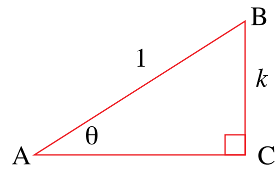
EXAMPLE 10.4

Given that $\sin\theta = k$ and $0 < \theta < \frac{\pi}{2}$ find

(a) $\tan\theta$ (b) $\mathrm{cosec}\,\theta$ (c) $\sec(\pi + \theta)$

Solution

As we are looking for trigonometric ratios based solely on that of the sine ratio, we start by constructing a right-angled triangle satisfying the relationship, $\sin\theta = k$



In this case, as $\sin\theta = \frac{\text{opp}}{\text{hyp}} = k \Rightarrow \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} = \frac{k}{1}$

(using the simplest ratio).

Using Pythagoras's theorem, we have

$$1^2 = k^2 + AC^2 \Leftrightarrow AC = \pm\sqrt{1-k^2}$$

(a) $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{k}{\pm\sqrt{1-k^2}}$.

However, as $0 < \theta < \frac{\pi}{2}$, $\tan\theta > 0$ $\therefore \tan\theta = \frac{k}{\sqrt{1-k^2}}$.

(b) $\mathrm{cosec}\,\theta = \frac{1}{\sin\theta}$ $\therefore \mathrm{cosec}\,\theta = \frac{1}{k}$.

(c) $\sec(\pi+\theta) = \frac{1}{\cos(\pi+\theta)} = -\frac{1}{\cos\theta}$.

But, $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\pm\sqrt{1-k^2}}{1} = \pm\sqrt{1-k^2}$.

However, as $0 < \theta < \frac{\pi}{2}$, $\cos\theta > 0$ $\therefore \cos\theta = \sqrt{1-k^2}$.

Therefore, $\sec(\pi+\theta) = -\frac{1}{\sqrt{1-k^2}}$.

EXAMPLE 10.5

Find the exact values of

(a) $\sec 45°$ (b) $\mathrm{cosec}\,150°$ (c) $\cot\frac{11\pi}{6}$ (d) $\sec 0$

Solution

(a) $\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}$

(b) $\mathrm{cosec}\,150° = \frac{1}{\sin 150°} = \frac{1}{\sin 30°} = \frac{1}{\left(\frac{1}{2}\right)} = 2$

(c) $\cot\frac{11\pi}{6} = \frac{1}{\tan\left(\frac{11\pi}{6}\right)} = \frac{1}{\tan\left(-\frac{\pi}{6}\right)} = \frac{1}{-\tan\frac{\pi}{6}} = \frac{1}{-\left(\frac{1}{\sqrt{3}}\right)} = -\sqrt{3}$

(c) $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

EXAMPLE 10.6

Find (a) $\sin\theta = \frac{1}{2}, 0° < \theta < 360°$

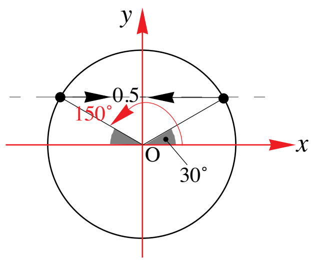
(b) $\tan\theta = -\sqrt{3}, 0 < \theta < 2\pi$

(c) $\sec\theta = 1, 0 < \theta < 2\pi$

Solution

(a) This time we are searching for those values of θ for which $\sin\theta = \frac{1}{2}$.

To do this we make use of the unit circle:
From the unit circle, we draw a horizontal chord passing the y–axis at $y = 0.5$.
Then, in the first quadrant, from our table of exact values we have that $\theta = 30°$.



However, by symmetry, we also have that $\sin 150° = \frac{1}{2}$.

Therefore, $\sin\theta = \frac{1}{2}, 0° < \theta < 360°$ if $\theta = 30°$ or $150°$.

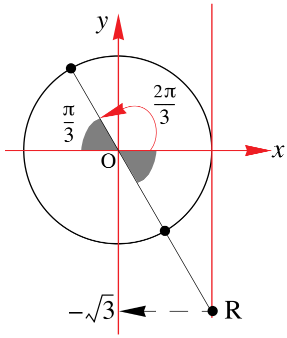
(b) This time we are searching for those values of θ for which $\tan\theta = -\sqrt{3}$.
From the unit circle, we extend the ray OP so that it cuts the tangent line at R.

Using the exact values, we have $\tan\left(\pi - \frac{\pi}{3}\right) = \tan\frac{2\pi}{3} = -\sqrt{3}$

(as our first value).

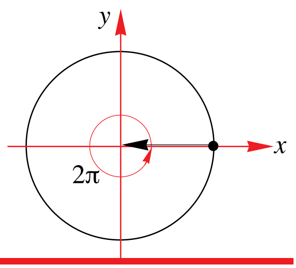
And, by symmetry, we also have that $\tan\left(2\pi - \frac{\pi}{3}\right) = -\sqrt{3}$.

Therefore, $\theta = \frac{2\pi}{3}$ or $\theta = \frac{5\pi}{3}$.



(c) $\sec\theta = 1 \Leftrightarrow \frac{1}{\cos\theta} = 1 \Leftrightarrow \cos\theta = 1$

Therefore, $\theta = 0$ or $\theta = 2\pi$



EXAMPLE 10.7 Simplify

(a) $\dfrac{\sin(\pi+\theta)}{\cos(2\pi-\theta)}$ (b) $\dfrac{\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)}{\cos(\pi+\theta)}$, where $0<\theta<\frac{\pi}{2}$.

Solution

(a) $\dfrac{\sin(\pi+\theta)}{\cos(2\pi-\theta)} = \dfrac{-\sin\theta}{\cos\theta}$

$= -\tan\theta$

(b) $\dfrac{\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)}{\cos(\pi+\theta)} = \dfrac{\cos\theta\sin\theta}{-\cos\theta}$

$= -\sin\theta$

EXERCISES 10.1

1. Convert the following angles to degrees.

(a) $\frac{2\pi}{3}$ (b) $\frac{3\pi}{5}$ (c) $\frac{12\pi}{10}$ (d) $\frac{5\pi}{18}$

2. Convert the following angles to radians.

(a) 180˚ (b) 270˚ (c) 140˚ (d) 320˚

3. Find the exact value of

(a)	$\sin 120°$	(b)	$\cos 120°$	(c)	$\tan 120°$	(d)	$\sec 120°$
(e)	$\sin 210°$	(f)	$\cos 210°$	(g)	$\tan 210°$	(h)	$\cot 210°$
(i)	$\sin 225°$	(j)	$\cos 225°$	(k)	$\tan 225°$	(l)	$\text{cosec } 225°$
(m)	$\sin 315°$	(n)	$\cos 315°$	(o)	$\tan 315°$	(p)	$\sec 315°$
(q)	$\sin 360°$	(r)	$\cos 360°$	(s)	$\tan 360°$	(t)	$\text{cosec } 360°$

4. Find the exact value of

(a)	$\sin\pi$	(b)	$\cos\pi$	(c)	$\tan\pi$	(d)	$\sec\pi$
(e)	$\sin\frac{3\pi}{4}$	(f)	$\cos\frac{3\pi}{4}$	(g)	$\tan\frac{3\pi}{4}$	(h)	$\text{cosec}\frac{3\pi}{4}$
(i)	$\sin\frac{7\pi}{6}$	(j)	$\cos\frac{7\pi}{6}$	(k)	$\tan\frac{7\pi}{6}$	(l)	$\cot\frac{7\pi}{6}$
(m)	$\sin\frac{5\pi}{3}$	(n)	$\cos\frac{5\pi}{3}$	(o)	$\tan\frac{5\pi}{3}$	(p)	$\sec\frac{5\pi}{3}$
(q)	$\sin\frac{7\pi}{4}$	(r)	$\cos\frac{7\pi}{4}$	(s)	$\tan\frac{7\pi}{4}$	(t)	$\text{cosec}\frac{7\pi}{4}$

5. Find the exact value of

(a) $\sin(-210^\circ)$ (b) $\cos(-30^\circ)$ (c) $\tan(-135^\circ)$ (d) $\cos(-420^\circ)$
(e) $\cot(-60^\circ)$ (f) $\sin(-150^\circ)$ (g) $\sec(-135^\circ)$ (h) $\operatorname{cosec}(-120^\circ)$

6. Find the exact value of

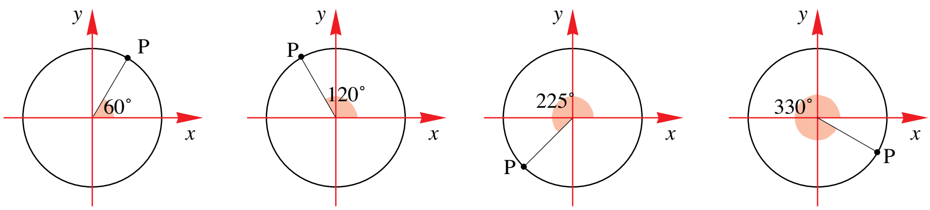
(a) $\sin\left(-\frac{\pi}{6}\right)$ (b) $\cos\left(-\frac{3\pi}{4}\right)$ (c) $\tan\left(-\frac{2\pi}{3}\right)$ (d) $\sec\left(-\frac{4\pi}{3}\right)$

(e) $\cot\left(-\frac{3\pi}{4}\right)$ (f) $\sin\left(-\frac{7\pi}{6}\right)$ (g) $\cot\left(-\frac{\pi}{3}\right)$ (h) $\cos\left(-\frac{7\pi}{6}\right)$

(i) $\operatorname{cosec}\left(-\frac{2\pi}{3}\right)$ (j) $\tan\left(-\frac{11\pi}{6}\right)$ (k) $\sec\left(-\frac{13\pi}{6}\right)$ (l) $\sin\left(-\frac{7\pi}{3}\right)$

7. Find the coordinates of the point P on the following unit circles.

(a) (b) (c) (d)



8. Find the exact value of

(a) $\sin\frac{11\pi}{6}\cos\frac{5\pi}{6} - \sin\frac{5\pi}{6}\cos\frac{11\pi}{6}$ (b) $2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

(c) $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ (d) $\cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

9. Show that the following relationships are true

(a) $\sin 2\theta = 2\sin\theta\cos\theta$, where $\theta = \frac{\pi}{3}$

(b) $\cos 2\theta = 2\cos^2\theta - 1$, where $\theta = \frac{\pi}{6}$.

(c) $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, where $\theta = \frac{2\pi}{3}$.

(d) $\sin(\theta - \phi) = \sin\theta\cos\phi - \sin\phi\cos\theta$, where $\theta = \frac{2\pi}{3}$ and $\phi = -\frac{\pi}{3}$.

10. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$ find

(a) $\sin(\pi + \theta)$ (b) $\sin(2\pi - \theta)$ (c) $\cos\left(\frac{\pi}{2} + \theta\right)$

11. Given that $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$ find

(a) $\cos(\pi - \theta)$ (b) $\sec\theta$ (c) $\sin\left(\frac{\pi}{2} - \theta\right)$

12. Given that $\tan\theta = k$ and $0 < \theta < \frac{\pi}{2}$ find

(a) $\tan(\pi + \theta)$ (b) $\tan\left(\frac{\pi}{2} + \theta\right)$ (c) $\tan(-\theta)$

13. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$ find

(a) $\cos\theta$ (b) $\sec\theta$ (c) $\cos(\pi + \theta)$

14. Given that $\cos\theta = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$ find

(a) $\sin\theta$ (b) $\tan\theta$ (c) $\cos(\pi + \theta)$

15. Given that $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$ find

(a) $\sin\theta$ (b) $\tan\left(\frac{\pi}{2} + \theta\right)$ (c) $\sec\theta$

16. Given that $\cos\theta = k$ and $\frac{3\pi}{2} < \theta < 2\pi$ find

(a) $\cos(\pi - \theta)$ (b) $\sin\theta$ (c) $\cot\theta$

17. Given that $\sin\theta = -k$ and $\pi < \theta < \frac{3\pi}{2}$ find

(a) $\cos\theta$ (b) $\tan\theta$ (c) $\text{cosec}\left(\frac{\pi}{2} + \theta\right)$

18. Simplify the following

(a) $\dfrac{\sin(\pi - \theta)\cos\left(\frac{\pi}{2} + \theta\right)}{\sin(\pi + \theta)}$ (b) $\dfrac{\sin\left(\frac{\pi}{2} + \theta\right)\cos\left(\frac{\pi}{2} - \theta\right)}{\sin^2\theta}$ (c) $\dfrac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\theta}$

(d) $\tan(\pi + \theta)\cot\theta$ (e) $\cos(2\pi - \theta)\text{cosec}\theta$ (f) $\dfrac{\sec\theta}{\text{cosec}\theta}$

19. If $0 \le \theta \le 2\pi$, find all values of x such that

(a) $\sin x = \frac{\sqrt{3}}{2}$ (b) $\cos x = \frac{1}{2}$ (c) $\tan x = \sqrt{3}$

(d) $\cos x = -\frac{\sqrt{3}}{2}$ (e) $\tan x = -\frac{1}{\sqrt{3}}$ (f) $\sin x = -\frac{1}{2}$

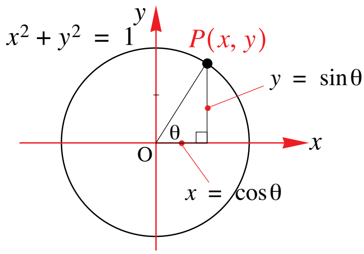
10.2 TRIGONOMETRIC IDENTITIES

10.2.1 THE FUNDAMENTAL IDENTITY

We have seen a number of important relationships between trigonometric ratios. Relationships that are true for all values of θ are known as **identities**. To signal an identity (as opposed to an equation) the **equivalence** symbol is used, i.e., ‘ $\equiv$ ’.

For example, we can write $(x+1)^2 \equiv x^2 + 2x + 1$, as this statement is true for all values of x. However, we would have to write $(x+1)^2 = x+1$, as this relationship is only true for some values of x (which need to be determined).

One trigonometric identity is based on the unit circle



Consider the point $P(x, y)$ on the unit circle,

$$x^2 + y^2 = 1 \quad \text{– (1)}$$

From the previous section, we know that

$$x = \cos\theta \quad \text{– (2)}$$

$$y = \sin\theta \quad \text{– (3)}$$

Substituting (2) and (3) into (1) we have: $(\cos\theta)^2 + (\sin\theta)^2 = 1$

or

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{– (4)}$$

This is known as the fundamental trigonometric identity. Note that we have not used the identity symbol, i.e., we have not written $\sin^2\theta + \cos^2\theta \equiv 1$. This is because more often than not, it will be ‘obvious’ from the setting as to whether a relationship is an identity or an equation. And so, there is a tendency to forgo the formal use of the identity statement and restrict ourselves to the equality statement.

By rearranging the identity we have that $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$. Similarly we obtain the following two new identities:

Divide both sides of (4) by $\cos^2\theta$: $\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \Leftrightarrow \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$

$$\Leftrightarrow \tan^2\theta + 1 = \sec^2\theta \quad \text{– (5)}$$

Divide both sides of (4) by $\sin^2\theta$: $\dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta} \Leftrightarrow \dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$

$$\Leftrightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta \quad \text{– (6)}$$

In summary we have:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \operatorname{cosec}^2\theta \end{aligned}$$

EXAMPLE 10.8

If $\cos\theta = -\frac{3}{5}$, where $\pi \le \theta \le \frac{3\pi}{2}$, find (a) $\sin\theta$ (b) $\tan\theta$

Solution

(a) Although we solved problems like this in §10.1 by making use of a right-angled triangle, we now solve this question by making use of the trigonometric identities we have just developed.

From $\sin^2\theta + \cos^2\theta = 1$ we have $\sin^2\theta + \left(-\frac{3}{5}\right)^2 = 1 \Leftrightarrow \sin^2\theta + \frac{9}{25} = 1$

$$\Leftrightarrow \sin^2\theta = \frac{16}{25}$$

$$\therefore \sin\theta = \pm\frac{4}{5}$$

Now, as $\pi \le \theta \le \frac{3\pi}{2}$, this means the angle is in the third quadrant, where the sine value is negative. Therefore, we have that $\sin\theta = -\frac{4}{5}$

(b) Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have $\tan\theta = \frac{(-4/5)}{(-3/5)} = \frac{4}{3}$.

EXAMPLE 10.9

If $\tan\theta = \frac{5}{12}$, where $\pi \le \theta \le \frac{3\pi}{2}$, find (a) $\cos\theta$ (b) $\operatorname{cosec}\theta$

Solution

(a) From the identity $\tan^2\theta + 1 = \sec^2\theta$ we have $\left(\frac{5}{12}\right)^2 + 1 = \sec^2\theta \Leftrightarrow \sec^2\theta = \frac{25}{144} + 1$

$$\therefore \sec^2\theta = \frac{169}{144}$$

$$\therefore \sec\theta = \pm\frac{13}{12}$$

Therefore, as $\cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \pm\frac{12}{13}$. However, $\pi \le \theta \le \frac{3\pi}{2}$, meaning that θ is in the third quadrant. And so, the cosine is negative. That is, $\cos\theta = -\frac{12}{13}$.

(b) Now, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, but, $\tan\theta = \frac{\sin\theta}{\cos\theta} \Leftrightarrow \sin\theta = \tan\theta\cos\theta \therefore \sin\theta = \frac{5}{12} \times -\frac{12}{13}$

$$= -\frac{5}{13}$$

Therefore, $\operatorname{cosec}\theta = \frac{1}{(-5/13)} = -\frac{13}{5}$.

EXAMPLE 10.10

Simplify the following expressions

(a) $\cos\theta + \tan\theta\sin\theta$ (b) $\dfrac{\cos\theta}{1+\sin\theta} - \dfrac{1-\sin\theta}{\cos\theta}$

Solution

(a)

$$\begin{aligned}\cos\theta + \tan\theta\sin\theta &= \cos\theta + \frac{\sin\theta}{\cos\theta}\sin\theta \\ &= \cos\theta + \frac{\sin^2\theta}{\cos\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} \\ &= \frac{1}{\cos\theta} \\ &= \sec\theta\end{aligned}$$

(b)

$$\begin{aligned}\frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta} &= \frac{\cos^2\theta}{(1+\sin\theta)\cos\theta} - \frac{(1-\sin\theta)(1+\sin\theta)}{(1+\sin\theta)\cos\theta} \\ &= \frac{\cos^2\theta}{(1+\sin\theta)\cos\theta} - \frac{1-\sin^2\theta}{(1+\sin\theta)\cos\theta} \\ &= \frac{\cos^2\theta - 1 + \sin^2\theta}{(1+\sin\theta)\cos\theta} \\ &= \frac{(\cos^2\theta + \sin^2\theta) - 1}{(1+\sin\theta)\cos\theta} \\ &= \frac{1-1}{(1+\sin\theta)\cos\theta} \\ &= 0\end{aligned}$$

EXAMPLE 10.11

Show that $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta - \cot\theta$.

Solution

$$\begin{aligned}\text{R.H.S} &= \tan\theta - \cot\theta \\ &= \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} \\ &= \frac{(1-\cos^2\theta) - \cos^2\theta}{\sin\theta\cos\theta} \\ &= \frac{1-2\cos^2\theta}{\sin\theta\cos\theta} \\ &= \text{L.H.S}\end{aligned}$$

EXERCISES 10.2.1

1. Prove the identity

(a) $\sin\theta + \cot\theta\cos\theta = \operatorname{cosec}\theta$

(b) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

(c) $\dfrac{\sin^2\theta}{1-\cos\theta} = 1+\cos\theta$

(d) $3\cos^2 x - 2 = 1 - 3\sin^2 x$

(e) $\tan^2 x\cos^2 x + \cot^2 x\sin^2 x = 1$

(f) $\sec\theta - \sec\theta\sin^2\theta = \cos\theta$

(g) $\sin^2\theta(1+\cot^2\theta) - 1 = 0$

(h) $\dfrac{1}{1-\sin\phi} + \dfrac{1}{1+\sin\phi} = 2\sec^2\phi$

(i) $\dfrac{\cos\theta}{1+\sin\theta} + \tan\theta = \sec\theta$

(j) $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

(k) $\dfrac{1}{\sec x + \tan x} = \sec x - \tan x$

(l) $\sin x + \dfrac{\cos^2 x}{1+\sin x} = 1$

(m) $\dfrac{\sec\phi + \operatorname{cosec}\phi}{\tan\phi + \cot\phi} = \sin\phi + \cos\phi$

(n) $\dfrac{\sin x + 1}{\cos x} = \dfrac{\sin x - \cos x + 1}{\sin x + \cos x - 1}$

(o) $\tan x + \sec x = \dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$

2. Prove the following

(a) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

(b) $\sec^2\theta\operatorname{cosec}^2\theta = \sec^2\theta + \operatorname{cosec}^2\theta$

(c) $\sin^4 x - \cos^4 x = (\sin x + \cos x)(\sin x - \cos x)$

(d) $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

(e) $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x\cos x$

(f) $(\cot x - \operatorname{cosec} x)^2 = \dfrac{\sec x - 1}{\sec x + 1}$

(g) $(2b\sin x\cos x)^2 + b^2(\cos^2 x - \sin^2 x)^2 = b^2$

3. Eliminate θ from each of the following pairs

(a) $x = k\sin\theta,\ y = k\cos\theta$

(b) $x = b\sin\theta,\ y = a\cos\theta$

(c) $x = 1+\sin\theta,\ y = 2-\cos\theta$

(d) $x = 1-b\sin\theta,\ y = 2+a\cos\theta$

(e) $x = \sin\theta + 2\cos\theta,\ y = \sin\theta - 2\cos\theta$

4. (a) If $\tan\theta = \dfrac{3}{4}, \pi \le \theta \le \dfrac{3\pi}{2}$, find i. $\cos\theta$ ii. $\operatorname{cosec}\theta$

(b) If $\sin\theta = -\dfrac{3}{4}, \dfrac{3\pi}{2} \le \theta \le 2\pi$, find i. $\sec\theta$ ii. $\cot\theta$

5. Solve the following, where $0 \le \theta \le 2\pi$

(a) $4\sin\theta = 3\operatorname{cosec}\theta$

(b) $2\cos^2\theta + \sin\theta - 1 = 0$

(c) $2 - \sin\theta = 2\cos^2\theta$

(d) $2\sin^2\theta = 2 + 3\cos\theta$

6. Prove $\sin^2x(1+n\cot^2x)+\cos^2x(1+n\tan^2x) = \sin^2x(n+\cot^2x)+\cos^2x(n+\tan^2x)$.

7. If $k\sec\phi = m\tan\phi$, prove that $\sec\phi\tan\phi = \dfrac{mk}{m^2-k^2}$.

8. If $x = k\sec^2\phi + m\tan^2\phi$ and $y = l\sec^2\phi + n\tan^2\phi$, prove that $\dfrac{x-k}{k+m} = \dfrac{y-l}{l+n}$.

9. Given that $\tan\theta = \dfrac{2a}{a^2-1}, 0<\theta<\dfrac{\pi}{2}$, find (a) $\sin\theta$ (b) $\cos\theta$

10. (a) If $\sin x + \cos x = 1$, find the values of

i. $\sin^3x+\cos^3x$ ii. $\sin^4x+\cos^4x$

(b) Hence, deduce the value of $\sin^kx+\cos^kx$, where k is a positive integer.

11. If $\tan\phi = -\dfrac{1}{\sqrt{x^2-1}}, \dfrac{\pi}{2}<\phi<\pi$, find, in terms of x,

(a) $\sin\phi+\cos\phi$ (b) $\sin\phi-\cos\phi$ (c) $\sin^4\phi-\cos^4\phi$

12. Find (a) the maximum value of (b) the minimum value of

i. $\cos^2\theta+5$ ii. $\dfrac{5}{3\sin^2\theta+2}$ iii. $2\cos^2\theta+\sin\theta-1$

13. (a) Given that $b\sin\phi = 1$ and $b\cos\phi = \sqrt{3}$, find b.

(b) Hence, find all values of ϕ that satisfy the relationship described in (a).

14. Find (a) the maximum value of (b) the minimum value of

i. $5^{3\sin\theta-1}$ ii. $3^{1-2\cos\theta}$

15. Given that $\sin\theta\cos\theta = k$, find (a) $(\sin\theta+\cos\theta)^2$, $\sin\theta+\cos\theta>0$.

(b) $\sin^3\theta+\cos^3\theta$, $\sin\theta+\cos\theta>0$.

16. (a) Given that $\sin\phi = \dfrac{1-a}{1+a}, 0<\phi<\dfrac{\pi}{2}$, find $\tan\phi$.

(b) Given that $\sin\phi = 1-a, \dfrac{\pi}{2}<\phi<\pi$, find i. $2-\cos\phi$

ii. $\cot\phi$

17. Find

(a) the value(s) of $\cos x$, where $\cot x = 4(\operatorname{cosec}x-\tan x)$, $0<x<\pi$.

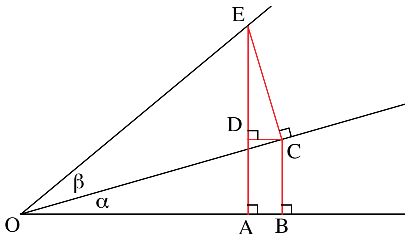
(b) the values of $\sin x$, where $3\cos x = 2+\dfrac{1}{\cos x}, 0\le x\le 2\pi$.

18. Given that $\sin 2x = 2\sin x\cos x$, find all values of x, such that $2\sin 2x = \tan x, 0\le x\le\pi$.

10.2.2 COMPOUND ANGLE IDENTITIES

As we have seen in the previous section, there are numerous trigonometric identities. However, they were all derived from the fundamental identities. In this section we develop some more fundamental identities (which will lead us to more identities). This set of fundamental identities are known as compound angle identities. That is, identities that involve the sine, cosine and tangent of the sum and difference of two angles.

We start with the sine of the sum of two angles, $\sin(\alpha+\beta)$:
A commonly given proof of these identities is again only valid for acute angles:



In the figure, $\angle AOE = \alpha+\beta$. The construction lines are drawn with the right angles indicated. Since $\angle DCO = \alpha$ (alternate angles) and $\angle DCE = 90^\circ - \alpha$, it follows that $\angle AOE = \alpha$. Therefore, we have,

$$\begin{aligned}
\sin(\alpha+\beta) = \sin AOE &= \frac{AE}{OE} \\
&= \frac{AD+DE}{OE} \\
&= \frac{AD}{OE}+\frac{DE}{OE} \\
&= \frac{BC}{OE}+\frac{DE}{OE} \\
&= \frac{BC}{OC}\times\frac{OC}{OE}+\frac{DE}{EC}\times\frac{EC}{OE} \\
&= \sin\alpha\times\cos\beta+\cos\alpha\times\sin\beta
\end{aligned}$$

It is now possible to prove the difference formula, replacing β by $-\beta$ we have

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) \\
&= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)\ (\cos(-\beta) = \cos\beta \text{ and } \sin(-\beta) = -\sin\beta) \\
&= \sin\alpha\cos\beta-\cos\alpha\sin\beta
\end{aligned}$$

And so we have the addition and difference identities for sine:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta
\end{aligned}$$

A similar identity can be derived for the cosine function (using the same diagram):

$$\begin{aligned}
\cos(\alpha+\beta) = \frac{OA}{OE} &= \frac{OB-AB}{OE} \\
&= \frac{OB}{OE}-\frac{AB}{OE} \\
&= \frac{OB}{OE}-\frac{CD}{OE} \\
&= \frac{OB}{OC}\times\frac{OC}{OE}-\frac{CD}{EC}\times\frac{EC}{OE} \\
&= \cos\alpha\cos\beta-\sin\alpha\sin\beta
\end{aligned}$$

Similarly, from this and replacing β by $-\beta$ we have that $\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$

And so we have the addition and difference identities for cosine:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Also, the tangent addition identity can be proved as follows:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \quad \text{using } \frac{\sin\theta}{\cos\theta} = \tan\theta \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\end{aligned}$$

Again, if we replace β by $-\beta$ we have $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

And so we have the addition and difference identities for tangent:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

As a special case of the compound identities we have obtained so far, we have a set of identities known as the **double angle identities**.

Using the substitution $\theta = \alpha = \beta$ we obtain the identities:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

i.e., substituting $\theta = \alpha = \beta$ into $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ we obtain

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \therefore \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}$$

Similarly, substituting $\theta = \alpha = \beta$ into $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ we obtain

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ \therefore \cos 2\theta &= \cos^2\theta - \sin^2\theta\end{aligned}$$

The second of these can be further developed to give:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$

and

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

Finally, we have a double angle identity for the tangent; $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

Summary of double angle identities:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

We have seen how trigonometric identities can be used to solve equations, simplify expressions and to prove further identities. We now illustrate this using the new set of identities.

EXAMPLE 10.12

Simplify the expression: $\dfrac{\sin 3\alpha}{\sin\alpha} - \dfrac{\cos 3\alpha}{\cos\alpha}$.

Solution

$$\begin{aligned}\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} &= \frac{\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha}\\ &= \frac{\sin(3\alpha-\alpha)}{\sin\alpha\cos\alpha}\\ &= \frac{\sin 2\alpha}{\frac{1}{2}\sin 2\alpha}\\ &= 2\end{aligned}$$

EXAMPLE 10.13

Prove the identity $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.

Solution

$$\begin{aligned}\cos 3\alpha &= \cos(2\alpha+\alpha)\\ &= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha\\ &= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha\\ &= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha\\ &= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha\\ &= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha\\ &= 4\cos^3\alpha - 3\cos\alpha\end{aligned}$$

EXAMPLE 10.14

Using a compound identity, show that $\cos\left(\dfrac{3\pi}{2} - \theta\right) = -\sin\theta$

Solution

$$\text{L.H.S} = \cos\left(\frac{3\pi}{2} - \theta\right)$$

$$= \cos\left(\frac{3\pi}{2}\right)\cos\theta + \sin\left(\frac{3\pi}{2}\right)\sin\theta$$

$$= 0 \times \cos\theta + (-1) \times \sin\theta$$

$$= -\sin\theta$$

$$= \text{R.H.S}$$

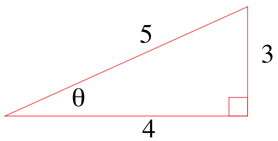
EXAMPLE 10.15

If $\sin\theta = \frac{3}{5}$ and $\cos\phi = -\frac{12}{13}$, where $0 \le \theta \le \frac{\pi}{2}$ and $\pi \le \phi \le \frac{3\pi}{2}$, find

(a) $\sin(\theta + \phi)$ (b) $\cos(\theta + \phi)$ (c) $\tan(\theta - \phi)$

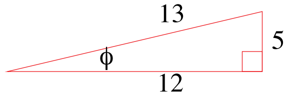
Solution

We start by drawing two right-angled triangles satisfying the given conditions:



$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$



$\sin\phi = \frac{5}{13}$

$\tan\phi = \frac{5}{12}$

(a) $\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$

However, we cannot simply substitute the above ratios into this expression as we now need to consider the sign of the ratios.

As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\sin(\theta + \phi) = \frac{3}{5} \times -\frac{12}{13} + -\frac{5}{13} \times \frac{4}{5} = -\frac{56}{65}$

(b) $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\cos(\theta + \phi) = \frac{4}{5} \times -\frac{12}{13} - \frac{3}{5} \times -\frac{5}{13} = -\frac{33}{65}$

(c) $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

As $0 \le \theta \le \frac{\pi}{2}$ then $\tan\theta = \frac{3}{4}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\tan\phi = \frac{5}{12}$.

Therefore, $\tan(\theta - \phi) = \dfrac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4} \times \frac{5}{12}} = \frac{16}{63}$

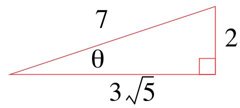
EXAMPLE 10.16

If $\sin\theta = \frac{2}{7}$, where $\frac{\pi}{2} \le \theta \le \pi$ find

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$

Solution

We start by drawing the relevant right-angled triangle:



(a) $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{2}{7} \times -\frac{3\sqrt{5}}{7}$

$$= -\frac{12\sqrt{5}}{49}$$

(b) $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2 \times \left(\frac{2}{7}\right)^2 = \frac{41}{49}$

(c) $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{12\sqrt{5}}{49}}{\frac{41}{49}} = -\frac{12\sqrt{5}}{41}$

EXAMPLE 10.17

Find the exact value of (a) $\cos 15°$ (b) $\tan\frac{5\pi}{12}$

Solution

(a) $\cos 15° = \cos(45° - 30°) = \cos 45°\cos 30° + \sin 45°\sin 30°$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(b) $\tan\frac{5\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \frac{\tan\frac{3\pi}{12} + \tan\frac{2\pi}{12}}{1 - \tan\frac{3\pi}{12}\tan\frac{2\pi}{12}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Note: we used the fact that $\frac{5\pi}{12} = 5 \times \frac{\pi}{12} = 5 \times 15° = 75° = 45° + 30° = 3 \times \frac{\pi}{12} + 2 \times \frac{\pi}{12}$ to 'break up' the angle.

EXAMPLE 10.18

Prove that $\dfrac{\sin 2\phi + \sin\phi}{\cos 2\phi + \cos\phi + 1} = \tan\phi$

Solution

(a) $$\text{L.H.S} = \frac{\sin 2\phi + \sin\phi}{\cos 2\phi + \cos\phi + 1} = \frac{2\sin\phi\cos\phi + \sin\phi}{2\cos^2\phi - 1 + \cos\phi + 1}$$
$$= \frac{\sin\phi(2\cos\phi + 1)}{\cos\phi(2\cos\phi + 1)}$$
$$= \frac{\sin\phi}{\cos\phi}$$
$$= \tan\phi$$
$$= \text{R.H.S}$$

EXAMPLE 10.19

Prove that (a) $\sin 2\alpha\tan\alpha + \cos 2\alpha = 1$

(b) $2\cot 2\beta = \cot\beta - \tan\beta$

Solution

(a) $$\text{L.H.S} = \sin 2\alpha\tan\alpha + \cos 2\alpha = 2\sin\alpha\cos\alpha \times \frac{\sin\alpha}{\cos\alpha} + (1 - 2\sin^2\alpha)$$
$$= 2\sin^2\alpha + 1 - 2\sin^2\alpha$$
$$= 1$$
$$= \text{R.H.S}$$

(b) $$\text{R.H.S} = \cot\beta - \tan\beta = \frac{\cos\beta}{\sin\beta} - \frac{\sin\beta}{\cos\beta}$$
$$= \frac{\cos^2\beta - \sin^2\beta}{\sin\beta\cos\beta}$$
$$= \frac{\cos 2\beta}{\frac{1}{2}\sin 2\beta}$$
$$= 2\frac{\cos 2\beta}{\sin 2\beta}$$
$$= 2\cot 2\beta$$
$$= \text{L.H.S}$$

Notice that when proving identities, when all else fails, then express everything in terms of sine and cosine. This will always lead to the desired result – even though sometimes the working seems like it will only grow and grow – eventually, it does simplify. Be persistent.

To prove a given identity, any one of the following approaches can be used:

1. Start with the L.H.S and then show that L.H.S = R.H.S
2. Start with the R.H.S and then show that R.H.S = L.H.S
3. Show that L.H.S = p, show that R.H.S = p $\Rightarrow$ L.H.S = R.H.S
4. Start with L.H.S = R.H.S $\Rightarrow$ L.H.S – R.H.S = 0 .

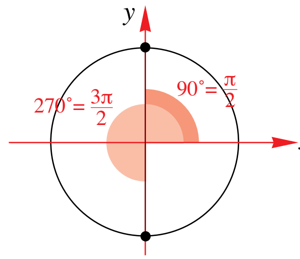
When using approaches 1., and 2., choose whichever side has more to work with.

EXAMPLE 10.20

Find all values of x, such that $\sin 2x = \cos x$, where $0 \le x \le 2\pi$.

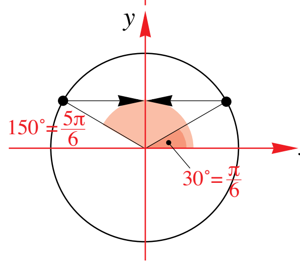
Solution

$$\sin 2x = \cos x \Leftrightarrow 2\sin x\cos x = \cos x$$
$$\Leftrightarrow 2\sin x\cos x - \cos x = 0$$
$$\Leftrightarrow \cos x(2\sin x - 1) = 0$$
$$\Leftrightarrow \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$



Now, $\cos x = 0, 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

and $\sin x = \frac{1}{2}, 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$



EXAMPLE 10.21

(a) Simplify $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

(b) Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are real numbers. Hence find the maximum value of $\cos\theta - \sin\theta$.

Solution

(a) $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\sin\theta \times \frac{1}{\sqrt{2}} + \cos\theta \times \frac{1}{\sqrt{2}}\right]$

$$= \sin\theta + \cos\theta$$

(b) In this instance, as the statement needs to be true for all values of θ we will determine the values of R and α by setting $R\cos(\theta + \alpha) \equiv \cos\theta - \sin\theta$.

Now, $R\cos(\theta + \alpha) = R[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Therefore, we have that $R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \equiv \cos\theta - \sin\theta$

$$\Rightarrow R\cos\theta\cos\alpha = \cos\theta \Leftrightarrow R\cos\alpha = 1 \quad \text{– (1)}$$
$$\Rightarrow R\sin\theta\sin\alpha = \cos\theta \Leftrightarrow R\sin\alpha = 1 \quad \text{– (2)}$$

Dividing (2) by (1) we have $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1} \Leftrightarrow \tan\alpha = 1 \therefore \alpha = \frac{\pi}{4}$

Substituting into (1) we have $R\cos\frac{\pi}{4} = 1 \Leftrightarrow R \times \frac{1}{\sqrt{2}} = 1 \therefore R = \sqrt{2}$.

Therefore, $\cos\theta - \sin\theta \equiv \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

Then, as the maximum value of the cosine is 1, the maximum of $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$ is $\sqrt{2}$.

EXAMPLE 10.22

Express $3\cos\theta - 4\sin\theta$ in the form $R\cos(\theta \pm \alpha)$.
Hence solve the equation $3\cos\theta - 4\sin\theta = 2$ on the interval [–1,1], to 2 decimal places.

Solution

We are required to find the values of R & θ such that $R.\cos(\theta - \alpha) \equiv 3\cos\theta - 4\sin\theta$.
Use the appropriate identity to expand the left hand side:

$$R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \equiv 3\cos\theta - 4\sin\theta \quad (R > 0)$$

The values of R & θ must give an identity that is true for all values of θ. This means that both sides of the identity must have the same amount of cosθ and sinθ on each side. We must next equate coefficients of these variables.

Equating the coefficients of cosθ gives: $R\cos\alpha = 3$ – (1)
Equating the coefficients of sinθ gives: $R\sin\alpha = -4$ – (2)

Squaring and adding gives:

$$\begin{aligned} R^2\cos^2\alpha + R^2\sin^2\alpha &= 3^2 + (-4)^2 \\ R^2(\cos^2\alpha + \sin^2\alpha) &= 9 + 16 \\ R^2 &= 25 \\ R &= 5 \end{aligned}$$

Dividing (2) by (1) gives:

$$\begin{aligned} \frac{R\sin\alpha}{R\cos\alpha} &= -\frac{4}{3} \\ \tan\alpha &= -\frac{4}{3} \\ \alpha &\approx -0.927^c \end{aligned}$$

It follows that $5.\cos(\theta + 0.927^c) \approx 3\cos\theta - 4\sin\theta$. This helps in finding the solution of the equation, which can be re-written as $5.\cos(\theta + 0.927^c) = 2$. The solution now follows:

$$\begin{aligned} 5.\cos(\theta + 0.927^c) &= 2 \\ \cos(\theta + 0.927^c) &= 0.4 \\ \theta + 0.927^c &= \pm 1.159^c\ldots \\ \theta &\approx 0.23 \text{ the only solution in the given interval.} \end{aligned}$$

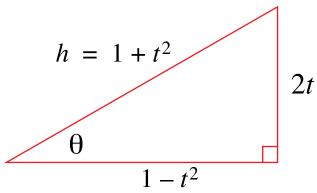
An extension of the compound and double angle identities is that of **half angle identities**, also referred to as ***t* substitution identities**.

Sometimes it is useful to make a substitution by letting $t = \tan\left(\frac{\theta}{2}\right)$.

If $\tan\frac{\theta}{2} \equiv t$ then $\sin\theta \equiv \frac{2t}{1+t^2}$ and $\cos\theta \equiv \frac{1-t^2}{1+t^2}$

$\tan 2\phi \equiv \dfrac{2\tan\phi}{1-\tan^2\phi}$ making the substitution $\phi = \dfrac{\theta}{2}$ gives: $\tan\theta \equiv \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \dfrac{2t}{1-t^2}$

This can be viewed (at least for acute angles) as a right–angled triangle



The hypotenuse of the triangle can be found using the theorem of Pythagoras:

$$h = \sqrt{(2t)^2+(1-t^2)^2} = \sqrt{4t^2+1-2t^2+t^4}$$
$$= \sqrt{1+2t^2+t^4}$$
$$= \sqrt{(1+t^2)^2}$$
$$= 1+t^2$$

Two further identities follow: $\sin\theta \equiv \dfrac{2t}{1+t^2}$ and $\cos\theta \equiv \dfrac{1-t^2}{1+t^2}$. These are useful in both calculus and trigonometry.

EXAMPLE 10.23

Solve the equation $\sin\theta + \cos\theta = 1, -\pi \le \theta \le \pi$.

Solution

$\sin\theta + \cos\theta = 1, -\pi \le \theta \le \pi$.

Make the substitution: $\tan\dfrac{\theta}{2} \equiv t \Rightarrow \sin\theta \equiv \dfrac{2t}{1+t^2}$ and $\cos\theta \equiv \dfrac{1-t^2}{1+t^2}$

The equation becomes: $\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2} = 1 \Leftrightarrow 2t+1-t^2 = 1+t^2$

$$0 = 2t^2 - 2t$$
$$t = 0, 1$$
$$\tan\frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = 0 \Rightarrow \theta = 0\,.$$
$$\tan\frac{\theta}{2} = 1 \Rightarrow \frac{\theta}{2} = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}$$

Other solutions are outside the interval.

Note: The section on half angle identities has been added as an extension to the prescribed H.L syllabus. You should not expect to have questions that refer directly to half-angle identities in your examinations, however, such questions can be approached indirectly.

EXERCISES 10.2.2

1. Expand the following

(a)	$\sin(\alpha+\phi)$	(b)	$\cos(3\alpha+2\beta)$	(c)	$\sin(2x-y)$
(d)	$\cos(\phi-2\alpha)$	(e)	$\tan(2\theta-\alpha)$	(f)	$\tan(\phi-3\omega)$

2. Simplify the following

(a) $\sin 2\alpha\cos 3\beta - \sin 3\beta\cos 2\alpha$ (b) $\cos 2\alpha\cos 5\beta - \sin 2\alpha\sin 5\beta$

(c) $\sin x\cos 2y + \sin 2y\cos x$ (d) $\cos x\cos 3y + \sin x\sin 3y$

(e) $\dfrac{\tan 2\alpha - \tan\beta}{1 + \tan 2\alpha\tan\beta}$ (f) $\dfrac{\tan(x-y) + \tan y}{1 - \tan(x-y)\tan y}$

(g) $\dfrac{1 - \tan\phi}{1 + \tan\phi}$ (h) $\dfrac{1}{\sqrt{2}}\sin(\alpha+\beta) + \dfrac{1}{\sqrt{2}}\cos(\alpha+\beta)$

(i) $\cos(x+2y)\sin(x-2y) + \sin(x+2y)\cos(x-2y)$

3. Given that $\sin\theta = \frac{4}{5}, 0 \le \theta \le \frac{\pi}{2}$ and $\cos\phi = -\frac{5}{13}, \pi \le \phi \le \frac{3\pi}{2}$, evaluate

(a) $\sin(\theta+\phi)$ (b) $\cos(\theta+\phi)$ (c) $\tan(\theta-\phi)$

4. Given that $\sin\theta = -\frac{3}{5}, \pi \le \theta \le \frac{3\pi}{2}$ and $\cos\phi = -\frac{12}{13}, \pi \le \phi \le \frac{3\pi}{2}$, evaluate

(a) $\sin(\theta-\phi)$ (b) $\cos(\theta-\phi)$ (c) $\tan(\theta+\phi)$

5. Given that $\sin\theta = -\frac{5}{6}, \frac{3\pi}{2} \le \theta \le 2\pi$, evaluate

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$ (d) $\sin 4\theta$

6. Given that $\tan x = -3, \frac{\pi}{2} \le x \le \pi$, evaluate

(a) $\sin 2x$ (b) $\cos 2x$ (c) $\tan 2x$ (d) $\tan 4x$

7. Find the exact value of

(a) $\sin\frac{5\pi}{12}$ (b) $\sin 105°$ (c) $\cos\frac{11\pi}{12}$ (d) $\tan 165°$

8. Given that $\tan x = \frac{a}{b}, \pi \le x \le \frac{3\pi}{2}$, evaluate

(a) $\sin 2x$ (b) $\text{cosec}\, 2x$ (c) $\cos 4x$ (d) $\tan 2x$

9. Prove the following identities:

(a) $\cot x - \cot 2x = \text{cosec}\, 2x$ (b) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

(c) $\sec^2 x = 1 + \tan^2 x$ (d) $\tan(\theta+\phi) + \tan(\theta-\phi) = \dfrac{2\sin 2\theta}{\cos 2\theta + \cos 2\phi}$

(e) $\cos^4\alpha - \sin^4\alpha = 1 - 2\sin^2\alpha$ (f) $\dfrac{1}{\sin y\cos y} - \dfrac{\cos y}{\sin y} = \tan y$

(g) $\dfrac{1+\cos 2y}{\sin 2y} = \dfrac{\sin 2y}{1 - \cos 2y}$ (h) $\csc\left(\theta + \frac{\pi}{2}\right) = \sec\theta$

(i) $\cos 3x = \cos x - 4\sin^2 x\cos x$ (j) $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

(k) $(\cot x + \csc x)^2 = \dfrac{1+\cos x}{1-\cos x}$ (l) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

(m) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ (n) $2\cot\theta\sin^2\theta = \sin 2\theta$

(o) $\tan\left(\dfrac{\phi}{2}\right) = \csc\phi - \cot\phi$ (p) $2\csc x = \tan\left(\dfrac{x}{2}\right) + \cot\left(\dfrac{x}{2}\right)$

(q) $\cos\beta + \sin\beta = \dfrac{\cos 2\beta}{\cos\beta - \sin\beta}$ (r) $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

(s) $\sin^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$ (t) $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \dfrac{1}{2}\sin 2x$

10. Prove that (a) $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x} = \cot\dfrac{1}{2}x$.

(b) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

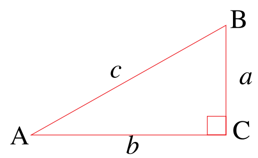
(c) $\sin^4\phi = \dfrac{3}{8} + \dfrac{1}{8}\cos 4\phi - \dfrac{1}{2}\cos 2\phi$

(d) $\sin x = \dfrac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}$

11. For the right-angled triangle shown, prove that

(a) $\sin 2\alpha = \dfrac{2ab}{c^2}$ (b) $\cos 2\alpha = \dfrac{b^2-a^2}{c^2}$

(c) $\sin\dfrac{1}{2}\alpha = \sqrt{\dfrac{c-b}{2c}}$ (d) $\cos\dfrac{1}{2}\alpha = \sqrt{\dfrac{c+b}{2c}}$



12. Find the exact value $\tan\dfrac{\pi}{8}$.

13. Given that $\alpha+\beta+\gamma = \pi$, prove that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

14. Solve the following for $0 \le x \le 2\pi$

(a) $\sin x = \sin 2x$ (b) $\sin x = \cos 2x$ (c) $\tan 2x = 4\tan x$

15. (a) Given that $a\sin\theta + b\cos\theta \equiv R\sin(\theta+\alpha)$, express R and α in terms of a and b.

(b) Find the maximum value of $5 + 4\sin\theta + 3\cos\theta$.

16. (a) Given that $a\cos\theta + b\sin\theta \equiv R\cos(\theta-\alpha)$, express R and α in terms of a and b.

(b) Find the minimum value of $2 + 12\cos\theta + 5\sin\theta$.

17. Using the t–substitution, prove that $\tan\left(\dfrac{\pi}{4} + \dfrac{1}{2}x\right) = \sec x + \tan x$.

18. Show that if $t = \tan\dfrac{\pi}{12}$ then $t^2 + 2\sqrt{3}t = 1$. Hence find the exact value of $\tan\dfrac{\pi}{12}$.

10.3 TRIGONOMETRIC FUNCTIONS

10.3.1 THE SINE, COSINE AND TANGENT FUNCTIONS

As we saw at the begining of this chapter, there is an infinite set of angles all of which give values (when they exist) for the main trigonometric ratios. We also noticed that the trigonometric ratios behave in such a way that values are repeated over and over. This is known as **periodic** behaviour. Many real world phenomena are periodic in the sense that the same patterns repeat over time. The **trigonometric functions** are often used to model such phenomena which include sound waves, light waves, alternating current electricity and other more approximately periodic events such as tides and even our moods (i.e., biorythms).

Notice how we have introduced the term trigonometric function, replacing the term trigonometric ratio. By doing this we can extend the use of the trigonometric ratios to a new field of problems described at the end of the previous paragraph.

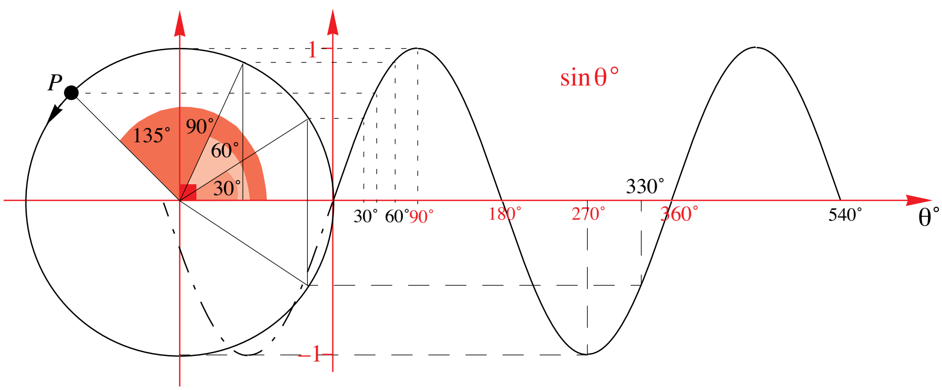
When the trigonometric functions are used for these purposes, the angles are almost always measured in **radians** (which is a different way of measuring angles). However there is no reason why we cannot use degrees. It will always be obvious from the equation as to which mode of angle we are using, an expression such as $\sin x$ will imply (by default) that the angle is measured in radians, otherwise it will be written as $\sin x°$, implying that the angle is measured in degrees.

What do trigonometric functions look like?

The sine and cosine values have displayed a periodic nature. Meaning that if we were to plot a graph of the sine values versus their angles or the cosine values versus their angles, we could expect their graphs to demonstrate periodic behaviour. We start by plotting points.

The sine function:

θ	0	30	45	60	90	120	135	150	180	. . .	330	360
$\sin\theta°$	0.0	0.5	0.71	0.87	1.0	0.86	0.71	0.5	0.0	. . .	–0.5	0.0



Notice that as the sine of angle θ corresponds to the y–value of a point P on the unit circle, as P moves around the circle in an anti–clockwise direction, we can measure the y–value and plot it on a graph as a function of θ (as shown above).

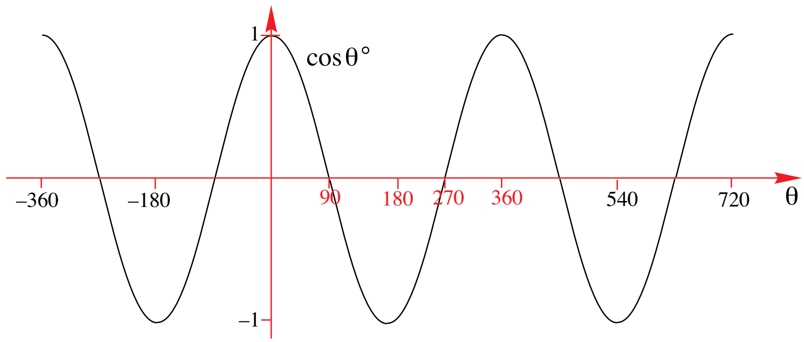
Feature of sine graph:

1. Maximum value = 1, Minimum value = –1
2. Period = 360˚ (i.e., graph repeats itself every 360˚)
3. If P moves in a clockwise direction, y–values continue in their periodic nature (see dashed part of graph).

The cosine function:

θ	0	30	45	60	90	120	135	150	180	. . .	330	360
$\cos\theta°$	1.0	0.87	0.71	0.5	0.0	–0.5	–0.71	–0.87	–1.0	. . .	0.87	1.0

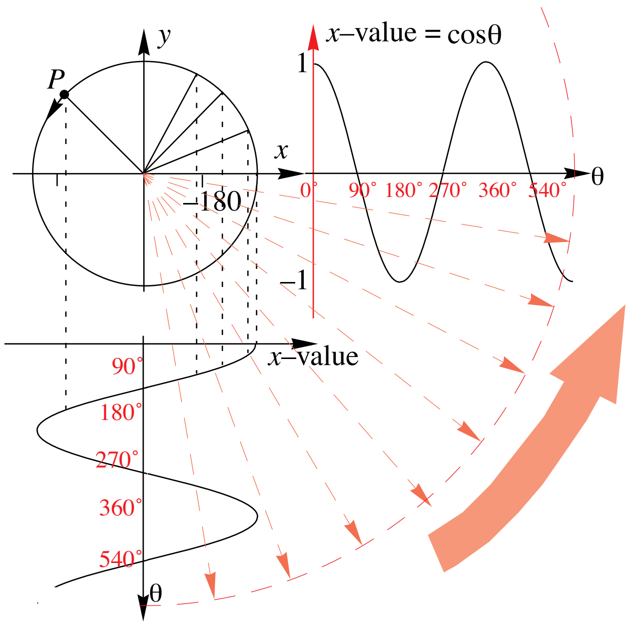
Plotting these points on a $\cos\theta°$ versus θ–axis, we have:



Feature of cosine graph:

1. Maximum value = 1, Minimum value = –1
2. Period = 360˚ (i.e., graph repeats itself every 360˚)
3. If P moves in a clockwise direction, x–values continue in their periodic nature.

There is a note to be made about using the second method (the one used to obtain the sine graph) when dealing with the cosine graph. As the cosine values correspond to the x–values on the unit circle, the actual cosine graph should have been plotted as shown below. However, for the sake of consistency, we convert the 'vertical graph' to the more standard 'horizontal graph':



There are some common observations to be made from these two graphs;

1. We have that the **period** of each of these functions is 360˚.
This is the length that it takes for the curve to start repeating itself.

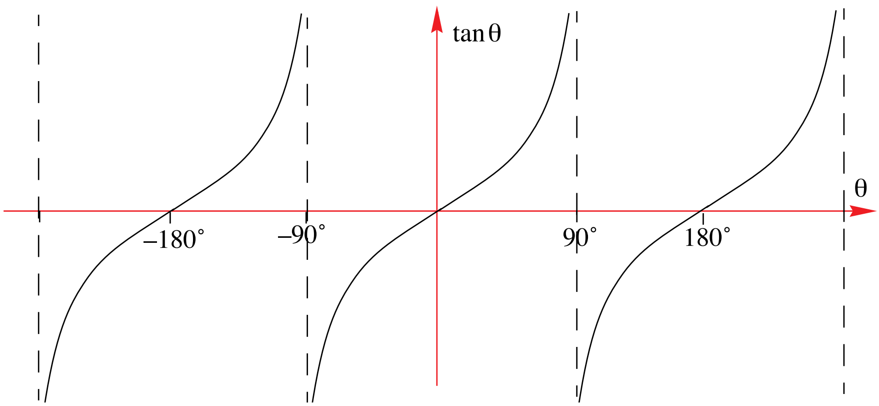
2. The **amplitude** of the function is the distance between the centre line (in this case the θ -axis) and one of the maximum points. In this case, the amplitude is 1.

The sine and cosine functions are useful for modelling wave phenomena such as sound, light, water waves etc.

The third trigonometric function (tangent) is defined as $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and so is defined for all angles for which the cosine function is non-zero.

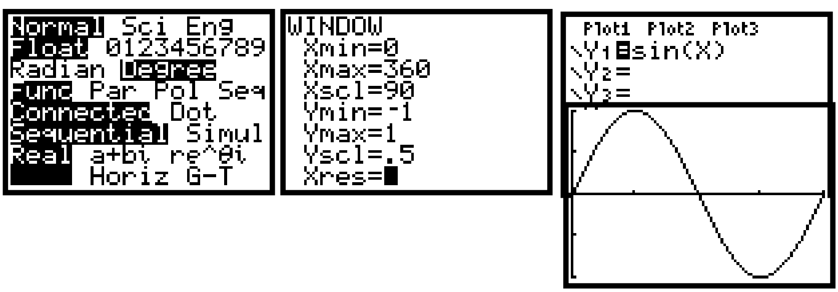
The angles for which the tangent function are not defined correspond to the *x*-axis intercepts of the cosine function which are $\pm 90°$, $\pm 270°$, $\pm 450°$, … . At these points the graph of the tangent function has **vertical asymptotes**.

The period of the tangent function is 180˚, which is half that of the sine and cosine functions. Since the tangent function has a vertical asymptote, it cannot be said to have an amplitude. It is also generally true that the tangent function is less useful than the sine and cosine functions for modelling applications. The graph of the basic tangent function is:



When sketching these graphs using the TI–83, be sure that the **WINDOW** settings are appropriate for the **MODE** setting. In the case of degrees we have:

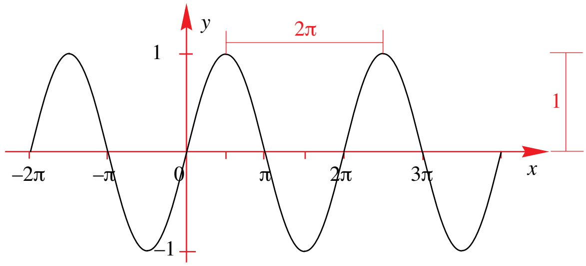
Step 1. Make sure that the calculator is in degree mode. Failure to do this could be disastrous!
Step 2. Select an appropriate range
Step 3. Enter the function rule.



As we mentioned at the start of this section, angles measured in radians are much more useful when modelling situations that are cyclic or repetitive. And, although we have sketched the graphs of the sine, cosine and tangent functions using angles measured in degrees, it is reassuring to know that the shape of the graph does not alter when radians are used. The only difference between the graphs then is that 90° would be replaced by $\frac{\pi}{2}$, 180° by π and so on. We provide these graphs with the corresponding observations.

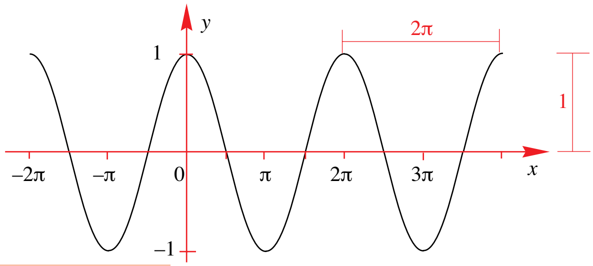
$f:\mathbb{R}\mapsto\mathbb{R},\ f(x) = \sin x$

period 2π
amplitude 1



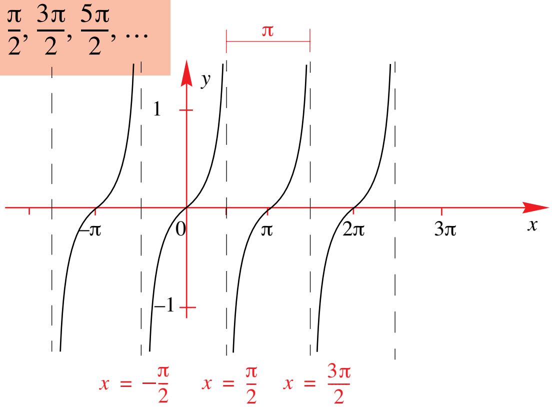
$f:\mathbb{R}\mapsto\mathbb{R},\ f(x) = \cos x$

period 2π
amplitude 1



$f:\mathbb{R}\mapsto\mathbb{R},\ f(x) = \tan x,\ x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

period π
no amplitude



10.3.2 TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

We now consider some of the possible transformations that can be applied to the standard sine and cosine function and look at how these transformations affect the basic properties of both these graphs

1. Vertical translations

Functions of the type $f(x) = \sin(x) + c$, $f(x) = \cos(x) + c$ and $f(x) = \tan(x) + c$ represent vertical translations of the curves of $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively. If $c > 0$ the graph is moved vertically up and if $c < 0$ the graph is moved vertically down.

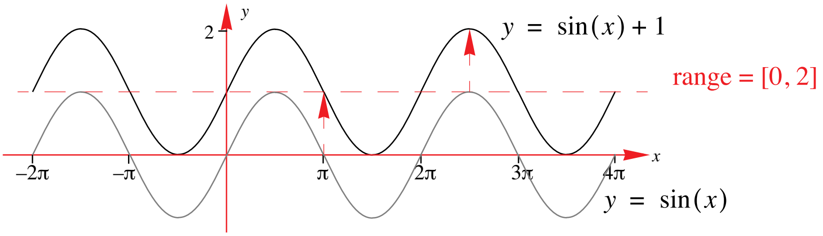
That is, adding or subtracting a fixed amount to a trigonometric function translates the graph parallel to the y-axis.

EXAMPLE 10.24 Sketch the graphs of the functions for x values in the range -2π to 4π.

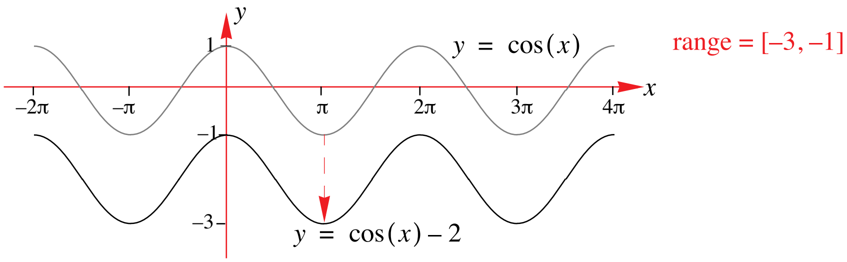
(a) $y = \sin(x) + 1$ (b) $y = \cos(x) - 2$

Solution

(a)



(b)



A graph sketch should show all the important features of a graph. In this case, the axes scales are important and should show the correct period (2π) and range $[-3,-1]$.

2. Horizontal translations

Functions of the type $f(x) = \sin(x \pm \alpha)$, $f(x) = \cos(x \pm \alpha)$ and $f(x) = \tan(x \pm \alpha)$ where $\alpha > 0$ are horizontal translations of the curves $\sin x$, $\cos x$ and $\tan x$ respectively.

So that $f(x) = \sin(x - \alpha)$, $f(x) = \cos(x - \alpha)$ and $f(x) = \tan(x - \alpha)$ are translations to the **right**.

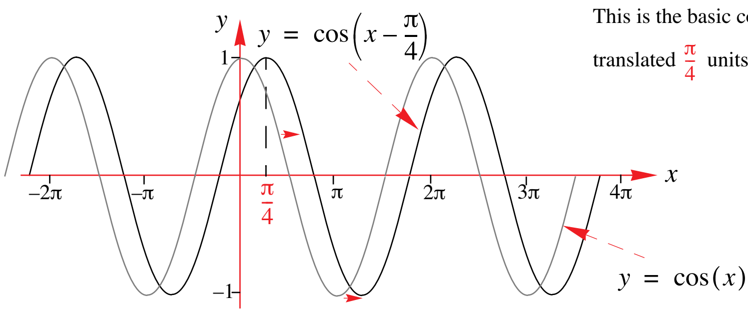
while that of $f(x) = \sin(x + \alpha)$, $f(x) = \cos(x + \alpha)$ and $f(x) = \tan(x + \alpha)$ are translations to the **left**.

EXAMPLE 10.25 For $-2\pi \le x \le 4\pi$, sketch the graphs of the curves with equations

(a) $y = \cos\left(x - \frac{\pi}{4}\right)$ (b) $y = \cos\left(x + \frac{\pi}{3}\right)$ (c) $y = \tan\left(x - \frac{\pi}{6}\right)$

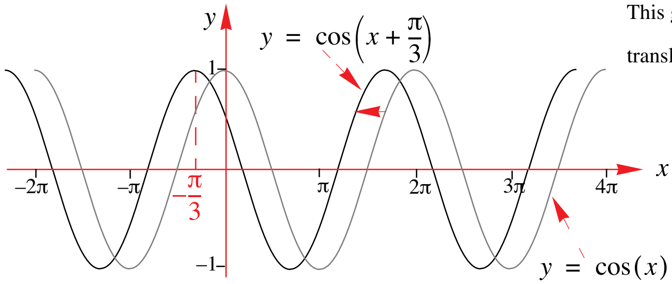
Solution

(a)



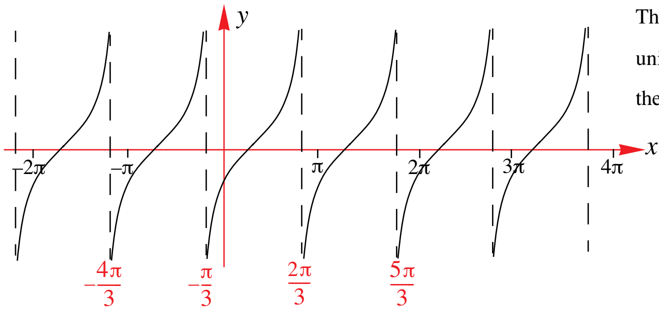
This is the basic cosine graph translated $\frac{\pi}{4}$ units to the **right**.

(b)



This graph is the basic cosine graph translated $\frac{\pi}{3}$ units to the **left**.

(c)



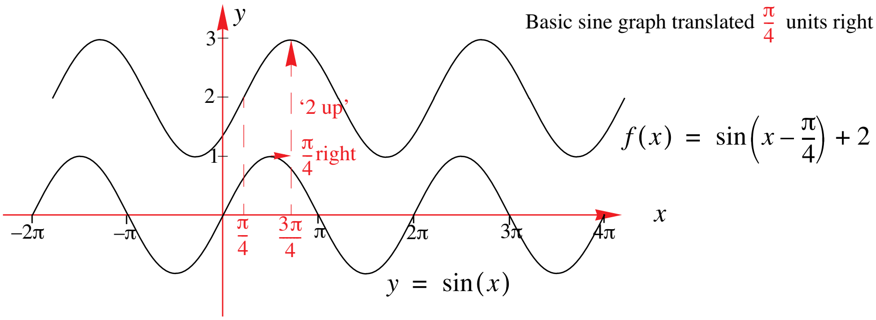
This is the tangent graph translated $\frac{\pi}{6}$ units to the **right**. This also translates the **asymptotes** $\frac{\pi}{6}$ units to the **right**.

Of course, it is also possible to combine vertical and horizontal translations, as the next example shows.

EXAMPLE 10.26

Sketch the graphs of the function $f(x) = \sin\left(x - \frac{\pi}{4}\right) + 2, -2\pi \le x \le 4\pi$.

Solution



Basic sine graph translated $\frac{\pi}{4}$ units right and 2 up.

3. Dilations

Functions of the form $f(x) = a\sin x$, $f(x) = a\cos(x)$ and $f(x) = a\tan(x)$ are **dilations** of the curves $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively, **parallel to the *y*-axis**.

In the case of the sine and cosine functions, the **amplitude** becomes $|a|$ and not 1. This dilation does not affect the shape of the graph. Also, as the tangent function extends indefinitely, the term amplitude has no relevance.

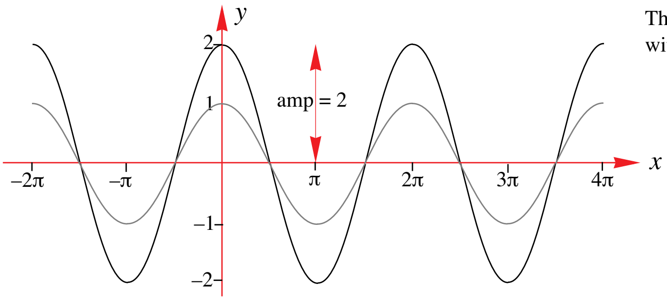
EXAMPLE 10.27

Sketch the graphs of the following functions for $-2\pi \le x \le 4\pi$.

(a) $f(x) = 2\cos x$ (b) $f(x) = \frac{1}{3}\sin x$

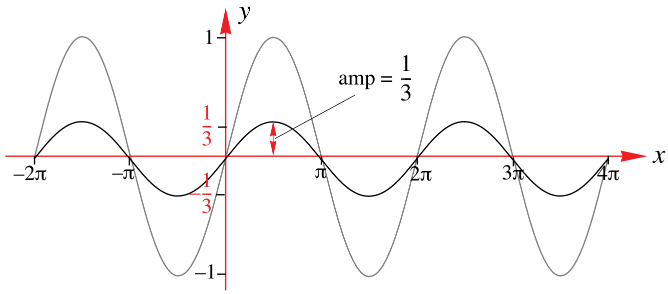
Solution

(a)



This is the cosine graph (broken line) with an amplitude of 2.

(b)



This is the sine graph (broken line) with an amplitude of $\frac{1}{3}$.

Functions of the form $f(x) = \cos(bx)$, $f(x) = \sin(bx)$ and $f(x) = \tan(bx)$ are **dilations** of the curves $\sin(x)$, $\cos(x)$ and $\tan(x)$ respectively, **parallel to the *x*-axis**.

This means that the period of the graph is altered. It can be valuable to remember and use the formula that relates the value of b to the period τ of the dilated function:

1. The graph of $f(x) = \cos(bx)$ will show b cycles in 2π radians, meaning that its period will be given by $\tau = \frac{2\pi}{b}$.

2. The graph of $f(x) = \sin(bx)$ will show b cycles in 2π radians, meaning that its period will be given by $\tau = \frac{2\pi}{b}$.

3. The graph of $f(x) = \tan(bx)$ will show b cycles in π radians, meaning that its period will be given by $\tau = \frac{\pi}{b}$.

Note: In the case of the tangent function, whose original period is π, the period is $\tau = \frac{\pi}{n}$.

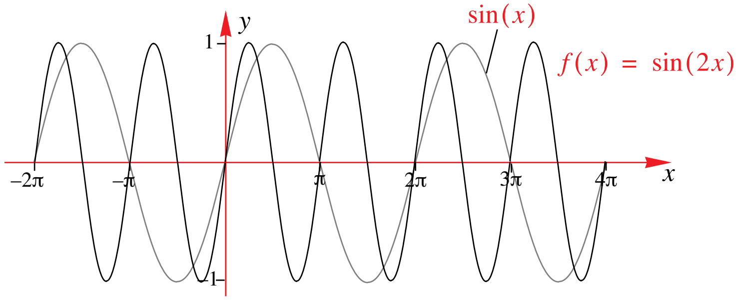
EXAMPLE 10.28

Sketch graphs of the following functions for x values in the range -2π to 4π.

(a) $f(x) = \sin(2x)$ (b) $f(x) = \cos\left(\frac{x}{2}\right)$ (c) $f(x) = \tan(2x)$

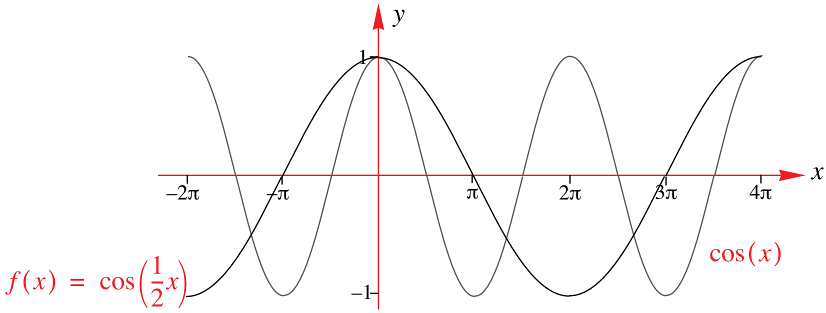
Solution

(a) $f(x) = \sin(2x)$. The value of n is 2 so the period is $\tau = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$. Note that this means that the period is **half** that of the basic sine function.

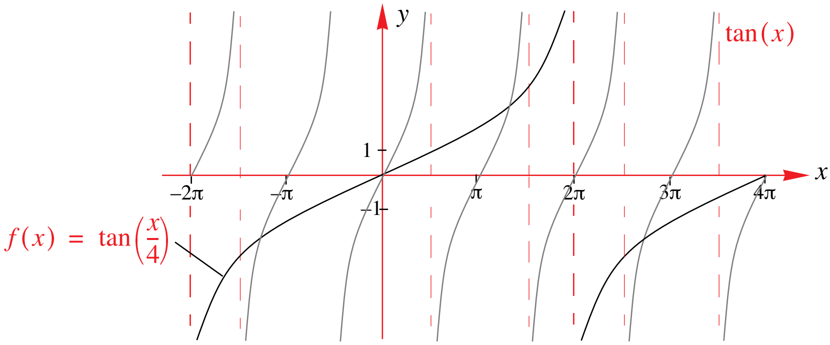


(b) $f(x) = \cos\left(\frac{x}{2}\right)$. In this case the value of $n = \frac{1}{2}$ and the period $\tau = \frac{2\pi}{n} = \frac{2\pi}{1/2} = 4\pi$.

The graph is effectively stretched to twice its original period.



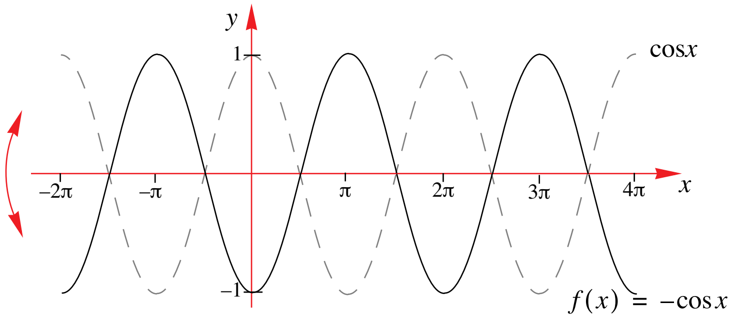
(c) In this case, with $f(x) = \tan\left(\frac{x}{4}\right)$, the value of $n = \frac{1}{4}$ and the period $\tau = \frac{\pi}{n} = 4\pi$.



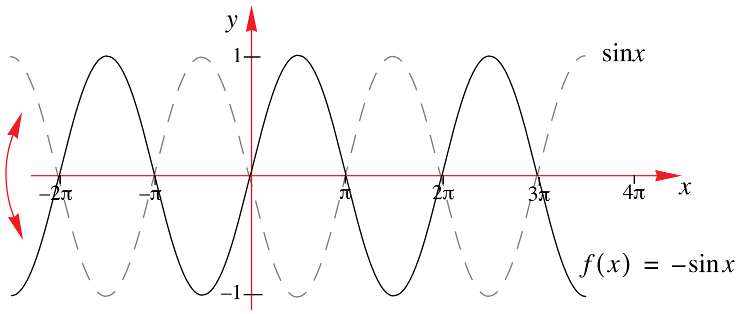
4. Reflections

Recall that the graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x–axis, while that of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y–axis.

$f(x) = -\cos x$ is the basic cosine graph (broken line) **reflected in the x-axis**.



$f(x) = \sin(-x)$ is the basic sine graph **reflected in the y-axis**.



5. Combined transformations

You may be required to combine some or all of these transformations in a single function. The functions of the type

$$f(x) = a\sin[b(x+c)]+d \quad \text{and} \quad f(x) = a\cos[b(x+c)]+d$$

have:

1. an amplitude of $|a|$ (i.e., the absolute value of a).
2. a period of $\frac{2\pi}{b}$
3. a horizontal translation of c units, $c > 0 \Rightarrow$ to the left, $c < 0 \Rightarrow$ to the right
4. a vertical translation of d units, $d > 0 \Rightarrow$ up, $d < 0 \Rightarrow$ down

Care must be taken with the horizontal translation.

For example, the function $f(x) = 2\cos\left(3x+\frac{\pi}{2}\right)-1$ has a horizontal translation of $\frac{\pi}{6}$ to the left, not $\frac{\pi}{2}$! This is because $f(x) = 2\cos\left(3x+\frac{\pi}{2}\right)-1 = 2\cos\left[3\left(x+\frac{\pi}{6}\right)\right]-1$. i.e., if the coefficient of x is not one, we must first express the function in the form $a\cos[b(x+c)]+d$.

Similarly we have

$$f(x) = a\tan[b(x+c)]+d$$

has:

1. no amplitude (as it is not appropriate for the tan function).
2. a period of $\frac{\pi}{b}$
3. a horizontal translation of c units, $c > 0 \Rightarrow$ to the left, $c < 0 \Rightarrow$ to the right
4. a vertical translation of d units, $d > 0 \Rightarrow$ up, $d < 0 \Rightarrow$ down

EXAMPLE 10.29

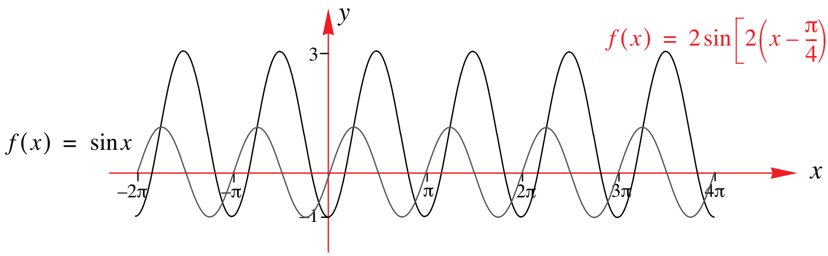
Sketch graphs of the following functions for x values in the range -2π to 4π.

(a) $f(x) = 2\sin\left[2\left(x-\frac{\pi}{4}\right)\right]+1$ (b) $f(x) = -\cos\frac{1}{2}\left(x-\frac{\pi}{3}\right)+2$

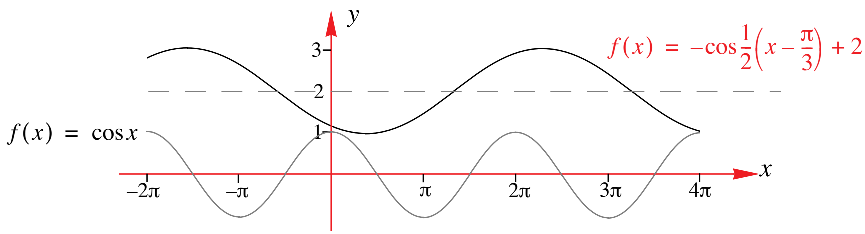
(c) $f(x) = -\frac{1}{2}\tan\frac{1}{2}\left(x-\frac{\pi}{2}\right)$

Solution

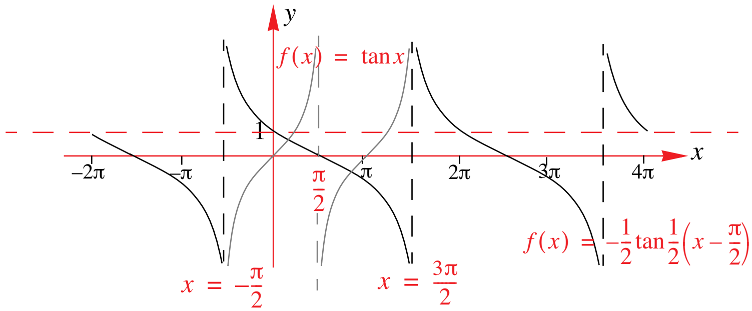
(a) $f(x) = 2\sin\left[2\left(x-\frac{\pi}{4}\right)\right]+1$. This graph has an amplitude of 2, a period of π, a horizontal translation of $\frac{\pi}{4}$ units to the right and a vertical translation of 1 unit up.



(b) $f(x) = -\cos\frac{1}{2}\left(x-\frac{\pi}{3}\right)+2$. The transformations are a reflection in the x-axis, a dilation of factor 2 parallel to the x-axis and a translation of $\frac{\pi}{3}$ right and 2 up.



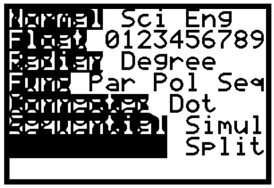
(b) $f(x) = -\frac{1}{2}\tan\frac{1}{2}\left(x - \frac{\pi}{2}\right)$. The transformations are a reflection in the x-axis, a vertical dilation with factor $\frac{1}{2}$, a horizontal dilation with factor 2 and a translation of $\frac{\pi}{2}$ to the right.



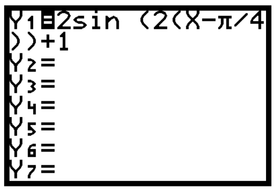
Again we see that a graphics calculator is very useful in such situations – in particular it allows for a checking process.

Example 10.28 (a) could be sketched as follows:

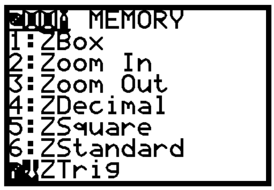
Step 1. Make sure that the calculator is in radian mode.
Failure to do this could be disastrous!



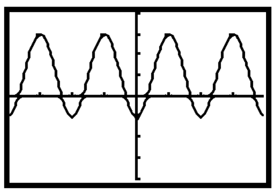
Step 2. Enter the function rule. Remember to use the π key where necessary.
Do not use approximations such as 3.14



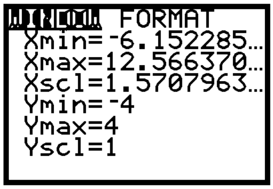
Step 3. Select an appropriate range. This may not be the **ZTrig** option (7) from the **ZOOM** menu.
However, this is probably a good place to start.



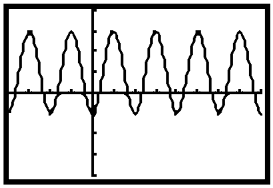
In this case, the viewing window is suitable, though it should be noted that it is not the specified range of x values $[-2\pi, 4\pi]$.



Step 4. Adjust the viewing window using the **WINDOW** menu. In this case it would be wise to select the correct set of x values.



Finally use **GRAPH** to display the graph.



EXERCISES 10.3

1. State the period of the following functions

(a) $f(x) = \sin\frac{1}{2}x$ (b) $f(x) = \cos 3x$

(c) $f(x) = \tan\frac{x}{3}$ (d) $g(x) = \cos\left(\frac{x}{2} - \pi\right)$

(e) $g(x) = 4\sin(\pi x + 2)$ (f) $g(x) = 3\tan\left(\frac{\pi}{2} - 2x\right)$

2. State the amplitude of the following functions

(a) $f(x) = 5\sin 2x$ (b) $g(x) = -3\cos\frac{x}{2}$

(c) $g(x) = 4 - 5\cos(2x)$ (d) $f(x) = \frac{1}{2}\sin(3x)$

3. Find the period and, where appropriate, the amplitude of the following functions.

(a) $y = 2\sin x$ (b) $y = 3\cos\frac{x}{3}$

(c) $y = 3\tan x$ (d) $2\tan(x - 2\pi)$

(e) $y = -4\sin\left[2\left(x + \frac{\pi}{6}\right)\right] + 1$ (f) $y = 2 - 3\cos(2x - \pi)$

(g) $y = -2\tan\frac{x}{6}$ (h) $y = \frac{1}{4}\cos\left[3\left(x - \frac{3\pi}{4}\right)\right] + 5$

(i) $y = 4\tan\left(\frac{x-4}{3}\right) - 3$ (j) $y = -\frac{2}{3}\cos\left(\frac{3}{4}\left(x + \frac{3\pi}{5}\right)\right) + 5$

4. Sketch the graph of the curve with equation given by

(a) $y = 3\cos x, 0 \le x \le 2\pi$

(b) $y = \sin\frac{x}{2}, -\pi \le x \le \pi$

(c) $y = 2\cos\left(\frac{x}{3}\right), 0 \le x \le 3\pi$

(d) $y = -\frac{1}{2}\sin 3x, 0 \le x \le \pi$

(e) $y = 4\tan\left(\frac{x}{2}\right), 0 \le x \le 2\pi$

(f) $y = \tan(-2x), -\pi \le x \le \frac{\pi}{4}$

(g) $y = \frac{1}{3}\cos(-3x), -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

(h) $y = 3\sin(-2x), -\pi \le 0 \le \pi$

5. Sketch the graph of the curve with equation given by

(a) $y = 3\cos x + 3, 0 \le x \le 2\pi$

(b) $y = \sin\frac{x}{2} - 1, -\pi \le x \le \pi$

(c) $y = 2\cos\left(\frac{x}{3}\right) - 2, 0 \le x \le 3\pi$

(d) $y = -\frac{1}{2}\sin 3x + 2, 0 \le x \le \pi$

(e) $y = 4\tan\left(\frac{x}{2}\right) - 1, 0 \le x \le 2\pi$

(f) $y = \tan(-2x) + 2, -\pi \le x \le \frac{\pi}{4}$

(g) $y = \frac{1}{3}\cos(-3x) + \frac{1}{3}, -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

(h) $y = 3\sin(-2x) - 2, -\pi \le 0 \le \pi$

6. Sketch the graph of the curve with equation given by

(a) $y = 3\cos\left(x + \frac{\pi}{2}\right), 0 \le x \le 2\pi$

(b) $y = \sin\left(\frac{x}{2} - \pi\right), -\pi \le x \le \pi$

(c) $y = 2\cos\left(\frac{x}{3} + \frac{\pi}{6}\right), 0 \le x \le 3\pi$

(d) $y = -\frac{1}{2}\sin(3x + 3\pi), 0 \le x \le \pi$

(e) $y = 4\tan\left(\frac{x}{2} - \frac{\pi}{4}\right), 0 \le x \le 2\pi$

(f) $y = \tan(-2x + \pi), -\pi \le x \le \frac{\pi}{4}$

(g) $y = \frac{1}{3}\cos(-3x - \pi), -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

(h) $y = 3\sin\left(-2x - \frac{\pi}{2}\right), -\pi \le 0 \le \pi$

7. Sketch graphs of the following functions for x values in the interval $[-2\pi, 2\pi]$.

(a) $y = \sin(2x)$

(b) $y = -\cos\left(\frac{x}{2}\right)$

(c) $y = 3\tan\left(x - \frac{\pi}{4}\right)$

(d) $y = 2\sin\left(x - \frac{\pi}{2}\right)$

(e) $y = 1 - 2\sin(2x)$

(f) $y = -2\cos\left(\frac{x - \pi}{2}\right)$

(g) $y = 3\tan\left[\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right] - 3$

(h) $y = 3\cos\left(x + \frac{\pi}{4}\right)$

(i) $y = 2\sin\left[\frac{1}{3}\left(x + \frac{2\pi}{3}\right)\right] - 1$ (j) $y = 3\tan(2x + \pi)$

(k) $y = 4\sin\left(\frac{x + \frac{\pi}{2}}{2}\right)$ (l) $y = 2 - \sin\left(\frac{2(x - \pi)}{3}\right)$

(m) $y = 2\cos(\pi x)$ (n) $y = 2\sin[\pi(x + 1)]$

8. (a) i. Sketch one cycle of the graph of the function $f(x) = \sin x$.

ii. For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii. Hence, sketch one cycle of the graph of the function $g(x) = \operatorname{cosec} x$.

(b) i. Sketch one cycle of the graph of the function $f(x) = \cos x$.

ii. For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii. Hence, sketch one cycle of the graph of the function $g(x) = \sec x$.

(c) i. Sketch one cycle of the graph of the function $f(x) = \tan x$.

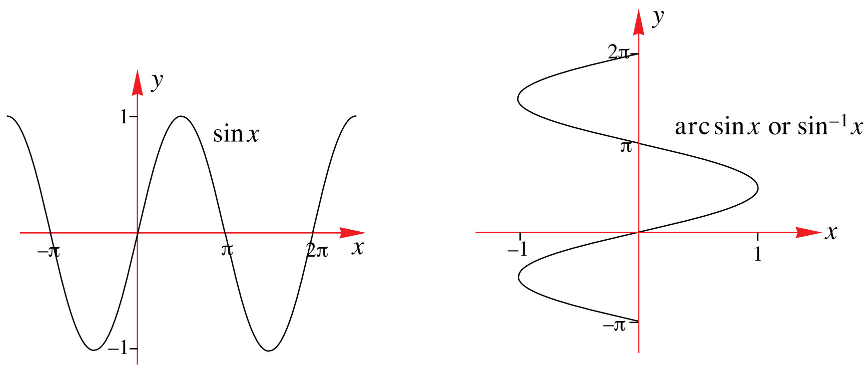
ii. For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii. Hence, sketch one cycle of graph of the function $g(x) = \cot x$.

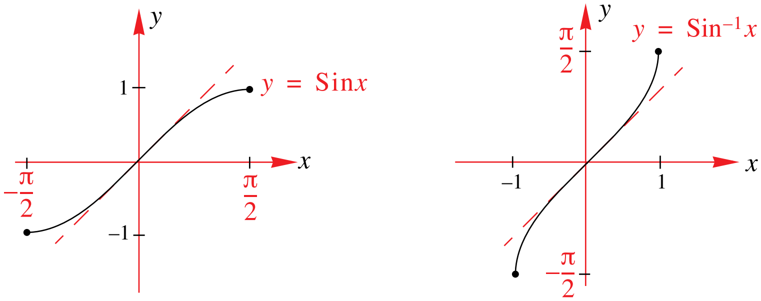
10.4 INVERSE TRIGONOMETRIC FUNCTIONS

10.4.1 THE INVERSE SINE FUNCTION

The trigonometric functions are many to one which means that, unless we are careful about defining domains, their inverses are not properly defined. The basic graphs of the sine function and its inverse (after reflection about the line $y = x$ for the arcsin$x$ function) are:



The inverse as depicted here is not a function (as it is one:many). This is inconvenient as the inverse trigonometric functions are useful. The most useful solution to this problem is to restrict the domain of the function to an interval over which it is one to one. In the case of the sine function, this is usually taken as $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, though this is not the only possible choice, it is one that allows for consistency to be maintained in literature and among mathematicians. The function thus defined is written with a capital letter: $f(x) = \mathrm{Sin}(x), x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The graphs are:



Notice then that the domain of $\mathrm{Sin}^{-1}x$ = range of $\mathrm{Sin}x$ = [–1, 1]

and the range of $\mathrm{Sin}^{-1}x$ = domain of $\mathrm{Sin}x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

When these restrictions are adhered to, we refer to $\mathrm{Sin}^{-1}x$ (which is sometimes denoted by $\mathrm{Arcsin}x$) as the **principal value** of $\arcsin x$. For example, $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $-\frac{7\pi}{6}$ or … , however, $\mathrm{Arcsin}\left(\frac{1}{2}\right)$ has only one value (the principal value), so that $\mathrm{Arcsin}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

From our fundamental indentity property of inverse functions, i.e., $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$, we have that

$$\mathrm{Sin}(\mathrm{Sin}^{-1}x) = x, -1 \le x \le 1 \text{ and } \mathrm{Sin}^{-1}(\mathrm{Sin}x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Therefore, $\mathrm{Sin}(\mathrm{Sin}^{-1}x) = x = \mathrm{Sin}^{-1}(\mathrm{Sin}x)$ only if $-1 \le x \le 1$.

This then means that sometimes we can provide a meaningful interpretation to expressions such as $\sin(\mathrm{Sin}^{-1}x)$ and $\mathrm{Sin}^{-1}(\sin x)$ – as long as we adhere to the relevant restrictions.

Notice also that $\mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}x$.

EXAMPLE 10.30

Give the exact value of

(a) $\mathrm{Sin}^{-1}\frac{1}{2}$ (b) $\mathrm{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$ (c) $\mathrm{Sin}^{-1}(1.3)$ (d) $\mathrm{Sin}^{-1}(\sin\pi)$

Solution

(a) As $\frac{1}{2} \in [-1, 1] \Rightarrow \mathrm{Sin}^{-1}\frac{1}{2}$ exists. Therefore, $\mathrm{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$.

(b) As $-\frac{\sqrt{3}}{2} \in [-1, 1] \Rightarrow \mathrm{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$ exists. Now, $\mathrm{Arcsin}\left(-\frac{\sqrt{3}}{2}\right) = -\mathrm{Arcsin}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

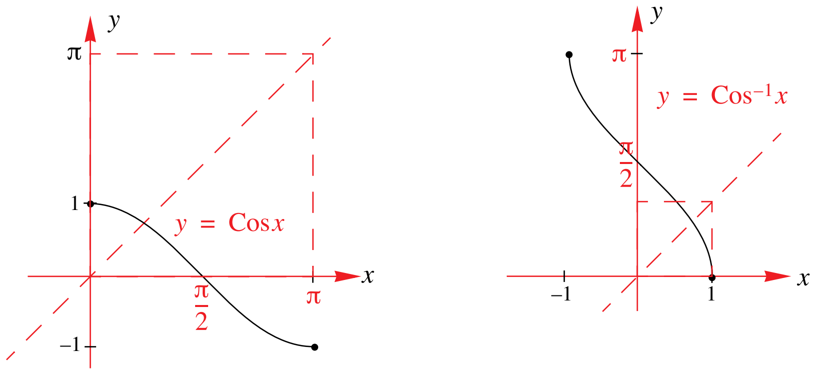
(c) As $1.3 \notin [-1, 1] \Rightarrow \mathrm{Sin}^{-1}(1.3)$ does **not** exist.

(d) As $\sin\pi \in [-1, 1] \Rightarrow \text{Sin}^{-1}(\sin\pi)$ exists. So, $\text{Sin}^{-1}(\sin\pi) = \text{Sin}^{-1}(0) = 0$.
Note that $\text{Sin}^{-1}(\sin\pi) \neq \pi$! Why?

10.4.2 THE INVERSE COSINE FUNCTION

For similar reasons as those for the sine function, the cosine function, $\cos x$, $x \in]-\infty, \infty[$ being a many to one function, with its inverse, $\arccos x$, $-1 \leq x \leq 1$ (or $\cos^{-1}x$, $-1 \leq x \leq 1$) needs to be restricted to the domain $[0,\pi]$, to produce a function that is one to one.

The function $y = \text{Cos}(x), x \in [0, \pi], -1 \leq y \leq 1$ (with a capital 'C') will have the inverse function defined as $f(x) = \text{Cos}^{-1}(x), -1 \leq x \leq 1, 0 \leq y \leq \pi$. The graphs of these functions are:



Notice then that the domain of $\text{Cos}^{-1}x$ = range of $\text{Cos}x$ = $[-1, 1]$
and the range of $\text{Cos}^{-1}x$ = domain of $\text{Cos}x$ = $[0, \pi]$.

When these restrictions are adhered to, we refer to $\text{Cos}^{-1}x$ (which is sometimes denoted by $\text{Arccos}x$) as the **principal value** of $\arccos x$.

From our fundamental indentity property of inverse functions, i.e., $f o f^{-1}(x) = f^{-1} o f(x) = x$, we have that

$$\text{Cos}(\text{Cos}^{-1}x) = x, -1 \leq x \leq 1 \text{ and } \text{Cos}^{-1}(\text{Cos}x) = x, 0 \leq x \leq \pi$$

Therefore, $\text{Cos}(\text{Cos}^{-1}x) = x = \text{Cos}^{-1}(\text{Cos}x)$ only if $0 \leq x \leq 1$.

This then means that sometimes we can provide a meaningful interpretation of expressions such as $\cos(\text{Cos}^{-1}x)$ and $\text{Cos}^{-1}(\cos x)$ – as long as we adhere to the relevant restrictions.
Note also that in this case, $\text{Cos}^{-1}(-x) \neq -\text{Cos}^{-1}(x)$.

EXAMPLE 10.31

Give the exact value of

(a) $\text{Cos}^{-1}\frac{1}{2}$ (b) $\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right)$ (c) $\text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$

Solution

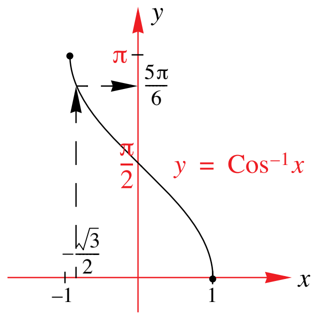
(a) As $\frac{1}{2} \in [-1, 1] \Rightarrow \text{Cos}^{-1}\frac{1}{2}$ exists. Therefore, $\text{Cos}^{-1}\frac{1}{2} = \frac{\pi}{3}$.

(b) As $-\frac{\sqrt{3}}{2} \in [-1, 1] \Rightarrow \text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ exists.

Let $y = \text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, then, $\text{Cos}\,y = -\frac{\sqrt{3}}{2}, 0 \le y \le \pi$.

$$\Leftrightarrow y = \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$



(c) As $\cos\left(\frac{3\pi}{2}\right) \in [-1, 1] \Rightarrow \text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$ exists. $\text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \text{Cos}^{-1}(0) = \frac{\pi}{2}$.

Notice that $\text{Cos}^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) \neq \frac{3\pi}{2}$.

EXAMPLE 10.32

Give the exact value of

(a) $\sin\left(\text{Arccos}\left(\frac{1}{\sqrt{2}}\right)\right)$ (b) $\cos\left(\text{Sin}^{-1}\left(\frac{1}{4}\right)\right)$ (c) $\sin\left(\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{3}{4}\right)\right)$

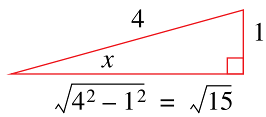
Solution

(a) Let $\text{Arccos}\left(\frac{1}{\sqrt{2}}\right) = x \therefore$ as $\frac{1}{\sqrt{2}} \in [0, 1] \Rightarrow \text{Arccos}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

Then, $\sin\left(\text{Arccos}\left(\frac{1}{\sqrt{2}}\right)\right) = \sin(x) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

(b) Let $\text{Sin}^{-1}\left(\frac{1}{4}\right) = x \therefore$ as $\frac{1}{4} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(\frac{1}{4}\right)$ exists.

However, this time we cannot obtain an exact value for x, so we make use of a right-angled triangle:



Therefore, from the triangle we have that $\cos x = \frac{\sqrt{15}}{4}$.

i.e., $\cos\left(\text{Sin}^{-1}\left(\frac{1}{4}\right)\right) = \cos x = \frac{\sqrt{15}}{4}$.

(c) Let $\text{Cos}^{-1}\left(\frac{3}{4}\right) = \theta \therefore$ as $\frac{3}{4} \in [-1, 1] \Rightarrow \text{Cos}^{-1}\left(\frac{3}{4}\right)$ exists.

Then, $\sin\left(\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{3}{4}\right)\right) = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Therefore, $\sin\left(\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{3}{4}\right)\right) = \cos\left(\text{Cos}^{-1}\left(\frac{3}{4}\right)\right) = \frac{3}{4}$

EXAMPLE 10.33 Give the exact value of

(a) $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right) + \text{Sin}^{-1}\left(\frac{3}{4}\right)\right)$ (b) $\cos\left(2\text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

Solution

(a) As $\frac{1}{3} \in [-1, 1]$ and $\frac{3}{4} \in [-1, 1]$ then both $\text{Cos}^{-1}\left(\frac{1}{3}\right)$ and $\text{Sin}^{-1}\left(\frac{3}{4}\right)$ exist.

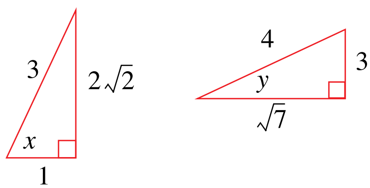
Using the compound angle formula for sine, we have that

$\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right) + \text{Sin}^{-1}\left(\frac{3}{4}\right)\right)$

$= \sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) + \sin\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)\cos\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)$

$= \sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) + \frac{3}{4} \times \frac{1}{3}$

However, we now need to construct two right-angled triangles to evaluate the first part of the expression. Let $x = \text{Cos}^{-1}\left(\frac{1}{3}\right)$ and $y = \text{Sin}^{-1}\left(\frac{3}{4}\right)$ so that we have the following triangles:



Meaning that $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right) = \sin x = \frac{2\sqrt{2}}{3}$

and $\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) = \cos y = \frac{\sqrt{7}}{4}$

Therefore, $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right) + \text{Sin}^{-1}\left(\frac{3}{4}\right)\right) = \frac{2\sqrt{2}}{3} \times \frac{\sqrt{7}}{4} + \frac{3}{4} \times \frac{1}{3} = \frac{2\sqrt{14} + 3}{12}$

(b) As $\frac{2}{\sqrt{5}} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)$ exists. Next, let $\theta = \text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)$, i.e., $\text{Sin}\theta = \frac{2}{\sqrt{5}}$.

Then, using the double angle formula for cosine, we have

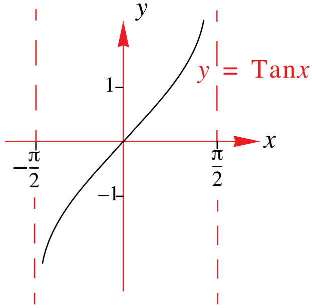
$$\cos\left(2\text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \cos(2\theta) = 1 - 2\sin^2\theta$$
$$= 1 - 2[\sin\theta]^2$$
$$= 1 - 2\left[\frac{2}{\sqrt{5}}\right]^2$$
$$= -\frac{3}{5}$$

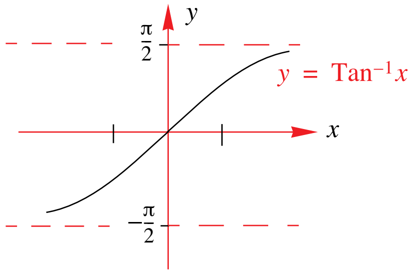
10.4.3 THE INVERSE TANGENT FUNCTION

The tangent function can be made one:one by restricting its domain to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$f(x) = \text{Tan}(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The function $y = \text{Tan}(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), -\infty < y < \infty$, (with a capital 'T') will have the inverse function defined as $f(x) = \text{Tan}^{-1}(x), -\infty < x < \infty$. The graphs of these functions are:





Notice then that the domain of $\text{Tan}^{-1}x$ = range of $\text{Tan}x = (-\infty, \infty)$

and the range of $\text{Tan}^{-1}x$ = domain of $\text{Tan}x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

When these restrictions are adhered to, we refer to $\text{Tan}^{-1}x$ (which is sometimes denoted by $\text{Arctan}x$) as the **principal value** of $\arctan x$.

From our fundamental indentity property of inverse functions, i.e., $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$, we have that

$$\text{Tan}(\text{Tan}^{-1}x) = x, -\infty < x < \infty \text{ and } \text{Tan}^{-1}(\text{Tan}x) = x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Therefore, $\text{Tan}(\text{Tan}^{-1}x) = x = \text{Tan}^{-1}(\text{Tan}x)$ only if $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

As we saw with the sine and cosine functions, it **may** also be possible to evaluate expressions such as $\tan(\text{Tan}^{-1}x)$ and $\text{Tan}^{-1}(\tan x)$.

For example, $\tan(\text{Tan}^{-1}1) = \tan\left(\frac{\pi}{4}\right) = 1$, however, $\text{Tan}^{-1}\left(\tan\frac{2\pi}{3}\right) = \text{Tan}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

Note also that $\text{Tan}^{-1}(-x) = -\text{Tan}(x)$

EXAMPLE 10.34

Give the exact value of

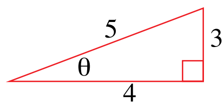
(a) $\tan\left(\text{Sin}^{-1}\left(-\frac{3}{5}\right)\right)$ (b) $\sin\left(2\text{Tan}^{-1}\left(\frac{1}{3}\right)\right)$

Solution

(a) As $-\frac{3}{5} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(-\frac{3}{5}\right)$ exists. Then, we let $\theta = \text{Sin}^{-1}\left(\frac{3}{5}\right)$, so that $\text{Sin}\theta = \frac{3}{5}$.

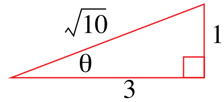
Next we construct an appropriate right-angled triangle:

So, $\tan\left(\text{Sin}^{-1}\left(-\frac{3}{5}\right)\right) = \tan\left(-\text{Sin}^{-1}\left(\frac{3}{5}\right)\right) = \tan(-\theta) = -\tan\theta = -\frac{3}{4}$



(b) As $\frac{1}{3} \in (-\infty, \infty) \Rightarrow \text{Tan}^{-1}\left(\frac{1}{3}\right)$ exists. Let $\text{Tan}^{-1}\left(\frac{1}{3}\right) = \theta \therefore \text{Tan}\theta = \frac{1}{3}$.

Next we construct an appropriate right-angled triangle:



Then, $\sin\left(2\text{Tan}^{-1}\left(\frac{1}{3}\right)\right) = \sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times \frac{1}{\sqrt{10}} \times \frac{3}{\sqrt{10}}$$

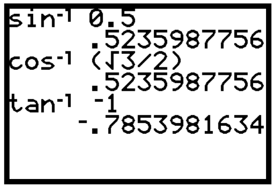
$$= \frac{3}{5}$$

It is these restricted functions that are programmed into most calculators, spreadsheets etc.
If the calculator is set in radian mode, some sample calculations are:

$$\text{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$\text{Cos}^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$$\text{Tan}^{-1}-1 = -\frac{\pi}{4}$$

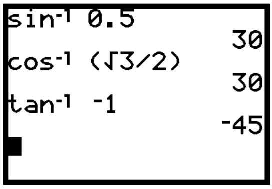


If the calculator is set in degree mode, the results are:

$$\text{Sin}^{-1}\frac{1}{2} = 30°$$

$$\text{Cos}^{-1}\frac{\sqrt{3}}{2} = 30°$$

$$\text{Tan}^{-1}-1 = -45°$$



Note that, when using a calculator in radian mode, answers that are multiples of π can be difficult to find. Not many of us know that 0.5235987756 is $\frac{\pi}{6}$. It can be a good idea to work on the calculator in degrees, even when the final answers are required in radians. Thus, in the above example, $\text{Cos}^{-1}\frac{\sqrt{3}}{2} = 30° \Rightarrow \text{Cos}^{-1}\frac{\sqrt{3}}{2} = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ could be a good way to infer the exact radian answer. It is not a good idea to work with approximations, unless you are told to do so.

EXAMPLE 10.35

Find the principal values of the following

(a) $\text{Sin}^{-1}\left(-\frac{1}{2}\right)$ (b) $\text{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (c) $\text{Tan}^{-1}\sqrt{3}$

Solution

(a) $\text{Sin}^{-1}\left(-\frac{1}{2}\right) = -30°$ from a calculator in degree mode. The radian answer is $-\frac{\pi}{6}$.

(b) $\text{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135°$ from a calculator in degree mode. The radian answer is $\frac{3\pi}{4}$.

(c) $\text{Tan}^{-1}\sqrt{3} = 60°$ from a calculator in degree mode. The radian answer is $\frac{\pi}{3}$.

EXERCISES 10.4

1. Find the principal values of the following, giving answers in radians:

(a) $\text{Tan}^{-1}1$ (b) $\text{Arcsin}1$ (c) $\text{Arccos}-1$

(d) $\text{Sin}^{-1}\frac{\sqrt{3}}{2}$ (e) $\text{Cos}^{-1}\frac{1}{\sqrt{2}}$ (f) $\text{Tan}^{-1}-\sqrt{3}$

(g) $\text{Tan}^{-1}2$ (h) $\text{Sin}^{-1}-0.7$ (i) $\text{Arctan}0.1$

(j) $\text{Arccos}0.3$ (k) $\text{Sin}^{-1}-0.6$ (l) $\text{Tan}^{-1}5$

(m) $\text{Cos}^{-1}3$ (n) $\text{Tan}^{-1}-30$ (o) $\text{Sin}^{-1}\left(\frac{7}{8}\right)$

2. Solve the following equations, giving exact answers

(a) $\text{Arctan}x = \frac{3\pi}{4}$ (b) $\text{Arcsin}(2x) = \frac{\pi}{3}$ (c) $\text{Arccos}(3x) = \frac{5\pi}{4}$

3. Prove (a) $\text{Arctan}(4) - \text{Arctan}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

(b) $\text{Sin}^{-1}\left(\frac{4}{5}\right) + \text{Sin}^{-1}\left(-\frac{4}{5}\right) = 0$

(c) $\text{Sin}^{-1}\left(\frac{3}{5}\right) + \text{Tan}^{-1}\left(\frac{7}{24}\right) = \text{Cos}^{-1}\left(\frac{3}{5}\right)$

4. Solve for x, where $\text{Arctan}(3x) - \text{Arctan}(2x) = \text{Arctan}\left(\frac{1}{5}\right)$

5. Find the exact value of

(a) $\sin\left[\frac{\pi}{2} - \text{Cos}^{-1}\left(\frac{2}{3}\right)\right]$ (b) $\cos\left[\frac{\pi}{2} + \text{Sin}^{-1}\left(-\frac{1}{3}\right)\right]$

(c) $\cos[\text{Tan}^{-1}(-\sqrt{3})]$ (d) $\tan\left(\text{Cos}^{-1}\left(\frac{4}{5}\right)\right)$

(e) $\sec\left(\text{Sin}^{-1}\left(-\frac{1}{3}\right)\right)$ (f) $\cot(\text{Tan}^{-1}(-1))$

6. Find the exact value of

(a) $\sin\left[\text{Sin}^{-1}\left(\frac{3}{5}\right) + \text{Sin}^{-1}\left(\frac{4}{5}\right)\right]$ (b) $\sin\left[\text{Sin}^{-1}\left(\frac{3}{5}\right) + \text{Sin}^{-1}\left(-\frac{4}{5}\right)\right]$

(c) $\cos\left[\text{Tan}^{-1}\left(\frac{4}{3}\right) - \text{Cos}^{-1}\left(\frac{5}{13}\right)\right]$ (d) $\tan\left[\text{Tan}^{-1}\left(\frac{4}{3}\right) + \text{Tan}^{-1}\left(\frac{3}{4}\right)\right]$

(e) $\sin\left(2\text{Arcsin}\left(\frac{2}{3}\right)\right)$ (f) $\cos\left(2\text{Tan}^{-1}\left(-\frac{1}{2}\right)\right)$

(g) $\tan\left(2\text{Cos}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ (h) $\cos\left(2\text{Sin}^{-1}\left(-\frac{1}{2}\right)\right)$

7. Prove that (a) $\text{Sin}^{-1}\left(\frac{7}{25}\right) = \text{Cos}^{-1}\left(\frac{24}{25}\right)$ (b) $\text{Tan}^{-1}\left(\frac{1}{2}\right) + \text{Tan}^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

(c) $\text{Tan}^{-1}(1) + \text{Tan}^{-1}(2) + \text{Tan}^{-1}(3) = \pi$

8. Prove that (a) $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2}, 0 \le x \le 1$

(b) $\text{Cos}^{-1}x = \text{Sin}^{-1}\sqrt{1-x^2}, 0 \le x \le 1$

(c) $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for all real x and y, $xy \ne 1$.

(d) $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$, for all real x and y, $xy \ne -1$.

9. Find (a) $\tan(\text{Cos}^{-1}k)$, where $-1 \le k \le 1, k \ne 0$.

(b) $\cos(\text{Tan}^{-1}k)$, where k is a real number.

10. (a) State the implied domain of the following functions and sketch their graphs

(a) $f(x) = \text{Sin}^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{2}$ (b) $f(x) = \text{Cos}^{-1}(2x) - \pi$

(c) $g(x) = 2\text{Sin}^{-1}(x-1)$ (d) $h(x) = \text{Cos}^{-1}(x+2) - \frac{\pi}{2}$

11. (a) Prove that $\text{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \text{Tan}^{-1}x$, for all real x.

(b) Prove that $\text{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \text{Tan}^{-1}x$, for all real x.

12. (a) On the same set of axes sketch the graphs of $y = \text{Cos}^{-1}x$ and $y = \text{Sin}^{-1}x$.

(b) Hence, deduce the value of k, where

i. $\text{Cos}^{-1}x + \text{Sin}^{-1}x = k, -1 \le x \le 0$

ii. $\text{Cos}^{-1}x + \text{Sin}^{-1}x = k, 0 \le x \le 1$

(c) On a separate set of axes, sketch the graph of $y = \text{Cos}^{-1}x + \text{Sin}^{-1}x, -1 \le x \le 1$.

13. Prove that if $n > 1$ then $\text{Arctan}\left(\frac{1}{n}\right) - \text{Arctan}\left(\frac{1}{n+1}\right) = \text{Arctan}\left(\frac{1}{1+n(n+1)}\right)$.

Hence, find $\sum_{i=1}^{n} \text{Arctan}\left(\frac{1}{1+i(i+1)}\right)$.

10.5 TRIGONOMETRIC EQUATIONS

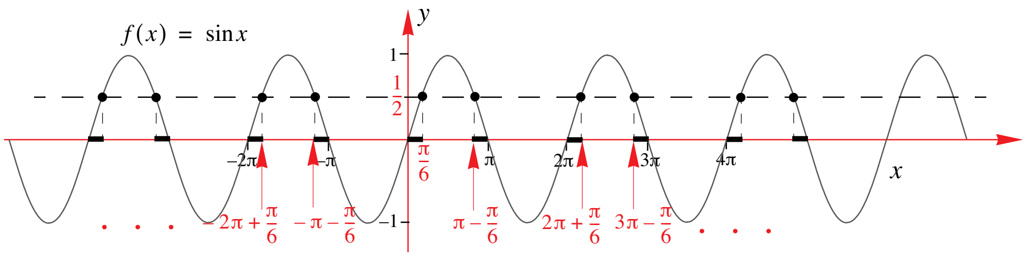
We have already encountered solutions to trigonometric equations as part of a general observation in this chapter. There are two basic methods that can be used when solving trigonometric equations;

Method 1. Use the unit circle as a visual aid.
Method 2. Use the graph of the function as a visual aid.

The method you choose depends entirely on what you feel comfortable with. However, it is recommended that you become familiar with both methods.

10.5.1 SOLUTION TO $\sin x = a$, $\cos x = a$ & $\tan x = a$

The equation $\sin x = \frac{1}{2}$ produces an infinite number of solutions. This can be seen from the graph of the sine function.



Using the **principal angle** $\left(\frac{\pi}{6}\right)$ and **symmetry**, the solutions generated are

For $x \geq 0$: $x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, \ldots$

$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

For $x < 0$: $x = -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}, -3\pi - \frac{\pi}{6}, \ldots$

$= -\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, \ldots$

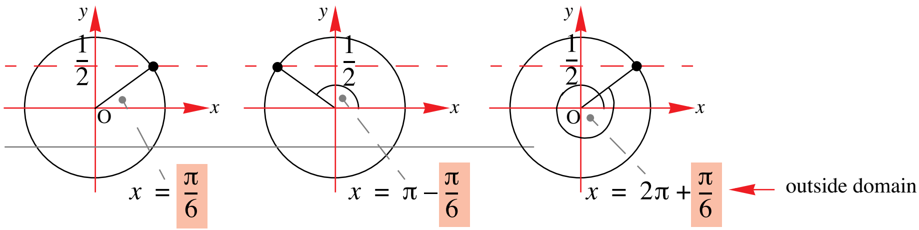
That is, $\sin x = \frac{1}{2} \Leftrightarrow x = \ldots, -\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{19\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

or, $x = n\pi + (-1)^n \times \frac{\pi}{6}$, where n is an integer (including zero).

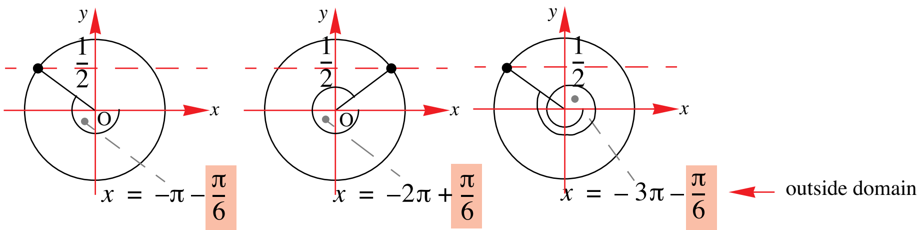
The solution $x = n\pi + (-1)^n \times \frac{\pi}{6}$ is known as the **general solution**. However, in this course, there will always be some restriction on the domain. For example, solve $\sin x = \frac{1}{2}$, $-2\pi < x < \pi$, which would then give $x = -\frac{7\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$ as its solutions.

The same problem could have been solved using Method 1. We start by drawing a unit circle and we continue to move around the circle until we have all the required solutions within the domain restriction. Again, the use of symmetry plays an important role in solving these equations.

For $x \geq 0$:



For $x < 0$:



Again, for the restricted domain $-2\pi < x < \pi$, we have $\sin x = \frac{1}{2}$ if $x = -\frac{7\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$.

The process is identical for the cosine and tangent function.

EXAMPLE 10.36

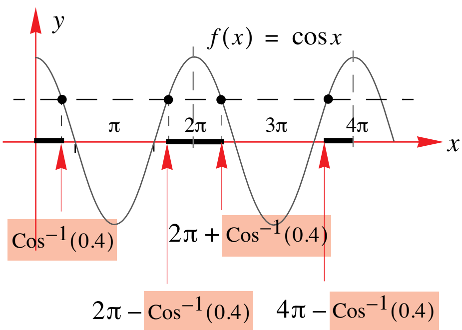
Solve the following, for $0 \leq x \leq 4\pi$, giving answers to 4 decimal places if no exact answers are available.

(a) $\cos x = 0.4$ (b) $\tan x = -1$ (c) $5\cos x - 2 = 0$

Solution

(a) **Step 1:** Find the **reference angle**: $x = \text{Cos}^{-1}(0.4) = 1.1593$.

Step 2: Sketch the cosine graph (or use the unit circle):



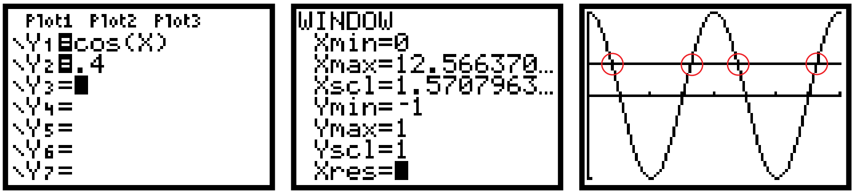
Step 3: Use the reference angle and symmetry to obtain solutions.

Therefore, solutions are, $x = \text{Cos}^{-1}(0.4), 2\pi - \text{Cos}^{-1}(0.4), 2\pi + \text{Cos}^{-1}(0.4), 4\pi - \text{Cos}^{-1}(0.4)$

$= 1.1593, 5.1239, 7.4425, 11.4071$

Step 4: Check that
i. all solutions are within the domain
ii. you have obtained all the solutions in the domain. (Use the graphics calculator to check).

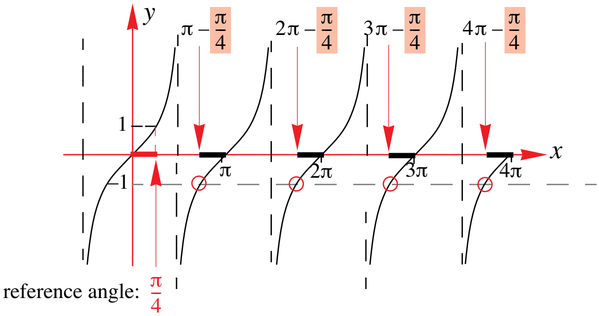
Using the TI–83 we count the number of intersections for $y = 0.4$ and $y = \cos x$:



4 solutions are required.

(b) **Step 1:** Find reference angle (in first quadrant): $\text{Tan}^{-1}(1) = \frac{\pi}{4}$.

Step 2: Sketch the tangent graph (or use the unit circle):



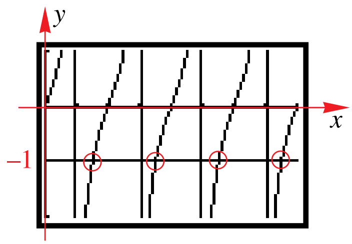
Step 3: Use the reference angle and symmetry to obtain solutions.

Reference angle is $\frac{\pi}{4}$.

Therefore solutions are $x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}$

$$= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

From the TI–83 display, we see that there are four solutions. Then, as the four solutions obtained all lie in the interval $[0, 4\pi]$ Step 4 is satisfied.



(c) $5\cos x - 2 = 0 \Leftrightarrow 5\cos x = 2 \Leftrightarrow \cos x = \frac{2}{5}$

i.e., $\cos x = 0.4$

Which is in fact, identical to the equation in part (a) and so, we have that

$$x = 1.1593, 5.1239, 7.4425, 11.4071$$

Part (c) in Example 10.36 highlights the fact that it is possible to transpose a trigonometric equation into a simpler form, which can readily be solved. Rather than remembering (or tryingt to commit to memory) the different possible forms of trigonometric equations and their specific solution processes, the four steps used (with possibly some algrebraic manupulation) will always transform a (seemingly) difficult equation into one having a simpler form, as in Example 10.36.

Some forms of trigonometric equations are:
$\sin(kx) = a$, $\cos(x+c) = a$, $\tan(kx+c) = a$, $b\cos(kx+c) = a$, $b\sin(kx+c)+d = a$

And of course, then there are equations involving the secant, cosecant and cotangent functions. But even the most involved of these equations, e.g., $b\sin(kx+c)+d = a$, can be reduced to a simpler form:

1. Transpose:

$$b\sin(kx+c)+d = a \Leftrightarrow b\sin(kx+c) = a-d$$
$$\Leftrightarrow \sin(kx+c) = \frac{a-d}{b}$$

2. Substitute:

Then, setting $kx+c = \theta$ and $\frac{a-d}{b} = m$, we have $\sin\theta = m$ which can be readily solved.

3. Solve for new variable:

So that the solutions to $\sin\theta = m$ are $\theta = \theta_1, \theta_2, \theta_3, \ldots$.

4. Solve for original variable:

We substitute back for θ and solve for x :

$$kx+c = \theta_1, \theta_2, \theta_3, \ldots$$
$$\Leftrightarrow kx = \theta_1-c, \theta_2-c, \theta_3-c, \ldots$$
$$\Leftrightarrow x = \frac{\theta_1-c}{k}, \frac{\theta_2-c}{k}, \frac{\theta_3-c}{k}, \ldots$$

All that remains is to check that all the solutions have been obtained and that they all lie in the restricted domain.

The best way to see how this works is through a number of examples.

EXAMPLE 10.37

Solve the following, for $0 \le x \le 2\pi$

(a) $\cos(2x) = 0.4$ (b) $\tan\left(\frac{1}{2}x\right) = -1$

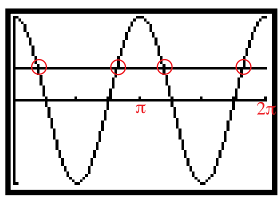
Solution

(a) Let $\theta = 2x$, so that we now solve the equation $\cos\theta = 0.4$.
From Example 10.36 (a) we already have the solutions, namely;

$$\theta = \mathrm{Cos}^{-1}(0.4), 2\pi - \mathrm{Cos}^{-1}(0.4), 2\pi + \mathrm{Cos}^{-1}(0.4), 4\pi - \mathrm{Cos}^{-1}(0.4)$$
$$= 1.1593, 5.1239, 7.4425, 11.4071$$

However, we want to solve for x not θ . So, we substitute back for x:

i.e., $2x = 1.1593, 5.1239, 7.4425, 11.4071$
$\therefore x = 0.5796, 2.5620, 3.7212, 5.7045$



To check that we have all the solutions, we sketch the graphs of $y = \cos(2x)$ and $y = 0.4$ over the domain $0 \le x \le 2\pi$.
The diagram shows that there should be four solutions.

(b) This time, to solve $\tan\left(\frac{1}{2}x\right) = -1$ we first let $\theta = \frac{1}{2}x$ so that we now need to solve the simpler equation $\tan\theta = -1$.

From Example 10.35 (b) we have that

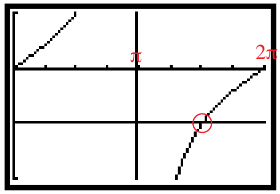
$$\theta = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

However, we want to solve for x, not θ. So, we substitute for x:

$$\frac{1}{2}x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$\therefore x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}$$



To check that we have all the solutions, we sketch the graphs of $y = \tan\left(\frac{1}{2}x\right)$ and $y = -1$ over the domain $0 \le x \le 2\pi$.

The diagram shows that there should be only one solution.

Therefore, the only solution is $x = \frac{3\pi}{2}$.

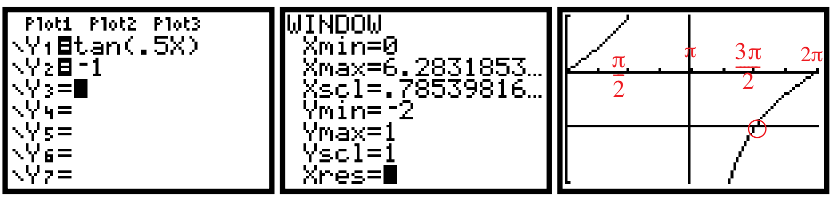
There is of course another step that could be used to help us predetermine which solutions are valid. This requires that we make a substitution not only into the equation, but also into the restricted domain statement.

In Example 10.36 (b), after setting $\theta = \frac{1}{2}x$ to give $\tan\theta = -1$, we next adjust the restricted domain as follows: $\theta = \frac{1}{2}x \Leftrightarrow x = 2\theta$.

So, from $0 \le x \le 2\pi$ we now have $0 \le 2\theta \le 2\pi \Leftrightarrow 0 \le \theta \le \pi$.

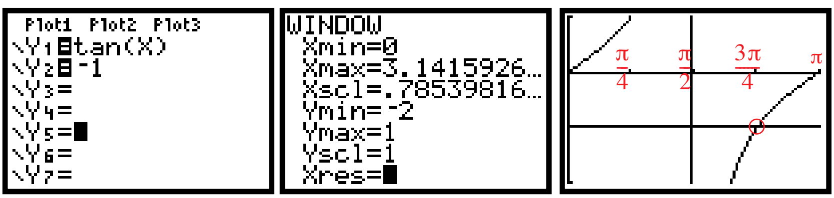
That is, we have the equivalent equations:

$\tan\left(\frac{1}{2}x\right) = -1, 0 \le x \le 2\pi$:



$\therefore x = \frac{3\pi}{2}$

$\tan\theta = -1, 0 \le \theta \le \pi$:



$\theta = \frac{3\pi}{4} \therefore \frac{1}{2}x = \frac{3\pi}{4}$

$\therefore x = \frac{3\pi}{2}$

EXAMPLE 10.38

Solve $4\sin(3x) = 2$, $0° \le x \le 360°$

Solution

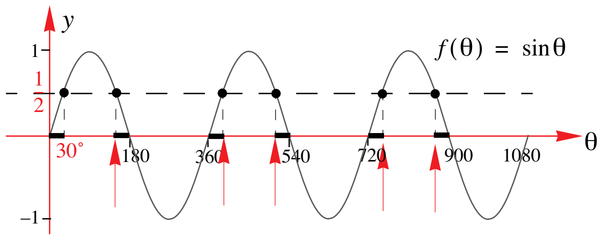
(a) $4\sin(3x) = 2 \Leftrightarrow \sin(3x) = 0.5, 0° \le x \le 360°$.

Let $\theta = 3x \Rightarrow \sin\theta = 0.5$.

New domain: $0° \le x \le 360° \Leftrightarrow 0° \le \frac{\theta}{3} \le 360° \Leftrightarrow 0° \le \theta \le 1080°$

Therefore, we have, $\sin\theta = 0.5, 0° \le \theta \le 1080°$

The reference angle is $30°$, then, by symmetry we have



$$\therefore \theta = 30°, 180° - 30°, 360° + 30°, 540° - 30°, 720° + 30°, 900° - 30°$$

$$\therefore 3x = 30°, 150°, 390°, 510°, 750°, 870°$$

$$\therefore x = 10°, 50°, 130°, 170°, 250°, 290°$$

All solutions lie within the **original** specified domain, $0° \le x \le 360°$.

EXAMPLE 10.39

Solve $2\cos\left(\frac{x}{2} + \frac{\pi}{2}\right) - \sqrt{3} = 0$, for $-\pi \le x \le 4\pi$.

Solution

$$2\cos\left(\frac{x}{2} + \frac{\pi}{2}\right) - \sqrt{3} = 0 \Leftrightarrow \cos\left(\frac{x}{2} + \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}, -\pi \le x \le 4\pi.$$

Let $\frac{x}{2} + \frac{\pi}{2} = \theta \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

Next, $\frac{x}{2} + \frac{\pi}{2} = \theta \Leftrightarrow \frac{x}{2} = \theta - \frac{\pi}{2} \Leftrightarrow x = 2\theta - \pi$. [Obtain x in terms of θ]

New domain: $-\pi \le x \le 4\pi \Leftrightarrow -\pi \le 2\theta - \pi \le 4\pi \Leftrightarrow 0 \le 2\theta \le 5\pi$

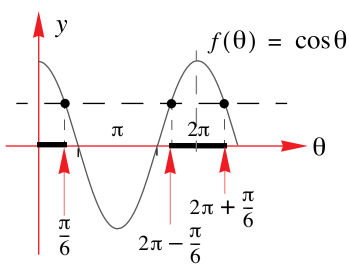
$$\Leftrightarrow 0 \le \theta \le \frac{5\pi}{2}$$

Therefore, our equivalent statement is .

$$\cos\theta = \frac{\sqrt{3}}{2}, 0 \le \theta \le \frac{5\pi}{2}$$

$$\therefore \theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}$$

$$= \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$$



But we still need to find the x-values:

Therefore, substituting $\theta = \frac{x}{2} + \frac{\pi}{2}$ back into the solution set, we have:

$$\frac{x}{2} + \frac{\pi}{2} = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$$

$$\Leftrightarrow x + \pi = \frac{\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$$

$$\Leftrightarrow x = -\frac{2\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

And, we notice that all solutions lie within the **original** specified domain, $-\pi \le x \le 4\pi$.

EXAMPLE 10.40

Solve the following, for $0 \le x \le 2\pi$.

(a) $2\sin x = 3\cos x$ (b) $2\sin 2x = 3\cos x$ (c) $\sin x = \cos 2x$

Solution

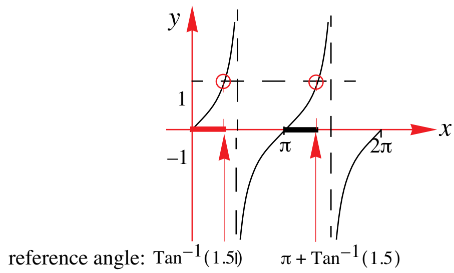
(a) $2\sin x = 3\cos x \Leftrightarrow \frac{\sin x}{\cos x} = \frac{3}{2}, 0 \le x \le 2\pi$

$\Leftrightarrow \tan x = 1.5, 0 \le x \le 2\pi$

The reference angle is $\mathrm{Tan}^{-1}(1.5) \approx 0.9828$

Therefore, $x = \mathrm{Tan}^{-1}(1.5), \pi + \mathrm{Tan}^{-1}(1.5)$

$\approx 0.9828, 4.1244$



(b) In this case we make use of the double angle identity, $\sin 2x = 2\sin x\cos x$.

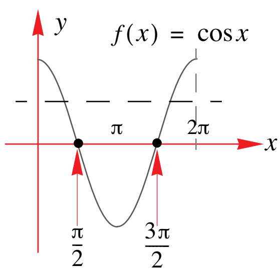
$\therefore 2\sin 2x = 3\cos x \Leftrightarrow 2(2\sin x\cos x) = 3\cos x$

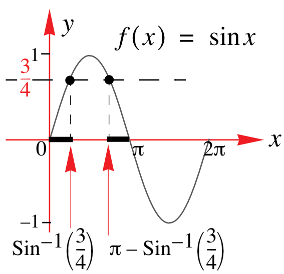
$\Leftrightarrow 4\sin x\cos x - 3\cos x = 0$

$\Leftrightarrow \cos x(4\sin x - 3) = 0$

$\cos x = 0, \sin x = \frac{3}{4}$

Solving for $\cos x = 0 : x = \frac{\pi}{2}, \frac{3\pi}{2}$. Solving for $\sin x = \frac{3}{4} : x = \mathrm{Sin}^{-1}\left(\frac{3}{4}\right), \pi - \mathrm{Sin}^{-1}\left(\frac{3}{4}\right)$





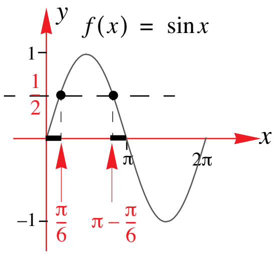
We solved two separate equations, giving the solution, $x = 0, \mathrm{Sin}^{-1}\left(\frac{3}{4}\right), \frac{\pi}{2}, \pi - \mathrm{Sin}^{-1}\left(\frac{3}{4}\right), \frac{3\pi}{2}$.

(c) This time we make use of the cosine double angle formula, $\cos 2x = 1 - 2\sin^2 x$.

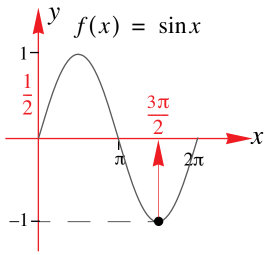
$$\therefore \sin x = \cos 2x$$
$$\Leftrightarrow \sin x = 1 - 2\sin^2 x$$
$$\Leftrightarrow 2\sin^2 x + \sin x - 1 = 0$$
$$\Leftrightarrow (2\sin x - 1)(\sin x + 1) = 0$$
$$\Leftrightarrow \sin x = \frac{1}{2} \text{ or } \sin x = -1$$

Again, we have two equations to solve.

Solving for $\sin x = \frac{1}{2}$:



Solving for $\sin x = -1$:



Therefore, solution set is $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

As the next example shows, the amount of working required to solve trigonometric equations can be significantly reduced, especially if you know the exact values (for the basic trigonometric angles) as well as the symmetry properties (without making use of a graph). However, we encourage you to use a visual aid when solving such equations.

EXAMPLE 10.41

Solve the equation $4\sin x\cos x = \sqrt{3}, -2\pi \le x \le 2\pi$

Solution

$$4\sin x\cos x = \sqrt{3}, -2\pi \le x \le 2\pi$$
$$2\sin 2x = \sqrt{3} \quad \text{using} \quad \sin 2\theta \equiv 2\sin\theta\cos\theta$$
$$\sin 2x = \frac{\sqrt{3}}{2}$$
$$2x = \ldots\frac{-11\pi}{3}, \frac{-10\pi}{3}, \frac{-5\pi}{3}, \frac{-4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \ldots$$
$$x = \frac{-11\pi}{6}, \frac{-5\pi}{3}, \frac{-5\pi}{6}, \frac{-2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$

EXERCISES 10.5

1. If $0 \le x \le 2\pi$, find:

(a) $\sin x = \frac{1}{\sqrt{2}}$ (b) $\sin x = -\frac{1}{2}$ (c) $\sin x = \frac{\sqrt{3}}{2}$

(d) $\sin 3x = \frac{1}{2}$ (e) $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ (f) $\sin(\pi x) = -\frac{\sqrt{2}}{2}$

2. If $0 \le x \le 2\pi$, find:

(a) $\cos x = \frac{1}{\sqrt{2}}$ (b) $\cos x = -\frac{1}{2}$ (c) $\cos x = \frac{\sqrt{3}}{2}$

(d) $\cos\left(\frac{x}{3}\right) = \frac{1}{2}$ (e) $\cos(2x) = \frac{1}{2}$ (f) $\cos\left(\frac{\pi}{2}x\right) = -\frac{\sqrt{2}}{2}$

3. If $0 \le x \le 2\pi$, find:

(a) $\tan x = \frac{1}{\sqrt{3}}$ (b) $\tan x = -1$ (c) $\tan x = \sqrt{3}$

(d) $\tan\left(\frac{x}{4}\right) = 2$ (e) $\tan(2x) = -\sqrt{3}$ (f) $\tan\left(\frac{\pi}{4}x\right) = -1$

4. If $0 \le x \le 2\pi$ or $0 \le x \le 360$, find:

(a) $\sin(x° + 60°) = \frac{1}{2}$ (b) $\cos(x° - 30°) = -\frac{\sqrt{3}}{2}$

(c) $\tan(x° + 45°) = -1$ (d) $\sin(x° - 20°) = \frac{1}{\sqrt{2}}$

(e) $\cos\left(2x - \frac{\pi}{2}\right) = \frac{1}{2}$ (f) $\tan\left(\frac{\pi}{4} - x\right) = 1$

(g) $\sec(2x + \pi) = 2$ (h) $\cot\left(2x + \frac{\pi}{2}\right) = 1$

5. If $0 \le x \le 2\pi$ or $0 \le x \le 360$, find:

(a) $\cos x° = \frac{1}{2}$ (b) $2\sin x + \sqrt{3} = 0$ (c) $\sqrt{3}\tan x = 1$

(d) $5\sin x° = 2$ (e) $4\sin^2 x - 3 = 0$ (f) $\frac{1}{\sqrt{3}}\tan x + 1 = 0$

(g) $2\sin\left(x + \frac{\pi}{3}\right) = -1$ (h) $5\cos(x + 2) - 3 = 0$ (i) $\tan\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

(j) $2\cos 2x + 1 = 0$ (k) $\tan 2x - \sqrt{3} = 0$ (l) $2\sin x° = 5\cos x°$

(m) $2\operatorname{cosec}\left(\frac{x}{2}\right) = 4$ (n) $\frac{1}{2}\cot(2x) = 0$ (o) $\sec\left(\frac{x}{3}\right) = -\sqrt{2}$

6. Solve the following equations for the intervals indicated, giving exact answers:

(a) $\sin\theta\cos\theta = \frac{1}{2}, -\pi \le \theta \le \pi$ (b) $\cos^2\theta - \sin^2\theta = -\frac{1}{2}, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(c) $\tan A = \frac{1 - \tan^2 A}{2}, -\pi \le A \le \pi$ (d) $\frac{\sin\theta}{1 + \cos\theta} = -1, -\pi \le \theta \le \pi$

(e) $\cos^2 x = 2\cos x, -\pi \le x \le \pi$ (f) $\sec 2x = \sqrt{2}, 0 \le x \le 2\pi$

(g) $2\sin^2 x - 3\cos x = 2, 0 \le x \le 2\pi$ (h) $\sin 2x = 3\cos x, 0 \le x \le 2\pi$

7. Find (a) $3\tan^2 x + \tan x = 2, 0 \le x \le 2\pi$.

(b) $\tan^3 x + \tan^2 x = 3\tan x + 3, 0 \le x \le 2\pi$.

8. If $0 \le x \le 2\pi$ find:

(a) $\sin^2 2x - \frac{1}{4} = 0$ (b) $\tan^2\left(\frac{x}{2}\right) - 3 = 0$ (c) $\cos^2(\pi x) = 1$

9. If $0 \le x \le 2\pi$ find:

(a) $\sec^2 x + 2\sec x = 8$ (b) $\sec^2 x = 2\tan x + 4$

(c) $\cot^2 x - \sqrt{3}\cot x = 0$ (d) $6\operatorname{cosec}^2 x = 8 + \cot x$

10. (a) Express $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x + \alpha)$.

(b) Solve $\sqrt{3}\sin x + \cos x = 1, 0 \le x \le 2\pi$.

11. (a) Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$.

(b) Solve $\sin x - \sqrt{3}\cos x = -1, 0 \le x \le 2\pi$.

12. Find x if $2\sin\left(x + \frac{\pi}{3}\right) + 2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{3}, 0 \le x \le 2\pi$.

13. (a) Sketch the graph of $f(x) = \sin x, 0 \le x \le 4\pi$.

(b) Hence, find i. $\left\{x \mid \sin x > \frac{1}{2}\right\} \cap \{x \mid 0 < x < 4\pi\}$.

ii. $\{x \mid \sqrt{3}\sin x < -1\} \cap \{x \mid 0 < x < 4\pi\}$.

14. (a) i. On the same set of axes sketch the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ for $0 \le x \le 2\pi$.

ii. Find $\{x \mid \sin x < \cos x, 0 \le x \le 2\pi\}$.

(b) i. On the same set of axes sketch the graphs of $f(x) = \sin 2x$ and $g(x) = \cos x$ for $0 \le x \le 2\pi$.

ii. Find $\{x \mid \sin 2x < \cos x, 0 \le x \le 2\pi\}$.

15. Show that $\{x : \sqrt{3}\cos x - \sin x = 1, x \in \mathbb{R}\} = \left\{x : x = 2n\pi + \frac{\pi}{6}\right\} \cup \left\{x : x = 2n\pi - \frac{\pi}{2}\right\}$,

where n is an integer.

16. Find (a) i. $\left\{x : \sin x = \sin\alpha, x \in \mathbb{R}, 0 < \alpha < \frac{\pi}{2}\right\}$.

ii. $\left\{x : \sin x \geq \sin\alpha, x \in \mathbb{R}, 0 < \alpha < \frac{\pi}{2}\right\}$

(b) $\{x : \sin 3x = \sin 2x, x \in \mathbb{R}\}$.

(c) $\{x : \cos 3x = \sin 2x, x \in \mathbb{R}\}$.

17. Given that the quadratic equation $x^2 - \sqrt{8}\cos\theta x + 3\cos\theta = 1$ has equal roots, find

(a) $\theta, 0 \leq \theta \leq 2\pi$.

(b) the roots of the quadratic.

18. (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) Using the substitution $t = 2\cos\theta$, show that $t^3 = 3t + 1$ becomes $\cos 3\theta = \frac{1}{2}$.

(c) Hence find the exact values of the roots of the equation $x^3 - 3x - 1 = 0$.

19. Find $\{x | \cos x + \cos 2x + \cos 3x = 0, -\pi < x < \pi\}$

20. (a) Given that $2\sin 2x + \cos 2x = a$, show that $(1 + a)\tan^2 x - 4\tan x + a = 1$.

(b) Hence, or otherwise show that if $\tan x_1$ and $\tan x_2$ are the roots of the quadratic in (a), then $\tan(x_1 + x_2) = 2$

21. (a) Solve $\left\{x° : 3\sin x° - \frac{1}{\sin x°} = 2, 0 \leq x \leq 360\right\}$

(b) Hence, find $\left\{x° : 3\sin x° < \frac{1}{\sin x°} + 2, 0 \leq x \leq 360\right\}$.

22. Prove that if $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \frac{\pi}{2}$, $\tan\alpha_1 < \frac{\sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_n}{\cos\alpha_1 + \cos\alpha_2 + \ldots + \cos\alpha_n} < \tan\alpha_n$.

23. Using the inequality $\tan\frac{x}{2} > \frac{x}{2}$, prove that $\sin x > x - \frac{1}{4}x^3$, where $0 < x < \frac{\pi}{2}$.

24. Find all real values of x and y such that $\sin^4 x + \cos^4 y + 2 = 4\sin x\cos y$.

[Hint: let $u = \sin x, v = \cos y$ and show that $(u^2 - 1)^2 + (v^2 - 1)^2 + 2(u - v)^2 = 0$].

10.6 APPLICATIONS

Functions of the type considered in the previous section are useful for modelling periodic phenomena. These sorts of applications usually start with data that has been measured in an experiment. The next task is to find a function that 'models' the data in the sense that it produces function values that are similar to the experimental data. Once this has been done, the function can be used to predict values that are missing from the measured data (interpolation) or values that lie outside the experimental data set (extrapolation).

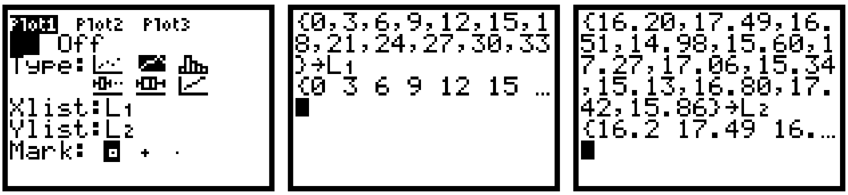
EXAMPLE 10.42 The table shows the depth of water at the end of a pier at various times (measured, in hours after midnight on the first day of the month.)

t (hr)	0	3	6	9	12	15	18	21	24	27	30	33
d (m)	16.20	17.49	16.51	14.98	15.60	17.27	17.06	15.34	15.13	16.80	17.42	15.89

Plot the data as a graph. Use your results to find a rule that models the depth data. Use your model to predict the time of the next high tide.

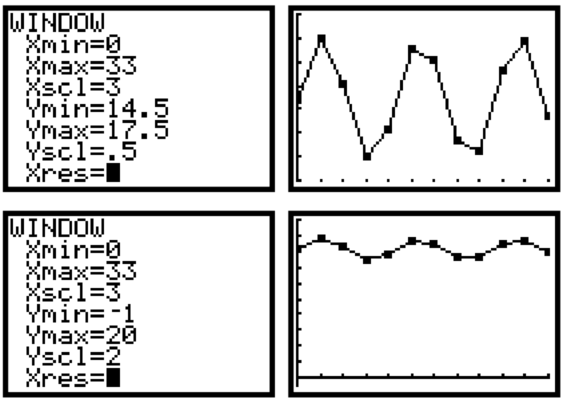
Solution

We start by entering the data as lists and then plotting them using the TI–83:



Using two different scales can provide graphs that do not really resemble each other.

Although both display a periodic nature, the second seems to reflect the need for a circular trigonometric function to model the behaviour.



This does suggest that the depth is varying periodically. It appears that the period is approximately 13 hours. This is found by looking at the time between successive high tides. This is not as easy as it sounds as the measurements do not appear to have been made exactly at the high tides. This means that an estimate will need to be made based upon the observation that successive high tides appear to have happened after 3, 16 and 32 hours. Next, we look at the amplitude and vertical translation. Again, because we do not have exact readings at high and low tides, these will need to be estimated. The lowest tide recorded is 14.98 and the highest is 17.49.

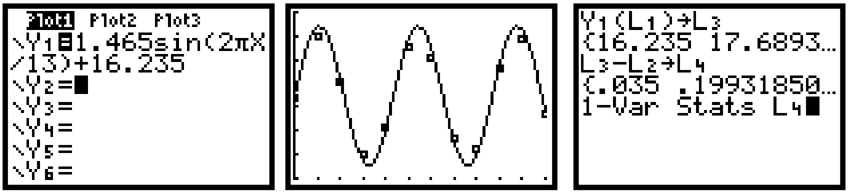
A first estimate of the vertical translation is $\frac{17.49 + 14.98}{2} = 16.235$ and the amplitude is

$17.7 - 16.235 = 1.465$. Since the graph starts near the mean depth and moves up it seems likely that the first model to try might be: $y = 1.465 \times \sin\left(\frac{2\pi t}{13}\right) + 16.235$

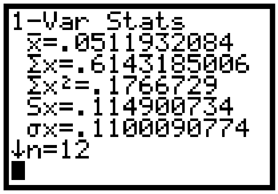
Notice that the dilation factor (along the x–axis) is found by using the result that if

$$\tau = 13 \Rightarrow \frac{2\pi}{n} = 13 \therefore n = \frac{2\pi}{13}.$$

The model should now be 'evaluated' which means testing how well it fits the data. This can be done by making tables of values of the data and the values predicted by the model and working to make the differences between these as small as possible. This can be done using a scientific or graphics calculator.



The model shown is quite good in that the errors are small with some being positive and some being negative. The function used is $d = 1.465 \times \sin\left(\frac{2\pi t}{13}\right) + 16.235$ and this can now be used to predict the depth for times that measurements were not made. Also, the graph of the modelling function can be added to the graph of the data (as shown).



The modelling function can also be used to predict depths into the future (extrapolation). The next high tide, for example can be expected to be 13 hours after the previous high tide at about 29.3 hours. This is after 42.3 hours.

EXAMPLE 10.43

During the summer months, a reservoir supplies water to an outer suburb based on the water demand, $D(t) = 120 + 60\sin\left(\frac{\pi}{90}t\right), 0 \le t \le 90$, where t measures the number of days from the start of Summer (which lasts for 90 days).

(a) Sketch the graph of $D(t)$.

(b) What are the maximum and minimum demands made by the community over this period?

Solution

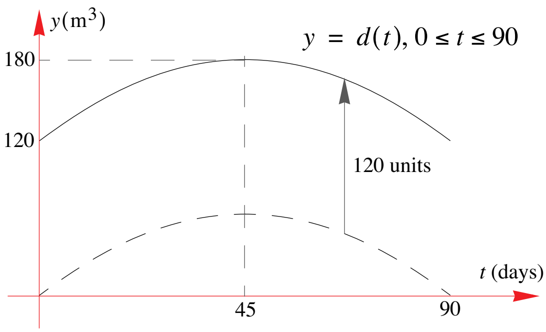
(a) The features of this function are:

Period $= \frac{2\pi}{\left(\frac{\pi}{90}\right)} = 180$ days

Amplitude $= 60$

Translation $= 120$ units up.

We 'pencil in' the graph of $y = 60\sin\left(\frac{\pi}{90}t\right)$ and then move it up 120 units:



(b) The minimum is 120 m^3 and the maximum is 180 m^3.

EXAMPLE 10.44 When a person is at rest, the blood pressure, P millimetres of mercury at any time t seconds can be approximately modelled by the equation

$$P(t) = -20\cos t\left(\frac{5\pi}{3}t\right) + 100,\ t \geq 0$$

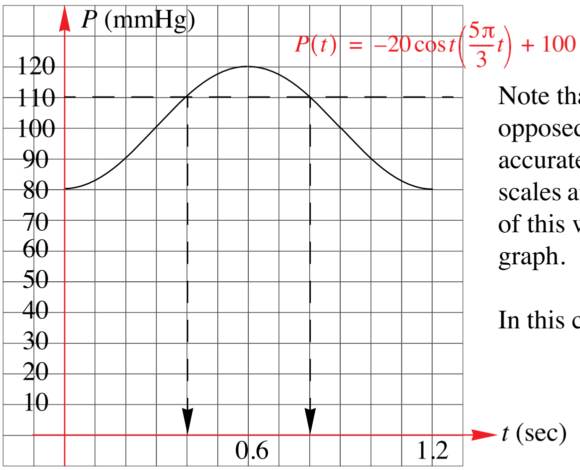
(a) Determine the amplitude and period of P.
(b) What is the maximum blood pressure reading that can be recorded for this person?
(c) Sketch the graph of $P(t)$, showing one full cycle.
(d) Find the first two times when the pressure reaches a reading of 110 mmHg.

Solution

(a) The amplitude is 20 mmHg and the period is given by $\dfrac{2\pi}{\left(\frac{5\pi}{3}\right)} = \dfrac{6}{5} = 1.2$ seconds.

(b) The maximum is given by (100 + amplitude) = 100 + 20 =120.

(c) One full cycle is 1.2 seconds long:



Note that the graph has been drawn as opposed to sketched. That is, it has been accurately sketched, meaning that the scales and the curve are accurate. Because of this we can read directly from the graph.

In this case, $P = 110$ when $t = 0.4$ and 0.8.

(d) Even though we have drawn the graph, we will now solve the relevant equation:

$$P(t) = 110 \Leftrightarrow 110 = -20\cos\left(\frac{5\pi}{3}t\right) + 100$$

$$\Leftrightarrow 10 = 20\cos\left(\frac{5\pi}{3}t\right)$$

$$\Leftrightarrow \cos\left(\frac{5\pi}{3}t\right) = -\frac{1}{2}$$

$$\therefore \frac{5\pi}{3}t = \pi - \mathrm{Cos}^{-1}\left(\frac{1}{2}\right), \pi + \mathrm{Cos}^{-1}\left(\frac{1}{2}\right) \quad \left[\text{Reference angle is } \mathrm{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}\right]$$

$$\Leftrightarrow \frac{5\pi}{3}t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\Leftrightarrow t = \frac{2}{5}, \frac{4}{5}$$

EXERCISES 10.6

1. The table shows the temperature in an office block over a 36 hour period.

t (hr)	0	3	6	9	12	15	18	21	24	27	30	33	36
T ˚C	18.3	15.0	14.1	16.0	19.7	23.0	23.9	22.0	18.3	15.0	14.1	16.0	19.7

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.
(c) Use your rule to predict the temperature after 40 hours.

2. The table shows the light level (L) during an experiment on dye fading.

t (hr)	0	1	2	3	4	5	6	7	8	9	10
L	6.6	4.0	7.0	10.0	7.5	4.1	6.1	9.8	8.3	4.4	5.3

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.

3. The table shows the value in $s of an industrial share over a 20 month period.

Month	0	2	4	6	8	10	12	14	16	18	20
Value	7.0	11.5	10.8	5.6	2.1	4.3	9.7	11.9	8.4	3.2	2.5

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.

4. The table shows the population (in thousands) of a species of fish in a lake over a 22 year period.

Year	0	2	4	6	8	10	12	14	16	18	20	22
Pop	11.2	12.1	13.0	12.7	11.6	11.0	11.6	12.7	13.0	12.1	11.2	11.2

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.

5. The table shows the average weekly sales (in thousands of $s) of a small company over a 15 year period.

Time	0	1.5	3	4.5	6	7.5	9	10.5	12	13.5	15
Sales	3.5	4.4	7.7	8.4	5.3	3.3	5.5	8.5	7.6	4.3	3.6

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.

6. The table shows the average annual rice production, P, (in thousands of tonnes) of a province over a 10 year period.

t (yr)	0	1	2	3	4	5	6	7	8	9	10
P	11.0	11.6	10.7	10.5	11.5	11.3	10.4	11.0	11.6	10.7	10.5

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.

7. The table shows the depth of water (D metres) over a 5 second period as waves pass the end of a pier.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
D	11.3	10.8	10.3	10.2	10.4	10.9	11.4	11.7	11.8	11.5	11.0

(a) Estimate the amplitude, period, horizontal and vertical translations.
(b) Find a rule that models the data.

8. The population (in thousands) of a species of butterfly in a nature sanctuary is modelled by the function:

$$P = 3 + 2\sin\left(\frac{3\pi t}{8}\right), 0 \le t \le 12$$

where t is the time in weeks after scientists first started making population estimates.

(a) What is the initial population?
(b) What are the largest and smallest populations?
(c) When does the population first reach 4 thousand butterflies?

9. A water wave passes a fixed point. As the wave passes, the depth of the water (D metres) at time t seconds is modelled by the function:

$$D = 7 + \frac{1}{2}\cos\left(\frac{2\pi t}{5}\right), t > 0$$

(a) What are the greatest and smallest depths?
(b) Find the first two times at which the depth is 6.8 metres.

10. The weekly sales (S) (in hundreds of cans) of a soft drink outlet is modelled by the function:

$$S = 13 + 5.5\cos\left(\frac{\pi t}{6} - 3\right), t > 0$$

t is the time in months with $t = 0$ corresponding to January 1st 1990,

(a) Find the minimum and maximum sales during 1990.
(b) Find the value of t for which the sales first exceed 1500 ($S = 15$).
(c) During which months do the weekly sales exceed 1500 cans?

11. The rabbit population, $R(t)$ thousands, in a northern region of South Australia is modelled by the equation $R(t) = 12 + 3\cos\left(\frac{\pi}{6}t\right), 0 \le t \le 24$, where t is measured in months after the first of January.

(a) What is the largest rabbit population predicted by this model?
(b) How long is it between the times when the population reaches consecutive peaks?
(c) Sketch the graph of $R(t)$ for $0 \le t \le 24$.
(d) Find the longest time span for which $R(t) \ge 13.5$.
(e) Give a possible explanation for the behaviour of this model.

12. A hill has its cross-section modelled by the function,

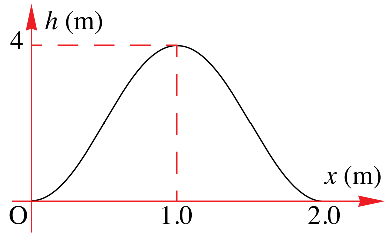
$$h : [0, 2] \mapsto \mathbb{R}, h(x) = a + b\cos(kx),$$

where $h(x)$ measures the height of the hill relative to the horizontal distance x m from O.

(a) Determine the values of
   i. k.
   ii. b.
   iii. a.

(b) How far, horizontally from O, would an ant climbing this hill from O be, when it first reaches a height of 1 metre?

(c) How much further, horizontally, will the ant have travelled when it reaches the same height of 1 metre once over the hill and on its way down?



13. A nursery has been infested by two insect pests: the Fruitfly and the Greatfly. These insects appear at about the same time that a particular plant starts to flower. The number of Fruitfly (in thousands), t weeks after flowering has started is modelled by the function

$$F(t) = 6 + 2\sin(\pi t), 0 \le t \le 4.$$

Whereas the number of Greatfly (in thousands), t weeks after flowering has started is modelled by the function

$$G(t) = 0.25t^2 + 4, 0 \le t \le 4$$

(a) Copy and complete the following table of values, giving your answers correct to the nearest hundred.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$F(t)$									
$G(t)$									

(b) On the same set of axes **draw** the graphs of
   i. $F(t) = 6 + 2\sin(\pi t), 0 \le t \le 4$.
   ii. $G(t) = 0.25t^2 + 4, 0 \le t \le 4$.

(c) On how many occassions will there be equal numbers of each insect?

(d) For what percentage of the time will there be more Greatflies than Fruitflies?

14. The depth, $d(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given

$$d(t) = 12 + 3\sin\left(\frac{\pi}{6}t\right), 0 \le t \le 24$$

(a) Sketch the graph of $d(t)$ for $0 \le t \le 24$.

(b) For what values of t will
   i. $d(t) = 10.5, 0 \le t \le 24$.
   ii. $d(t) \ge 10.5, 0 \le t \le 24$.

Boats requiring a minimum depth of b metres are only permitted to enter the harbour when the depth of water at the entrance of the harbour is at least b metres for a continuous period of one hour.

(c) Find the largest value of b, correct to two decimal place, which satisfies this condition.

5. (a) $I = \frac{a}{n^k}$ **6.** (a) 0.10 (b) $\lambda = \lambda_0 \times 10^{-kx}$ (c) 16.82% (d) $k = -\frac{1}{x}\log\left(\frac{\lambda}{\lambda_0}\right)$

EXERCISE 8.1.1

1. i. (b) 4 (c) $t_n = 4n-2$ ii. (b) –3 (c) $t_n = -3n+23$ iii. (b) –5 (c) $t_n = -5n+6$ iv. (b) 0.5 (c) $t_n = 0.5n$ v. (b) 2 (c) $t_n = y+2n-1$ vi. (b) –2 (c) $t_n = x-2n+4$ **2.** –28 **3.** 9,17 **4.** –43 **5.** 7 **6.** 7 **7.** –5 **8.** 0 **9.** (a) 41 (b) 31st **10.** 2, $\sqrt{3}$ **11.** (a) i. 2 ii. –3 (b) i. 4 ii. 11 **12.** $x-8y$ **13.** $t_n = 5+\frac{10}{3}(n-1)$ **14.** (a) –1 (b) 0

EXERCISE 8.1.2

1. (a) 145 (b) 300 (c) –170 **2.** (a) –18 (b) 690 (c) 70.4 **3.** (a) –105 (b) 507 (c) 224 **4.** (a) 126 (b) 3900 (c) 14th week **5.** 855 **6.** (a) 420 (b) –210 **7.** $a = 9, b = 7$

EXERCISE 8.1.3

1. 123 **2.** –3, –0.5, 2, 4.5, 7, 9.5, 12 **3.** 3.25 **4.** $a = 3$ $d = -0.05$ **5.** 10 000 **6.** 330 **7.** –20 **8.** 328 **9.** $725, 37wks **10.** i. $55 ii. 2750 **11.** (a) (i) 8m (ii) 40m (b) 84m (c) Dist $= 2n^2 - 2n = 2n(n-1)$ (d) 8 (e) 26 players, 1300m **12.** (a) 5050 (b) 10200 (c) 4233 **13.** (a) 145 (b) 390 (c) –1845 **14.** (b) $3n-2$

EXERCISE 8.2.1

1. (a) $r = 2, u_5 = 48, u_n = 3 \times 2^{n-1}$ (b) $r = \frac{1}{3}, u_5 = \frac{1}{27}, u_n = 3 \times \left(\frac{1}{3}\right)^{n-1}$ (c) $r = \frac{1}{5}, u_5 = \frac{2}{625}, u_n = 2 \times \left(\frac{1}{5}\right)^{n-1}$ (d) $r = -4, u_5 = -256, u_n = -1 \times (-4)^{n-1}$ (e) $r = \frac{1}{b}, u_5 = \frac{a}{b^3}, u_n = ab \times \left(\frac{1}{b}\right)^{n-1}$ (f) $r = \frac{b}{a}, u_5 = \frac{b^4}{a^2}, u_n = a^2 \times \left(\frac{b}{a}\right)^{n-1}$ **2.** (a) ±12 (b) $\frac{\pm\sqrt{5}}{2}$ **3.** (a) ±96 (b) 15th **4.** (a) $u_n = 10 \times \left(\frac{5}{6}\right)^{n-1}$ (b) $\frac{15625}{3888} \cong 4.02$ (c) $n = 5$ (4 times) **5.** $-2, \frac{4}{3}$ **6.** (a) i. $4096 ii. $2097.15 (b) 6.2 yrs **7.** $\left(u_n = \frac{1000}{169} \times \left(\frac{12}{5}\right)^{n-1}\right), \frac{1990656}{4225} \cong 471.16$ **8.** 2.5,5,10 or 10,5,2.5 **9.** 53757 **10.** 108 952 **11.** (a) $56 156 (b) $299 284

EXERCISE 8.2.2

1. (a) 3 (b) $\frac{1}{3}$ (c) –1 (d) $-\frac{1}{3}$ (e) 1.25 (f) $-\frac{2}{3}$ **2.** (a) 216513 (b) 1.6384×10^{-10} (c) $\frac{256}{729}$ (d) $\frac{729}{2401}$ (e) $-\frac{81}{1024}$ **3.** (a) 11; 354292 (b) 7; 473 (c) 8; 90.90909 (d) 8; 172.778 (e) 5; 2.256 (f) 13; 111.1111111111 **4.** (a) $\frac{127}{128}$ (b) $\frac{63}{8}$ (c) $\frac{130}{81}$ (d) 60 (e) $\frac{63}{64}$ **5.** 4; 118096 **6.** $2109.50 **7.** 9.28cm **8.** (a) $V_n = V_0 \times 0.7^n$ (b) 7 **9.** 54 **10.** 53.5gms; 50 weeks. **11.** 7 **12.** 9 **13.** –0.5, –0.7797 **14.** $r = 5$, 1.8×10^{10} **15.** $8407.35 **16.** 1.8×10^{19} or about 200 billion tonnes.

EXERCISE 8.2.3

1. Term 9 AP = 180, GP = 256. Sum to 11 terms AP = 1650, GP = 2047. **2.** 18. **3.** 12 **4.** 7, 12 **5.** 8 weeks (Ken $220 & Bo-Youn $255) **6.** (a) week 8 (b) week 12 **7.** (a) 1.618 (b) 121379 [~121400, depends on rounding errors]

EXERCISE 8.2.4

1. (i) $\frac{81}{2}$ (ii) $\frac{10}{13}$ (iii) 5000 (iv) $\frac{30}{11}$ **2.** $23\frac{23}{99}$ **3.** 6667 fish. [Nb: $t_{43} < 1$. If we use $n = 43$ then ans is 6660 fish]; 20 000 fish. Overfishing means that fewer fish are caught in the long run. [*An alternate estimate for the total catch is* 1665 *fish.*] **4.** 27 **5.** 48,12,3 or 16,12,9 **6.** (a) $\frac{11}{30}$ (b) $\frac{37}{99}$ (c) $\frac{191}{90}$ **7.** 128 cm **8.** $\frac{121}{9}$ **9.** $2 + \frac{4}{3}\sqrt{3}$ **10.** $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$ **11.** $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

EXERCISE 8.2.5

1. 3, –0.2 **2.** $\frac{2560}{93}$ **3.** $\frac{10}{3}$ **4.** (a) $\frac{43}{18}$ (b) $\frac{458}{99}$ (c) $\frac{413}{990}$ **5.** 9900 **6.** 3275 **7.** 3 **8.** $t_n = 6n-14$ **9.** 6 **10.** $-\frac{1}{6}$ **11.** i. 12 ii. 26 **12.** 9, 12 **13.** ±2 **14.** (5, 5, 5), (5, –10, 20) **15.** (a) 2, 7 (b) 2, 5, 8 (c) $3n-1$ **16.** (a) 5 (b) 2 m

EXERCISE 8.3

1. $2773.08 **2.** $4377.63 **3.** $1781.94 **4.** $12216 **5.** $35816.95 **6.** $40349.37 **7.** $64006.80 **8.** $276971.93, $281325.41 **9.** $63762.25 **10.** $98.62, $9467.14, interest $4467.14. Flat interest = $6000 **11.** $134.41, $3790.44, 0.602% /month (or 7.22% p.a)

EXERCISE 9.1

1.	*a* cm	*b* cm	*c* cm	*A*	*B*	*C*
1	3.8	4.1	1.6	67°	90°	23°
2	81.5	98.3	55.0	56°	90°	34°
3	32.7	47.1	33.9	44°	90°	46°
4	1.61	30.7	30.7	3°	90°	87°
5	2.3	2.74	1.49	57°	90°	33°
6	48.5	77	59.8	39°	90°	51°
7	44.4	81.6	68.4	33°	90°	57°
8	2.93	13.0	12.7	13°	90°	77°
9	74.4	94.4	58.1	52°	90°	38°
10	71.8	96.5	64.6	48°	90°	42°
11	23.3	34.1	24.9	43°	90°	47°
12	43.1	43.2	2.3	87°	90°	3°
13	71.5	80.2	36.4	63°	90°	27°
14	33.5	34.1	6.5	79°	90°	11°
15	6.1	7.2	3.82	58°	90°	32°
16	29.1	30	7.3	76°	90°	14°
17	29.0	29.1	2.0	86°	90°	4°
18	34.5	88.2	81.2	23°	90°	67°
19	24.0	29.7	17.5	54°	90°	36°
20	41.2	46.2	21.0	63°	90°	27°
21	59.6	72.9	41.8	55°	90°	35°
22	5.43	6.8	4.09	53°	90°	37°

23	13.0	19.8	14.9	41°	90°	49°
24	14.0	21.3	16.1	41°	90°	49°
25	82.4	88.9	33.3	68°	90°	22°

2. (a) $2\sqrt{3}$ (b) $5(1+\sqrt{3})$ (c) 4 (d) $2(1+\sqrt{3})$ (e) $\frac{4}{3}(3+\sqrt{3})$ (f) $\sqrt{106}-5$ **4.** (a) $25(1+\sqrt{3})$ (b) $\frac{40\sqrt{3}}{3}$

EXERCISE 9.2

1. (a) i. 030°T ii. 330°T iii. 195°T iv. 200°T (b) i. N25°E ii. S iii. S40°W iv. N10°W
2. 37.49m **3.** 18.94m **4.** 37° 18' **5.** $\frac{26}{9}$ m/s **6.** N58° 33'W, 37.23 km **7.** 199.82 m **8.** 10.58 m
9. 72.25 m **10.** 25.39 km **11.** 15.76 m **12.** (a) 3.01km N, 3.99km E (b) 2.87km E 0.88km S (c) 6.86km E 2.13km N (d) 7.19km 253°T **13.** 524m

EXERCISE 9.3

1. (a) 39°48' (b) 64°46' **2.** (a) 12.81 cm (b) 61.35 cm (c) 77°57' (d) 60.83 cm (e) 80° 32'
3. (a) 21°48' (b) 42°2' (c) 26°34' **4.** (a) 2274 (b) 12.7° **5.** 251.29 m **6.** (a) 103.5 m (b) 35.26° (c) 39.23° **7.** (b) 53.43 (c) 155.16 m (d) 145.68 m **8.** (b) 48.54 m **9.** (a) $\sqrt{(b-c)^2+h^2}$ (b) $\tan^{-1}\left(\frac{h}{a}\right)$ (c) $\tan^{-1}\left(\frac{h}{b-c}\right)$ (d) $2(b+c)\sqrt{h^2+a^2}+2a\sqrt{(b-c)^2+h^2}$ **10.** 82.80 m
11. (a) 40.61 m (b) 49.46 m **12.** (a) 10.61 cm (b) 75° 58' (c) 93° 22' **13.** (a) 1.44 m (b) 73° 13' (c) 62° 11'

EXERCISE 9.4

1. (a) 1999.2 cm^2 (b) 756.8 cm^2 (c) 3854.8 cm^2 (d) 2704.9 cm^2 (e) 538.0 cm^2 (f) 417.5 cm^2 (g) 549.4 cm^2 (h) 14.2 cm^2 (i) 516.2 cm^2 (j) 281.5 cm^2 (k) 918.8 cm^2 (l) 387.2 cm^2 (m) 139.0 cm^2 (n) 853.7 cm^2 (o) 314.6 cm^2 **2.** 69345 m^2 **3.** $100\pi - 6\sqrt{91}$ cm^2 **4.** 17.34 cm
5. (a) 36.77sq units (b) 14.70 sq units (c) 62.53 sq units **6.** 52.16 cm^2 **7.** 27° 2'
8. $\frac{(b+a\times\tan\theta)^2}{2\tan\theta}$ **9.** Area of $\Delta ACD = 101.78$ cm^2, Area of $\Delta ABC = 61.38$ cm^2

EXERCISE 9.5.1

	a cm	*b* cm	*c* cm	*A*	*B*	*C*
1	13.3	37.1	48.2	10°	29°	141°
2	2.7	1.2	2.8	74°	25°	81°
3	11.0	0.7	11.3	60°	3°	117°
4	31.9	39.1	51.7	38°	49°	93°
5	18.5	11.4	19.5	68°	35°	77°
6	14.6	15.0	5.3	75°	84°	21°
7	26.0	7.3	26.4	79°	16°	85°
8	21.6	10.1	28.5	39°	17°	124°
9	0.8	0.2	0.8	82°	16°	82°
10	27.7	7.4	33.3	36°	9°	135°
11	16.4	20.7	14.5	52°	84°	44°
12	21.4	45.6	64.3	11°	24°	145°
13	30.9	27.7	22.6	75°	60°	45°
14	29.3	45.6	59.1	29°	49°	102°
15	9.7	9.8	7.9	65°	67°	48°
16	21.5	36.6	54.2	16°	28°	136°
17	14.8	29.3	27.2	30°	83°	67°
18	10.5	0.7	10.9	52°	3°	125°
19	11.2	6.9	17.0	25°	15°	140°
20	25.8	18.5	40.1	30°	21°	129°

EXERCISE 9.5.2

	a	*b*	*c*	*A*°	*B*°	*C*°	*c**	*B**°	*C**°
1	7.40	18.10	21.06	20.00	56.78	103.22	12.95	123.22	36.78
2	13.30	19.50	31.36	14.00	20.77	145.23	6.49	159.23	6.77
3	13.50	17.00	25.90	28.00	36.24	115.76	4.12	143.76	8.24
4	10.20	17.00	25.62	15.00	25.55	139.45	7.22	154.45	10.55
5	7.40	15.20	19.55	20.00	44.63	115.37	9.02	135.37	24.63
6	10.70	14.10	21.41	26.00	35.29	118.71	3.94	144.71	9.29
7	11.50	12.60	22.94	17.00	18.68	144.32	1.16	161.32	1.68
8	8.30	13.70	18.67	24.00	42.17	113.83	6.36	137.83	18.17
9	13.70	17.80	30.28	14.00	18.32	147.68	4.27	161.68	4.32
10	13.40	17.80	26.19	28.00	38.58	113.42	5.24	141.42	10.58
11	12.10	16.80	25.63	23.00	32.85	124.15	5.30	147.15	9.85
12	12.00	14.50	24.35	21.00	25.66	133.34	2.72	154.34	4.66
13	12.10	19.20	29.34	16.00	25.94	138.06	7.57	154.06	9.94
14	7.20	13.10	19.01	15.00	28.09	136.91	6.30	151.91	13.09
15	12.20	17.70	23.73	30.00	46.50	103.50	6.93	133.50	16.50
16	9.20	20.90	27.97	14.00	33.34	132.66	12.59	146.66	19.34
17	10.50	13.30	21.96	20.00	25.67	134.33	3.03	154.33	5.67
18	9.20	19.20	26.29	15.00	32.69	132.31	10.80	147.31	17.69
19	7.20	13.30	18.33	19.00	36.97	124.03	6.82	143.03	17.97
20	13.50	20.40	25.96	31.00	51.10	97.90	9.01	128.90	20.10
21	10.80	20.80	24.48	26.00	57.59	96.41	12.91	122.41	31.59
22	13.00	12.20	23.91	19.00	17.79	143.21	0.84	162.21	1.21
23	13.60	20.40	22.92	36.00	61.85	82.15	10.09	118.15	25.85
24	11.40	12.50	22.88	16.00	17.59	146.41	1.15	162.41	1.59
25	8.00	16.80	23.99	10.00	21.39	148.61	9.10	158.61	11.39

26. (a-d) no triangles exist.

EXERCISE 9.5.3

1. 30.64 km **2.** 4.57 m **3.** 476.4 m **4.** 201°47'T **5.** 222.9 m **6.** (a) 3.40 m (b) 3.11 m
7. (b) 1.000 m (c) 1.715 m **8.** (a) 51.19 min (b)1 hr 15.96 min (c) 14.08 km **9.** \$4886 **10.**906 m

EXERCISE 9.5.4

	a cm	*b* cm	*c* cm	*A*	*B*	*C*
1	13.5	9.8	16.7	54°	36°	90°
2	8.9	10.8	15.2	35°	44°	101°
3	22.8	25.6	12.8	63°	87°	30°
4	21.1	4.4	21.0	85°	12°	83°
5	15.9	10.6	15.1	74°	40°	66°
6	8.8	13.6	20.3	20°	32°	128°
7	9.2	9.5	13.2	44°	46°	90°
8	23.4	62.5	58.4	22°	89°	69°

9	10.5	9.6	15.7	41°	37°	102°
10	21.7	36.0	36.2	35°	72°	73°
11	7.6	3.4	9.4	49°	20°	111°
12	7.2	15.2	14.3	28°	83°	69°
13	9.1	12.5	15.8	35°	52°	93°
14	14.9	11.2	16.2	63°	42°	75°
15	2.0	0.7	2.5	38°	13°	129°
16	7.6	3.7	9.0	56°	24°	100°
17	18.5	9.8	24.1	45°	22°	113°
18	20.7	16.3	13.6	87°	52°	41°
19	14.6	22.4	29.9	28°	46°	106°
20	7.0	6.6	9.9	45°	42°	93°
21	21.8	20.8	23.8	58°	54°	68°
22	1.1	1.7	1.3	41°	89°	50°
23	1.2	1.2	0.4	85°	76°	19°
24	23.7	27.2	29.7	49°	60°	71°
25	3.4	4.6	5.2	40°	60°	80°

EXERCISE 9.5.5

1. (a) 10.14 km (b) 121°T **2.** 7° 33' **3.** 4.12 cm **4.** 57.32 m **5.** 315.5 m **6.** (a) 124.3 km (b) W28° 47' S

EXERCISE 9.5.6

1. 39.60m 52.84m **2.** 30.2m **3.** 54°,42°, 84° **4.** 37° **5.** 028°T. **6.** 108.1cm **7.** (i) 135° (ii) 136cm **8.** 41°, 56°, 83° **9.** (i) 158° left (ii) 43.22km **10.** 264m **11.** 53.33cm **12.** 186m **13.** 50.12cm **14.** 5.17cm **15.** (a) 5950m (b) 13341m (c) 160° (d) 243° **17.** (a) 20.70° (b) 2.578 m (c) 1.994 m^3 **18.** (a) 4243 m^2 (b) 86 m (c) 101 m

EXERCISE 9.6

1. 5.36 cm **2.** 12.3 m **3.** 24 m **4.** 40.3 m, 48.2° **5.** 16.5 min, 8.9° **6.** ~10:49 am

7. (a) i. $\frac{d\sin\phi}{\sin(\phi-\theta)}$ ii. $\frac{d\sin\theta}{\sin(\phi-\theta)}$ (b) $\frac{d\sin\phi\tan\alpha}{\sin(\phi-\theta)}$ or $\frac{d\sin\theta\tan\beta}{\sin(\phi-\theta)}$ (c) $d\left(\frac{\sin\phi\cos\theta}{\sin(\phi-\theta)}-1\right)$

EXERCISE 9.7

1. (i) $\frac{169\pi}{150}\text{cm}^2$, $5.2+\frac{13\pi}{15}\text{cm}$ (ii) $\frac{529\pi}{32}\text{cm}^2$, $23+\frac{23\pi}{8}\text{cm}$ (iii) $242\pi\text{cm}^2$, $88+11\pi\text{cm}$

(iv) $\frac{1156\pi}{75}\text{m}^2$, $13.6+\frac{68\pi}{15}\text{m}$ (v) $\frac{96\pi}{625}\text{cm}^2$, $1.28+\frac{12\pi}{25}\text{cm}$ (vi) $\frac{361\pi}{15}\text{cm}^2$, $15.2+\frac{19\pi}{3}\text{cm}$

(vii) $5248.8\pi\text{m}^2$, $648+32.4\pi\text{cm}$ (viii) $\frac{12943\pi}{300}\text{cm}^2$, $17.2+\frac{301\pi}{30}\text{cm}$

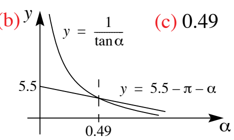
(ix) $\frac{1922\pi}{75}\text{cm}^2$, $12.4+\frac{124\pi}{15}\text{cm}$ (x) $\frac{15884\pi}{3}\text{cm}^2$, $152+\frac{418\pi}{3}\text{cm}$ (xi) $12\pi\text{cm}^2$, $24+2\pi\text{cm}$

(xii) $\frac{98\pi}{3}\text{cm}^2$, $28+\frac{14\pi}{3}\text{cm}$ (xiii) $\frac{196\pi}{75}\text{cm}^2$, $5.6+\frac{28\pi}{15}\text{cm}$ (xiv) $\frac{11532\pi}{25}\text{cm}^2$, $49.6+\frac{186\pi}{5}\text{cm}$

(xv) $\frac{3\pi}{50}\text{cm}^2$, $2.4+\frac{\pi}{10}\text{cm}$ **2.** 0.63^c, 36° **3.** 0.0942m^3 **4.** 1.64^c **5.** 79cm. **6.** 5.25cm^2

7. (a) 31.83m (b) 406.28m (c) 11° **8.** 1.11^c **9.** 0.75^c **10.** (a) 1.85^c (b) i. 37.09 cm ii. 88.57 cm (c) 370.92cm^2 **11.** 26.57 cm^2 **12.** 193.5 cm **13.** (a) 105.22 cm (b) 118.83 cm **14.** (a) 9 cm

(b) 12 cm (c) 36°52′ **15.** (b)



(c) 0.49 **16.** 1439.16 cm^2

EXERCISE 10.1

1. (a) 120° (b) 108° (c) 216° (d) 50° **2.** (a) π^c (b) $\frac{3\pi^c}{2}$ (c) $\frac{7\pi^c}{9}$ (d) $\frac{16\pi^c}{9}$ **3.** (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{2}$ (c) $-\sqrt{3}$ (d) –2 (e) $-\frac{1}{2}$ (f) $-\frac{\sqrt{3}}{2}$ (g) $\frac{1}{\sqrt{3}}$ (h) $\sqrt{3}$ (i) $-\frac{1}{\sqrt{2}}$ (j) $-\frac{1}{\sqrt{2}}$ (k) 1 (l) $-\sqrt{2}$ (m) $-\frac{1}{\sqrt{2}}$ (n) $\frac{1}{\sqrt{2}}$ (o) –1 (p) $\sqrt{2}$ (q) 0 (r) 1 (s) 0 (t) undefined **4.** (a) 0 (b) –1 (c) 0 (d) –1 (e) $\frac{1}{\sqrt{2}}$ (f) $-\frac{1}{\sqrt{2}}$ (g) –1 (h) $\sqrt{2}$ (i) $-\frac{1}{2}$ (j) $-\frac{\sqrt{3}}{2}$ (k) $\frac{1}{\sqrt{3}}$ (l) $\sqrt{3}$ (m) $-\frac{\sqrt{3}}{2}$ (n) $\frac{1}{2}$ (o) $-\sqrt{3}$ (p) 2 (q) $-\frac{1}{\sqrt{2}}$ (r) $\frac{1}{\sqrt{2}}$ (s) –1 (t) $-\sqrt{2}$ **5.** (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\frac{1}{2}$ (e) $-\frac{1}{\sqrt{3}}$ (f) $-\frac{1}{2}$ (g) $-\sqrt{2}$ (h) $-\frac{2}{\sqrt{3}}$ **6.** (a) $-\frac{1}{2}$ (b) $-\frac{1}{\sqrt{2}}$ (c) $\sqrt{3}$ (d) –2 (e) 1 (f) $\frac{1}{2}$ (g) $-\frac{1}{\sqrt{3}}$ (h) $-\frac{\sqrt{3}}{2}$ (i) $-\frac{2}{\sqrt{3}}$ (j) $\frac{1}{\sqrt{3}}$ (k) $\frac{2}{\sqrt{3}}$ (l) $-\frac{\sqrt{3}}{2}$ **7.** (a) $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ (b) $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ (c) $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ (d) $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$ **8.** (a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **10.** (a) $-\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $-\frac{2}{3}$ **11.** (a) $-\frac{2}{5}$ (b) $\frac{5}{2}$ (c) $\frac{2}{5}$ **12.** (a) k (b) $-\frac{1}{k}$ (c) $-k$ **13.** (a) $\frac{\sqrt{5}}{3}$ (b) $\frac{3}{\sqrt{5}}$ (c) $-\frac{\sqrt{5}}{3}$ **14.** (a) $-\frac{3}{5}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ **15.** (a) $\frac{4}{5}$ (b) $\frac{3}{4}$ (c) $-\frac{5}{3}$ **16.** (a) $-k$ (b) $-\sqrt{1-k^2}$ (c) $-\frac{k}{\sqrt{1-k^2}}$ **17.** (a) $-\sqrt{1-k^2}$ (b) $\frac{k}{\sqrt{1-k^2}}$ (c) $-\frac{1}{\sqrt{1-k^2}}$ **18.** (a) $\sin\theta$ (b) $\cot\theta$ (c) 1 (d) 1 (e) $\cot\theta$ (f) $\tan\theta$ **19.** (a) $\frac{\pi}{3},\frac{2\pi}{3}$ (b) $\frac{\pi}{3},\frac{5\pi}{3}$ (c) $\frac{\pi}{3},\frac{4\pi}{3}$ (d) $\frac{5\pi}{6},\frac{7\pi}{6}$ (e) $\frac{5\pi}{6},\frac{11\pi}{6}$ (f) $\frac{7\pi}{6},\frac{11\pi}{6}$

EXERCISE 10.2.1

3. (a) $x^2+y^2=k^2, -k\le x\le k$ (b) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, -b\le x\le b$ (c) $(x-1)^2+(2-y)^2=1, 0\le x\le 2$ (d) $\frac{(1-x)^2}{b^2}+\frac{(y-2)^2}{a^2}=1$ (e) $5x^2+5y^2+6xy=16$ **4.** (a) (i) $-\frac{4}{5}$ (ii) $-\frac{5}{3}$ (b) (i) $\frac{4}{\sqrt{7}}$ (ii) $-\frac{\sqrt{7}}{3}$ **5.** (a) $\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$ (b) $\frac{\pi}{2},\frac{7\pi}{6},\frac{11\pi}{6}$ (c) $0,\frac{\pi}{6},\frac{5\pi}{6},\pi,2\pi$ (d) $\frac{\pi}{2},\frac{3\pi}{2}$ **9.** (a) $\frac{2a}{a^2+1}$ (b) $\frac{a^2-1}{a^2+1}$ **10.** (a) i. 1 ii. 1 (b) 1 **11.** (a) $\frac{1-\sqrt{x^2-1}}{x}$ (b) $\frac{1+\sqrt{x^2-1}}{x}$ (c) $\frac{2}{x^2}-1$ **12.** (a) i. 6 ii. $\frac{5}{2}$ iii. $\frac{9}{8}$ (b) i. 5 ii. 1 iii. –2 **13.** (a) ±2 (b) $\frac{\pi}{6}+2k\pi, k\in\mathbb{Z}$ or $\frac{7\pi}{6}+2k\pi, k\in\mathbb{Z}$

14. (a) i. 25 ii. $\frac{1}{5^4}$ (b) i. 27 ii. $\frac{1}{3}$ **15.** (a) $1+2k$ (b) $(1-k)\sqrt{1+2k}$

16. (a) $\frac{1-a}{2\sqrt{a}}$ (b) i. $2+\sqrt{2a-a^2}$ ii. $\frac{-\sqrt{2a-a^2}}{1-a}$ **17.** (a) $\frac{2}{3}$ (b) $0, \pm\frac{2\sqrt{2}}{3}$ **18.** $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

EXERCISE 10.2.2

1. (a) $\sin\alpha\cos\phi + \cos\alpha\sin\phi$ (b) $\cos3\alpha\cos2\beta - \sin3\alpha\sin2\beta$ (c) $\sin2x\cos y - \cos2x\sin y$ (d) $\cos\phi\cos2\alpha + \sin\phi\sin2\alpha$ (e) $\frac{\tan2\theta - \tan\alpha}{1+\tan2\theta\tan\alpha}$ (f) $\frac{\tan\phi - \tan3\omega}{1+\tan\phi\tan3\omega}$ **2.** (a) $\sin(2\alpha-3\beta)$ (b) $\cos(2\alpha+5\beta)$ (c) $\sin(x+2y)$ (d) $\cos(x-3y)$ (e) $\tan(2\alpha-\beta)$ (f) $\tan x$ (g) $\tan\left(\frac{\pi}{4}-\phi\right)$ (h) $\sin\left(\frac{\pi}{4}+\alpha+\beta\right)$ (i) $\sin2x$ **3.** (a) $-\frac{56}{65}$ (b) $\frac{33}{65}$ (c) $-\frac{16}{63}$ **4.** (a) $\frac{16}{65}$ (b) $\frac{63}{65}$ (c) $\frac{56}{33}$

5. (a) $-\frac{5\sqrt{11}}{18}$ (b) $-\frac{7}{18}$ (c) $\frac{5\sqrt{11}}{7}$ (d) $\frac{35\sqrt{11}}{162}$ **6.** (a) $-\frac{3}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{3}{4}$ (d) $\frac{24}{7}$ **7.** (a) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ (b) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ (c) $-\frac{1+\sqrt{3}}{2\sqrt{2}}$ (d) $\sqrt{3}-2$ **8.** (a) $\frac{2ab}{a^2+b^2}$ (b) $\frac{a^2+b^2}{2ab}$ (c) $\frac{a^4-6a^2b^2+b^4}{(a^2+b^2)^2}$ (d) $\frac{2ab}{b^2-a^2}$

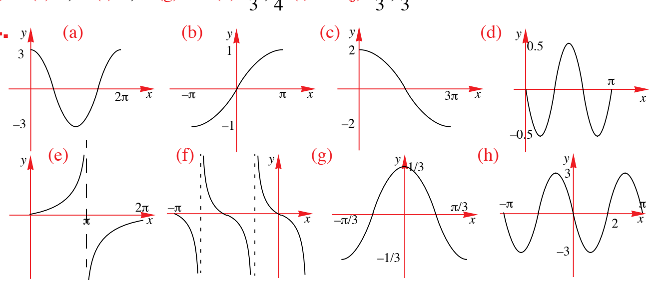
12. $\sqrt{2}-1$ **14.**(a) $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ (b) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ (c) $0, \pi, 2\pi, \alpha, \pi\pm\alpha, 2\pi-\alpha, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

15. (a) $R = \sqrt{a^2+b^2}, \tan\alpha = \frac{b}{a}$ (b) 10 **16.** (a) $R = \sqrt{a^2+b^2}, \tan\alpha = \frac{b}{a}$ (b) -11 **18.** $2-\sqrt{3}$

EXERCISE 10.3

1. (a) 4π (b) $\frac{2\pi}{3}$ (c) 3π (d) 4π (e) 2 (f) $\frac{\pi}{2}$ **2.** (a) 5 (b) 3 (c) 5 (d) 0.5 **3.** (a) 2π, 2 (b) 6π, 3 (c) π (d) π (e) π, 4 (f) π, 3 (g) 6π (h) $\frac{2\pi}{3}, \frac{1}{4}$ (i) 3π (j) $\frac{8\pi}{3}, \frac{2}{3}$

4.



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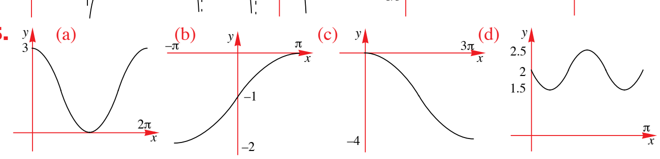

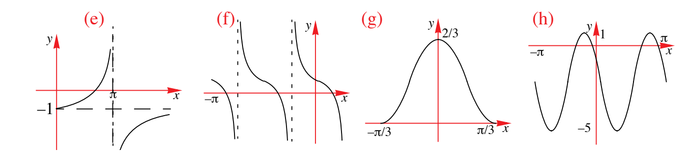

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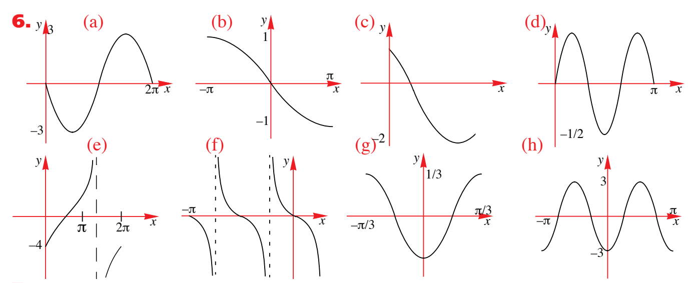

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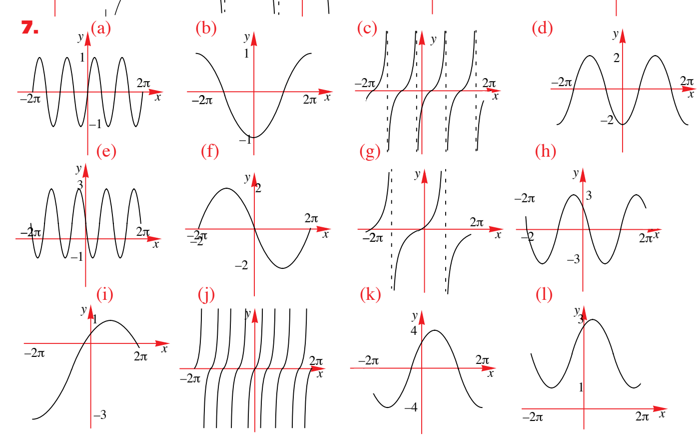
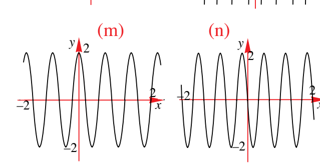
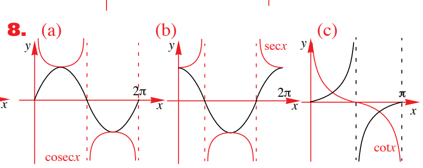




8.



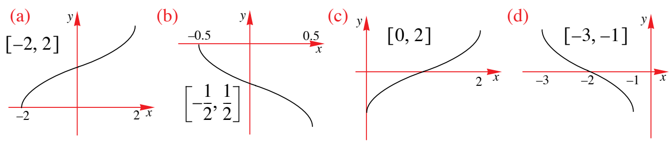
EXERCISE 10.4

1. (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{3}$ (e) $\frac{\pi}{4}$ (f) $-\frac{\pi}{3}$ (g) 1.1071^c (h) -0.7754^c (i) 0.0997^c (j) 1.2661^c (k) -0.6435^c (l) 1.3734^c (m) undefined (n) -1.5375^c (o) 1.0654^c **2.** (a) -1 (b) $\frac{\sqrt{3}}{4}$ (c) $-\frac{1}{3\sqrt{2}}$

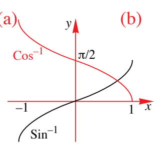
4. $\frac{1}{3}, \frac{1}{2}$ **5.** (a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$ (e) $\frac{3\sqrt{2}}{4}$ (f) -1 **6.** (a) 1 (b) $-\frac{7}{25}$ (c) $\frac{63}{65}$ (d) undefined

(e) $\frac{4\sqrt{5}}{9}$ (f) $\frac{3}{5}$ (g) $\frac{4}{3}$ (h) $\frac{1}{2}$ **9.** (a) $\frac{\sqrt{1-k^2}}{k}$ (b) $\frac{1}{\sqrt{1+k^2}}$

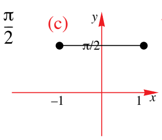
10.



12. (a)



(b) i. $\frac{\pi}{2}$ ii. $\frac{\pi}{2}$ (c)



13. $\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{n+1}\right)$

EXERCISE 10.5

1. (a) $\frac{\pi}{4}, \frac{3\pi}{4}$ (b) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (c) $\frac{\pi}{3}, \frac{2\pi}{3}$ (d) $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$ (e) $\frac{\pi}{3}, \frac{5\pi}{3}$

(f) $\frac{5}{4}, \frac{7}{4}, \frac{13}{4}, \frac{15}{4}, \frac{21}{4}, \frac{23}{4}$ **2.** (a) $\frac{\pi}{4}, \frac{7\pi}{4}$ (b) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (c) $\frac{\pi}{6}, \frac{11\pi}{6}$ (d) π (e) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ (f) $\frac{3}{2}, \frac{5}{2}, \frac{11}{2}$

3. (a) $\frac{\pi}{6}, \frac{7\pi}{6}$ (b) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (c) $\frac{\pi}{3}, \frac{4\pi}{3}$ (d) $4\tan^{-1}2$ (e) $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$ (f) 3 **4.** (a) 90°, 330°

(b) 180°,240° (c) 90°,270° (d) 65°,335° (e) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ (f) $0, \pi, 2\pi$ (g) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(h) $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$ **5.** (a) 60°,300° (b) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (c) $\frac{\pi}{6}, \frac{7\pi}{6}$ (d) 23°35',156°25' (e) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(f) $\frac{2\pi}{3}, \frac{5\pi}{3}$ (g) $\frac{5\pi}{6}, \frac{9\pi}{6}$ (h) $3.3559^c, 5.2105^c$ (i) $\frac{\pi}{3}, \frac{4\pi}{3}$ (j) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (k) $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

(l) 68°12',248°12' (m) $\frac{\pi}{3}, \frac{5\pi}{3}$ (n) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (o) Ø **6.** (a) $-\frac{3\pi}{4}, \frac{\pi}{4}$ (b) $\pm\frac{\pi}{3}$ (c) $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

(d) $-\frac{\pi}{2}$ (e) $\pm\frac{\pi}{2}$ (f) $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$ (g) $\frac{\pi}{2}, \frac{3\pi}{2}$ (h) $\frac{\pi}{2}, \frac{3\pi}{2}$ **7.** (a) $\frac{3\pi}{4}, \frac{7\pi}{4}, \tan^{-1}\left(\frac{2}{3}\right), \pi + \tan^{-1}\left(\frac{2}{3}\right)$

(b) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$ **8.** (a) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$ (b) $\frac{2\pi}{3}, \frac{4\pi}{3}$

(c) 0,1,2,3,4,5,6 **9.** (a) $\frac{\pi}{3}, \frac{5\pi}{3}, \pi \pm \cos^{-1}\left(\frac{1}{4}\right)$ (b) $\frac{3\pi}{4}, \frac{7\pi}{4}, \tan^{-1}(3), \pi + \tan^{-1}(3)$

(c) $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$ (d) $\tan^{-1}\left(\frac{3}{2}\right), \pi - \tan^{-1}(2), \pi + \tan^{-1}\left(\frac{3}{2}\right), 2\pi - \tan^{-1}(2)$ **10.** (a) $2\sin\left(x + \frac{\pi}{6}\right)$

(b) $0, \frac{2\pi}{3}, 2\pi$ **11.** (a) $2\sin\left(x - \frac{\pi}{3}\right)$ (b) $\frac{\pi}{6}, \frac{3\pi}{2}$ **12.** $\frac{\pi}{3}, \frac{2\pi}{3}$ **13.** i. $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \cup \left(\frac{13\pi}{6}, \frac{17\pi}{6}\right)$

ii. $\left(\pi + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), 2\pi - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) \cup \left(3\pi + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), 4\pi - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

14. (a) ii. $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ (b) ii. $\left[0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

16. (a) i. $\{x \mid x = k\pi + \alpha(-1)^k, k \in \mathbb{Z}\}$ ii. $\{x \mid 2k\pi + \alpha \le x \le (2k+1)\pi - \alpha, k \in \mathbb{Z}\}$

(b) $\left\{x \mid x = (2k+1)\frac{\pi}{5}\right\} \cup \{x \mid x = 2k\pi\}, k \in \mathbb{Z}$

(c) $\left\{x \mid x = \frac{2k\pi}{5} + \frac{\pi}{10}\right\} \cup \left\{x \mid x = 2k\pi - \frac{\pi}{2}\right\}, k \in \mathbb{Z}$ **17.** (a) $0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$ (b) $\sqrt{2}, \frac{\sqrt{2}}{2}$

18. (c) $2\cos\frac{\pi}{9}, 2\cos\frac{5\pi}{9}, 2\cos\frac{7\pi}{9}$ **19.** $\left\{\pm\frac{\pi}{4}, \pm\frac{2\pi}{3}, \pm\frac{3\pi}{4}\right\}$ **21.** (a) 90°,199°28',340°32'

(b) (199°28',340°32') **24.** $\left\{(x, y) \mid x = 2k\pi + \frac{\pi}{2}, y = 2k\pi\right\} \cup \left\{(x, y) \mid x = 2k\pi - \frac{\pi}{2}, y = 2k\pi + \pi\right\}, k \in \mathbb{Z}$

EXERCISE 10.6

1. (a) 5, 24, 11, 19 (b) $T = 5\sin\left(\frac{\pi t}{12} - 3\right) + 19$ (c) 23.6° **2.** (a) 3. 4.2, 2, 7

(b) $L = 3\sin\left(\frac{\pi t}{2.1} - 3\right) + 7$ **3.** (a) 5, 11, 0, 7 (b) $V = 5\sin\left(\frac{2\pi t}{11}\right) + 7$ **4.** (a) 1, 11, 1, 12

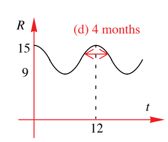
(b) $P = \sin\frac{2\pi}{11}(t-1) + 12$ **5.** (a) 2.6, 7, 2, 6 (b) $S = 2.6\sin\frac{2\pi}{7}(t-2) + 6$ **6.** (a) 0.6, 3.5, 0, 11

(b) $P = 0.6\sin\left(\frac{4\pi t}{7}\right) + 11$ **7.** (a) 0.8, 4.6, 2.7, 11 (b) $D = 0.8\sin\frac{\pi}{2.3}(t - 2.7) + 11$ **8.** (a) 3000

(b) 1000, 5000 (c) $\frac{4}{9}$ **9.** (a) 6.5 m, 7.5 m (b) 1.58 sec, 3.42 sec **10.** (a) 750, 1850 (b) 3.44

(c) Mid-April to End of August

11. (a) 15000 (b) 12 months (c)



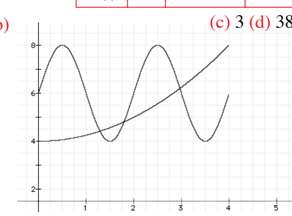
(d) 4 months

12. (a) $\pi, -2, 2$ (b) $\frac{1}{3}$ m (c) $\frac{4}{3}$ m

13. (a)

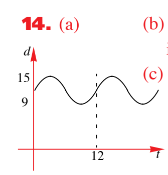
t	0	0.5	1	1.5	2	2.5	3	3.5	4
$F(t)$	6	8	6	4	6	8	6	4	6
$G(t)$	4	4.0625	4.25	4.5625	5	5.5625	6.25	7.0625	8

(b)



(c) 3 (d) 38.45%

14. (a)



(b) i. 7,11,19,23
ii. $[0, 7] \cup [11, 19] \cup [23, 24]$
(c) 14.9m