2.4 QUADRATICS

2.4.1 QUADRATIC EQUATION

A **quadratic equation** in the **variable** x (say) takes on the form $ax^2 + bx + c = 0$ where a, b and c are real constants. The equation is a quadratic because x is raised to the power of two. The solution(s) to such equations can be obtained in one of two ways.

Method 1: Factorise the quadratic and use the Null Factor Law.

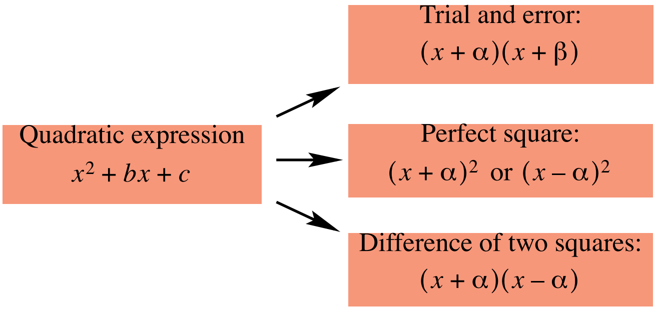
Method 2: Use the quadratic formula.

We look at each of these methods.

Method 1 Factorisation and the Null Factor Law

First of all we must have one side of the equation as 0, otherwise the Null Factor Law cannot be used. Next, when factorising the quadratic, you will need to rely on your ability to recognise the form of the quadratic and hence which approach to use. A summary of the factorisation process for quadratics is shown below:

Case 1: $a = 1$



Example

$x^2 + 12x + 32 = (x + 4)(x + 8)$

$x^2 + 12x - 28 = (x + 14)(x - 2)$

$x^2 + 6x + 9 = (x + 3)^2$

$x^2 - 4x + 4 = (x - 2)^2$

$x^2 - 16 = (x + 4)(x - 4)$

$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$

Note that sometimes you might need to use a perfect square approach to part of the quadratic and then complete the factorisation process by using the difference of two squares.

EXAMPLE 2.23

Solve the quadratic $x^2 + 6x + 7 = 0$

Solution

In this instance it is not obvious what the factors are and so trial and error is not appropriate. However, we notice that $x^2 + 6x + 7$ can be 'broken up' into $x^2 + 6x + 9 - 2$.
That is, part of the quadratic has been expressed as a perfect square, so that

$$x^2 + 6x + 9 - 2 = (x + 3)^2 - 2$$

Then, we are left with a difference of perfect squares: $(x + 3)^2 - 2 = (x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$.
Therefore, $x^2 + 6x + 7 = 0 \Leftrightarrow (x + 3 + \sqrt{2})(x + 3 - \sqrt{2}) = 0 \Leftrightarrow x = -3 \pm \sqrt{2}$.

Case 2: $a \neq 1$

Quadratic expression $ax^2 + bx + c$		Example
→	Trial and error:	$2x^2 + 3x + 1 = (2x + 1)(x + 1)$
→	Perfect square:	$4x^2 + 12x + 9 = (2x + 3)^2$
→	Difference of two squares:	$9x^2 - 25 = (3x + 5)(3x - 5)$

That is, the methods for the case $a \neq 1$ are the same as for when $a = 1$, they require a little more mental arithmetic to 'juggle' the correct numbers. Sometimes they can get messy, as the next example shows.

EXAMPLE 2.24

Solve the quadratic $3x^2 + 2x - 2 = 0$.

Solution

None of the above methods provides a quick solution, so, as for Example 2.23, we will combine the methods of perfect squares with the difference of two squares:

$$3x^2 + 2x - 2 = 3\left(x^2 + \frac{2}{3}x\right) - 2 = 3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - 3 \times \frac{1}{9} - 2 \quad \text{[add **and** subtract } \tfrac{3}{9}\text{]}$$

$$= 3\left(x + \frac{1}{3}\right)^2 - \frac{7}{3}$$

Part of the quadratic is now a perfect square. Next, we use the difference of perfect squares:

$$= 3\left(x + \frac{1}{3}\right)^2 - \frac{7}{3}$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 - \frac{7}{9}\right]$$

$$= 3\left(x + \frac{1}{3} - \frac{\sqrt{7}}{3}\right)\left(x + \frac{1}{3} + \frac{\sqrt{7}}{3}\right)$$

Therefore, $3x^2 + 2x - 2 = 0 \Leftrightarrow 3\left(x + \frac{1}{3} - \frac{\sqrt{7}}{3}\right)\left(x + \frac{1}{3} + \frac{\sqrt{7}}{3}\right) = 0 \Leftrightarrow x = -\frac{1}{3} \pm \frac{\sqrt{7}}{3}$.

Now, that was quite a bit of work!

Of course, coming up with the 'magic' number, $\frac{1}{9}$ did make life a little easier. The rest was simply being careful with the arithmetic. So, how did we pull $\frac{1}{9}$ out of the hat? Well, once we have made the coefficient of the x^2 one, i.e., by factorising the '3' out, we look at the coefficient of the x term. Then we halve it, square the result and add it. i.e., $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \rightarrow \left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

Then, so that the equation is unaltered, we subtract this result – but be careful, do not forget to multiply it by the factor '3' at the front of the brackets. The rest then follows.

The only way to be proficient with these methods is practice, practice and more practice. However, there is a short cut to solving quadratic equations. We look at this next.

Method 2 Quadratic Formula and the Discriminant

A formula that allows us to solve any quadratic equation $ax^2 + bx + c = 0$ (if real solutions exist), is given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. So that obtaining solutions requires that we make the appropriate substitution for a, b and c.

To derive this expression we proceed in exactly the same way as we did in Example 2.24. We show some of the key steps in obtaining this result and leave the proof for you to complete.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right)\right] - a \times \frac{b^2}{4a^2} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]$$

$$= a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)$$

Therefore, $ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE 2.25

Use the formula to solve the quadratic equations:

(a) $x^2 - x - 4 = 0$ (b) $2x^2 = 4 - x$

Solution

(a) For the equation $x^2 - x - 4 = 0$, we have $a = 1, b = -1$ & $c = -4$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times -4}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{17}}{2}$$

(b) Similarly, if $2x^2 = 4 - x$, then,

$2x^2 + x - 4 = 0$

so that $a = 2, b = 1$ and $c = -4$, so that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{(-1)^2 - 4 \times 2 \times -4}}{2 \times 2}$$

$$= \frac{-1 \pm \sqrt{33}}{4}$$

The Discriminant

Closer inspection of this formula indicates that much can be deduced from the term under the square root sign, i.e., $b^2 - 4ac$. The expression $b^2 - 4ac$ is known as the **discriminant** and is often represented by the delta symbol $\Delta = b^2 - 4ac$.

In particular, there are three cases to address:

Case 1. $b^2 - 4ac > 0$

Case 2. $b^2 - 4ac = 0$

Case 3. $b^2 - 4ac < 0$

Case 1. $b^2 - 4ac > 0$

In this case, the expression $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ produces **two real solutions**.

This is because taking the square root of a positive number will produce another positive real number. This in turn implies that there will be one solution corresponding to the '+' term and one solution corresponding to the '–' term.

That is, say that $\sqrt{b^2 - 4ac} = K$, where K is a real number. We then have that $x = \frac{-b \pm K}{2a}$, i.e., $x_1 = \frac{-b + K}{2a}$, $x_2 = \frac{-b - K}{2a}$, giving **two distinct real solutions**.

Case 2. $b^2 - 4ac = 0$

In this case, the expression $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ produces only **one real solution**.

This is because taking the square root of zero gives zero. This in turn implies that there will be only one solution because adding and subtracting '0' to the '$-b$' term in the numerator will not alter the answer.

That is, if $\sqrt{b^2 - 4ac} = 0$, we then have that $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}$ meaning that we have only **one real solution** (or two repeated solutions).

Case 3. $b^2 - 4ac < 0$

In this case, the expression $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ produces **no real solution**.

This is because the square root of a negative number will not produce a **real** number. This in turn implies that the formula cannot be utilised (if we are dealing with quadratic equations under the real numbers).

Summary

Discriminant $\Delta = b^2 - 4ac$	**Number of solutions for** $ax^2 + bx + c = 0$
$\Delta > 0$	Can be factorised to obtain **2 real** and unique solutions.
$\Delta = 0$	Can be factorised to obtain **1 real** (repeated) solution.
$\Delta < 0$	Cannot be factorised and so **no real** solutions exist.

EXAMPLE 2.26

(a) Find the value(s) of m for which the equation $2x^2 + mx + 1 = 0$ has one real solution.

(b) Find the value(s) of k for which the equation $x^2 + 4x + k = 0$ has two real solutions.

Solution

(a) For one real solution to exist, we must have that $\Delta = b^2 - 4ac = 0$.
For this quadratic we have that $a = 2, b = m$ and $c = 1$.
Therefore, we need that $m^2 - 4 \times 2 \times 1 = 0$

$$\Leftrightarrow m^2 - 8 = 0$$

$$\Leftrightarrow (m - \sqrt{8})(m + \sqrt{8}) = 0$$

$$\therefore m = 2\sqrt{2} \text{ or } m = -2\sqrt{2}$$

(b) For two real solutions, we must have that $\Delta = b^2 - 4ac > 0$.
For this quadratic we have that $a = 1, b = 4$ and $c = k$.
Therefore, we need that $4^2 - 4 \times 1 \times k > 0$

$$\Leftrightarrow 16 - 4k > 0$$

$$\Leftrightarrow 16 > 4k$$

$$\Leftrightarrow 4 > k.$$

i.e., the quadratic $x^2 + 4x + k = 0$ will have two real solutions as long as $k < 4$.

EXAMPLE 2.27

Find k if the equation $x^2 - (k+3)x + (k+6) = 0$ has

(a) 1 root (b) 2 real roots (c) no real root.

Solution

First we find an expression for the discriminant in terms of k:
Using the values $a = 1, b = k + 3$ and $c = k + 6$, we have:

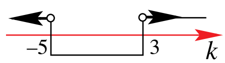
$$\Delta = b^2 - 4ac = (k+3)^2 - 4 \times 1 \times (k+6)$$

$$= k^2 + 2k - 15$$

$$= (k+5)(k-3)$$

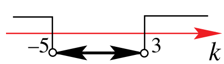
(a) For the equation to have 1 solution, the discriminant, $\Delta = 0$, thus,
$(k+5)(k-3) = 0 \Leftrightarrow k = -5$, or $k = 3$
That is, the solution set is $\{k : k = -5, 3\}$.

(b) For the equation to have 2 solutions, the discriminant, $\Delta > 0$, thus,
$(k+5)(k-3) > 0 \Leftrightarrow k < -5$ or $k > 3$ Using a sign diagram for k:



That is, the solution set is $\{k : k < -5\} \cup \{k : k > 3\}$.

(c) For the equation to have no real solutions, the discriminant, $\Delta < 0$, thus,
$(k+5)(k-3) < 0 \Leftrightarrow -5 < k < 3$ Using a sign diagam for k:



That is, the solution set is $\{k : -5 < k < 3\}$.

EXERCISES 2.4.1

1. By using a factorisation process, solve for the given variable.

(a) $x^2 + 10x + 25 = 0$ (b) $x^2 - 10x + 24 = 0$

(c) $3x^2 + 9x = 0$ (d) $x^2 - 4x + 3 = 0$

(e) $(3 - u)(u + 6) = 0$ (f) $3x^2 + x - 10 = 0$

(g) $3v^2 - 12v + 12 = 0$ (h) $y(y - 3) = 18$

(i) $(x + 3)(x + 2) = 12$ (j) $(2a - 1)(a - 1) = 1$

2. Without using the quadratic formula, solve for the given variable.

(a) $u + \frac{1}{u} = -2$ (b) $x + 2 = \frac{35}{x}$ (c) $5x - 13 = \frac{6}{x}$

(d) $\frac{x}{2} - \frac{1}{x+1} = 0$ (e) $y + 1 = \frac{4}{y+1}$ (f) $v + \frac{20}{v} = 9$

3. By completing the square, solve for the given variable.

(a) $x^2 + 2x = 5$ (b) $x^2 + 4 = 6x$ (c) $x^2 - 2x = 4$

(d) $4x^2 + x = 2$ (e) $2y^2 = 9y - 1$ (f) $3a^2 - a = 7$

4. Use the quadratic formula to solve these equations.

(a) $x^2 - 3x - 7 = 0$ (b) $x^2 - 5x = 2$

(c) $x^2 - 3x - 6 = 0$ (d) $x^2 = 7x + 2$

(e) $x(x + 7) = 4$ (f) $x^2 + 2x - 8 = 0$

(g) $x^2 + 2x - 7 = 0$ (h) $x^2 + 5x - 7 = 0$

(i) $x^2 - 3x - 7 = 0$ (j) $x^2 - 3x + 9 = 0$

(k) $x^2 + 9 = 8x$ (l) $4x^2 - 8x + 9 = 0$

(m) $4x^2 = 8x + 9$ (n) $5x^2 - 6x - 7 = 0$

(o) $5x^2 - 12x + 1 = 0$ (p) $7x^2 - 12x + 1 = 0$

5. For what value(s) of p does the equation $x^2 + px + 1 = 0$ have

(a) no real solutions

(b) one real solution

(c) two real solutions.

6. Find the values of m for which the quadratic $x^2 + 2x + m = 0$ has

(a) one real solution

(b) two real solutions

(c) no real solutions.

7. Find the values of m for which the quadratic $x^2 + mx + 2 = 0$ has

(a) one real solution

(b) two real solutions

(c) no real solutions.

8. Find the values of k for which the quadratic $2x^2 + kx + 9 = 0$ has
(a) one real solution
(b) two real solutions
(c) no real solutions.

9. Consider the equation $x^2 + 2x = 7$. Prove that this equation has two real roots.

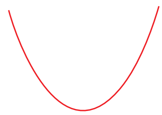
10. Find the value(s) of p such that the equation $px^2 - px + 1 = 0$ has exactly one real root.

11. Prove that the equation $kx^2 + 3x = k$ has two real solutions for all non-zero real values of k.

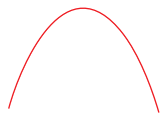
2.4.2 QUADRATIC FUNCTION

A **quadratic function** has the general form $f(x) = ax^2 + bxc, a \neq 0$ and $a, b, c \in \mathbb{R}$.
All quadratic functions have **parabolic graphs** and have a **vertical axis of symmetry**.

If $a > 0$, the parabola is concave up:



If $a < 0$ the parabola is concave down:



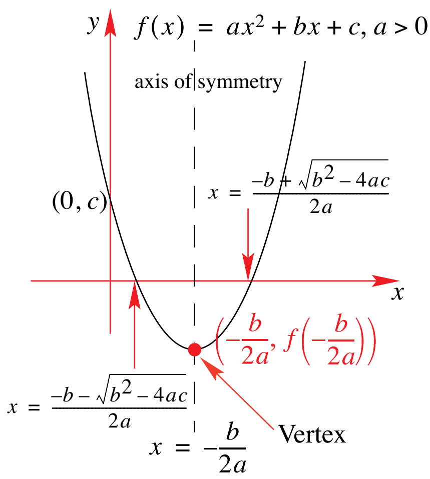
General Properties of the graph of $f(x) = ax^2 + bx + c, a \neq 0$

1. y–intercept

This occurs when $x = 0$, so that $y = f(0) = a(0)^2 + b(0) + c = c$.
That is, the curve passes through the point $(0, c)$

2. x–intercept(s)

This occurs where $f(x) = 0$.
Therefore we need to solve $ax^2 + bx + c = 0$.
To solve we either factorise and solve or use the quadratic formula, which would provide the solution(s) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.



3. Axis of symmetry

This occurs at $x = -\frac{b}{2a}$.

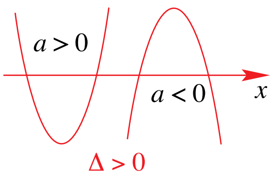
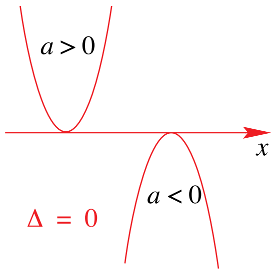
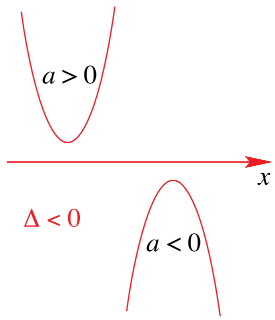
4. Vertex (turning point)

The vertex occurs when $x = -\frac{b}{2a}$. Then, to find the y–value, find $f\left(-\frac{b}{2a}\right)$

Geometrical Interpretation and The Discriminant

Just as it was the case for linear functions, solving the quadratic $ax^2 + bx + c = k$ is geometrically equivalent to finding where the parabola with function $f(x) = ax^2 + bx + c$ meets the graph (horizontal straight line) $y = k$. Then, when $k = 0$, we are finding where the parabola meets the line $y = 0$, i.e., we are finding the x–intercept(s). Based on our results of the discriminant about the number of solutions to the equation $ax^2 + bx + c = 0$, we can extend these results to the following:

The number of x–intercepts for the function $f(x) = ax^2 + bx + c$	**=**	The number of solutions to the equation $ax^2 + bx + c = 0$

If $\Delta = b^2 - 4ac > 0$, then there are **two x–intercepts**.	If $\Delta = b^2 - 4ac = 0$, then there is **one x–intercept**.	If $\Delta = b^2 - 4ac < 0$, then there are **no x–intercepts**.
		

Sketching the graph of a quadratic function

Two methods for sketching the graph of $f(x) = ax^2 + bx + c, a \neq 0$ are:

Method 1: The intercept method

i.e., expressing $f(x) = ax^2 + bx + c, a \neq 0$ in the form $f(x) = a(x-p)(x-q)$

This involves

Step 1 Finding the x–intercepts [by solving $ax^2 + bx + c = 0$]

Step 2 Finding the y–intercept [finding $f(0)$]

Step 3 Sketch parabola passing through the three points

Method 2: The turning-point form

i.e., expressing $f(x) = ax^2 + bx + c, a \neq 0$ in the form $f(x) = a(x-h)^2 + k$

This involves

Step 1 Expressing $f(x) = ax^2 + bx + c, a \neq 0$ in the form $a(x-h)^2 + k$ [by completing the square]

Step 2 Use the turning point (h, k)

Step 3 Finding the y–intercept [finding $f(0)$]

Step 4 Sketch the parabola passing through the two points

EXAMPLE 2.27 Sketch the graph of

(a) $f(x) = x^2 - 6x + 8$ (b) $f(x) = 3x^2 + 12x + 4$ using

i. the intercept method ii. the turning point method

Solution

(a) i. For the intercept method we start by factorising the quadratic:

$f(x) = x^2 - 6x + 8 = (x-4)(x-2)$.

Then, for the x–intercepts we have:

$$f(x) = 0 \Leftrightarrow (x-4)(x-2) = 0 \Leftrightarrow x = 2 \text{ or } x = 4$$

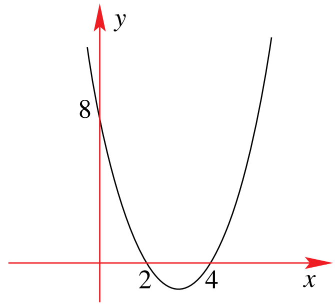
Next, the y–intercept:

$$x = 0 \Rightarrow f(0) = 0^2 - 6(0) + 8 = 8.$$

Summary: Parabola passes through $(2, 0)$, $(4, 0)$ and $(0, 8)$.

Note, the axis of symmetry is the midpoint of the x–intercepts,

i.e., $x = \frac{2+4}{2} = \frac{6}{2} = 3$. Or use $x = -\frac{b}{2a} = -\frac{-6}{2} = 3$



ii. We first complete the square:

$f(x) = x^2 - 6x + 8 = (x^2 - 6x + 9) - 1 = (x-3)^2 - 1$

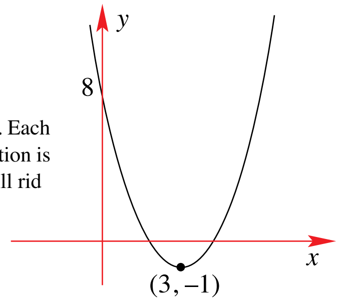
Therefore, the parabola has a turning point at $(3, -1)$

Then find the y–intercept:

$$x = 0 \Rightarrow f(0) = 0^2 - 6(0) + 8 = 8$$

Summary: Parabola passes through $(3, -1)$ and $(0, 8)$.

Notice that both graphs are the same but display different information. Each display has its strengths and weaknesses – it depends on what information is important for the question. Of course, including all the information will rid us of any problems.



(b) i. For the intercept method we start by factorising the quadratic:

$f(x) = 3x^2 + 12x + 4 = 3(x^2 + 4x + 4) - 12 + 4$ [completing the square]

$= 3(x+2)^2 - 8$ [next use difference of two squares]

$= [\sqrt{3}(x+2) - 2\sqrt{2}][\sqrt{3}(x+2) + 2\sqrt{2}]$.

Then, for the x–intercepts we have:

$$f(x) = 0 \Rightarrow [\sqrt{3}(x+2) - 2\sqrt{2}][\sqrt{3}(x+2) + 2\sqrt{2}] = 0$$

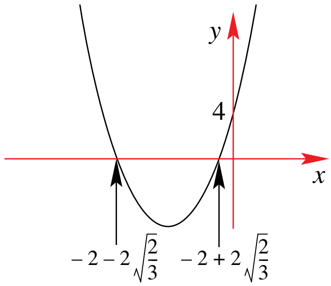
$$\therefore x = -2 + 2\sqrt{\frac{2}{3}} \text{ or } x = -2 - 2\sqrt{\frac{2}{3}}$$

Note: It would have been quicker to find the intercepts if we had used the quadratic formula!

Next, the y–intercept:

$$x = 0 \Rightarrow f(0) = 3(0)^2 + 12(0) = 4 = 4.$$

Summary: Parabola passes through $\left(-2+2\sqrt{\frac{2}{3}}, 0\right)$, $\left(-2-2\sqrt{\frac{2}{3}}, 0\right)$ and $(0, 4)$.



ii. We first complete the square:

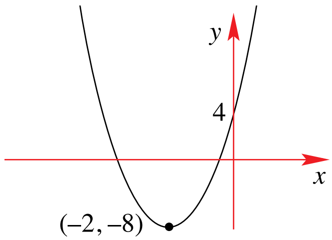
$$f(x) = 3x^2 + 12x + 4 = 3(x^2 + 4x + 4) - 12 + 4 \text{ [completing the square]}$$
$$= 3(x+2)^2 - 8$$

Therefore, the parabola has a turning point at (–2, –8)

Then find the y–intercept:

$$x = 0 \Rightarrow f(0) = 3(0)^2 + 12(0) = 4 = 4.$$

Summary: Parabola passes through (–2, –8) and (0, 4).



EXAMPLE 2.28

For what value(s) of k will the function $f(x) = x^2 + 6x + k$

(a) cut the x–axis twice (b) touch the x–axis (c) have no x–intercepts

Solution

We will need to use the relationship between the number of x–intercepts of the function $f(x) = x^2 + 6x + k$ and the number of solutions to the equation $x^2 + 6x + k = 0$. To do this we first find the discriminant: $\Delta = b^2 - 4ac = (6)^2 - 4 \times 1 \times k = 36 - 4k$.

(a) If the function cuts twice, then $\Delta > 0 \Rightarrow 36 - 4k > 0 \Leftrightarrow k < 9$.

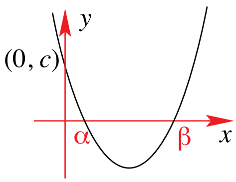
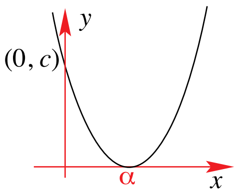
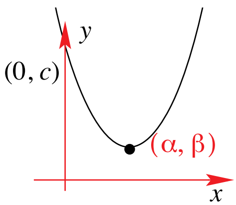
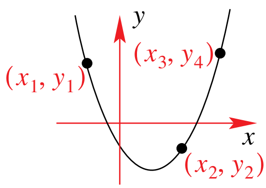
(b) If the function touches the x–axis, then we have repeated solutions (or roots), in this case we have only one solution, so $\Delta = 0 \Rightarrow 36 - 4k = 0 \Leftrightarrow k = 9$.

(c) If the function has no x–intercepts, then $\Delta < 0 \Rightarrow 36 - 4k < 0 \Leftrightarrow k > 9$.

So far we have sketched a graph from a given function, but what about finding the equation of a given graph?

Finding the equation from a graph

If sufficient information is provided on a graph, then it is possible to obtain the equation that corresponds to that graph. When dealing with quadratics there are some standard approaches that can be used (depending on the information provided).

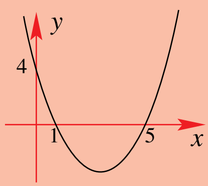
Information provided	Process
Graph cuts the *x*–axis at two points: 	Use the function $f(x) = k(x-\alpha)(x-\beta)$ and then use the point $(0, c)$ to solve for k.
Graph touches the *x*–axis at $x = \alpha$: 	Use the function $f(x) = k(x-\alpha)^2$ and then use the point $(0, c)$ to solve for k.
Graph does not meet the *x*–axis: 	Use the function $f(x) = k(x-\alpha)^2 + \beta$ and then use the point $(0, c)$ to solve for k.
Three arbitrary points are given: 	Use the function $f(x) = ax^2 + bx + c$ and then set up and solve the system of simultaneous equations by substituting each coordinate into the function: $ax_1^2 + bx_1 + c = y_1$ $ax_2^2 + bx_2 + c = y_2$ $ax_3^2 + bx_3 + c = y_3$

Note: the process is identical for a downward concave parabola.

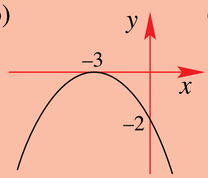
EXAMPLE 2.29

Find the equation defining each of the following graphs.

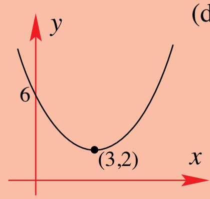
(a)



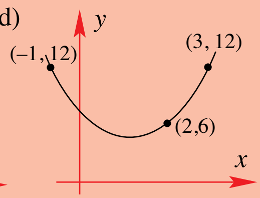
(b)



(c)



(d)



Solution

(a) Using the form $f(x) = k(x-\alpha)(x-\beta)$ we have $f(x) = k(x-1)(x-5)$.

Next, when $x = 0$, $y = 4$, therefore, $4 = k(0-1)(0-5) \Leftrightarrow 4 = 5k \Leftrightarrow k = \frac{4}{5}$.

Therefore, $f(x) = \frac{4}{5}(x-1)(x-5)$.

(b) Graph touches x–axis at $x = -3$, therefore use $f(x) = k(x+3)^2$.

As graph passes through $(0, -2)$, we have $-2 = k(0+3)^2 \Leftrightarrow -2 = 9k \Leftrightarrow k = -\frac{2}{9}$.

Therefore, $f(x) = -\frac{2}{9}(x+3)^2$.

(c) Graph shows turning point and another point, so use the form $f(x) = k(x-\alpha)^2 + \beta$.

So we have, $f(x) = k(x-3)^2 + 2$.

Then, as graph passes through $(0, 6)$, we have $6 = k(0-3)^2 + 2 \Leftrightarrow 4 = 9k \Leftrightarrow k = \frac{4}{9}$.

Therefore, $f(x) = \frac{4}{9}(x-3)^2 + 2$.

(d) As we are given three arbitrary points, we use the general equation $f(x) = ax^2 + bx + c$.

From $(-1, 12)$ we have $12 = a(-1)^2 + b(-1) + c$ i.e., $12 = a - b + c$ – (1)

From $(2, 6)$ we have $6 = a(2)^2 + b(2) + c$ i.e., $6 = 4a + 2b + c$ – (2)

From $(3, 12)$ we have $12 = a(3)^2 + b(3) + c$ i.e., $12 = 9a + 3b + c$ – (3)

Solving for a, b and c we have: (2) – (1): $-6 = 3a + 3b$

i.e., $-2 = a + b$ – (3)

(3) – (2): $6 = 5a + b$ – (4)

(4) – (3): $8 = 4a \Leftrightarrow a = 2$

Substitute $a = 2$ into (3): $-2 = 2 + b \Leftrightarrow b = -4$.

Substitute results into (1): $12 = 2 - (-4) + c \Leftrightarrow c = 6$.

Therefore function is $f(x) = 2x^2 - 4x + 6$

EXERCISES 2.4.2

1. Express the following functions in turning point form and hence sketch their graphs.

(a) $y = x^2 - 2x + 1$ (b) $y = x^2 + 4x + 2$ (c) $y = x^2 - 4x + 2$

(d) $y = x^2 + x - 1$ (e) $y = x^2 - x - 2$ (f) $y = x^2 + 3x + 1$

(g) $y = -x^2 + 2x + 1$ (h) $y = -x^2 - 2x + 2$ (i) $y = 2x^2 - 2x - 1$

(j) $y = -\frac{1}{2}x^2 + 3x - 2$ (k) $y = -\frac{x^2}{3} + x - 2$ (l) $y = 3x^2 - 2x + 1$

2. Find the axial intercepts of these quadratic functions (correct to 2 decimal places) and hence sketch their graphs.

(a)	$y = x^2 + 3x + 2$	(b)	$y = x^2 - x - 6$	(c)	$y = 2x^2 - 5x - 3$
(d)	$y = x^2 - 4$	(e)	$y = x^2 + x - 5$	(f)	$y = -x^2 + x + 6$
(g)	$y = -x^2 + x + 1$	(h)	$y = -2x^2 - 3x + 5$	(i)	$y = 2x^2 + 5x - 3$
(j)	$y = \frac{x^2}{3} - 2x + 3$	(k)	$y = -\frac{x^2}{2} + x + 4$	(l)	$y = 3x^2 - 2x - 4$

3. For the quadratic function $f(x) = 2x^2 - 4x + 1$, find
(a) the equation of the axis of symmetry.
(b) the coordinates of the vertex.
(c) the i. x–intercept(s) i i. y–intercept.
Hence, sketch the graph of the function.

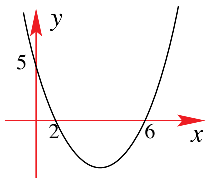
4. For the quadratic function $f(x) = 7 + 4x - 2x^2$, find
(a) the equation of the axis of symmetry.
(b) the coordinates of the vertex.
(c) the i. x–intercept(s) ii. y–intercept.
Hence, sketch the graph of the function.

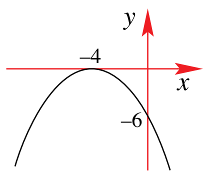
5. For what value(s) of k will the graph of $y = x^2 - 3x + k$
(a) touch the x–axis (b) cut the x–axis (c) never meet the x–axis?

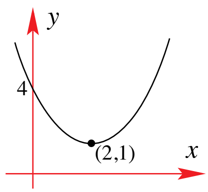
6. For what value(s) of k will the graph of $y = kx^2 + 5x + 2$
(a) touch the x–axis (b) cut the x–axis (c) never meet the x–axis?

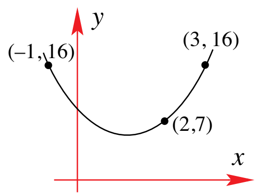
7. For what value(s) of k will the graph of $y = kx^2 - 2x + k$
(a) touch the x–axis (b) cut the x–axis (c) never meet the x–axis?

8. Find the equation of the quadratic function with graph

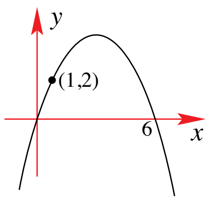
(a)


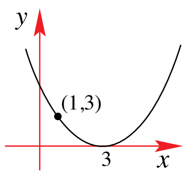
(b)


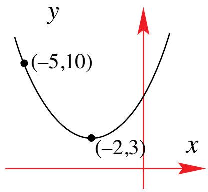
(c)


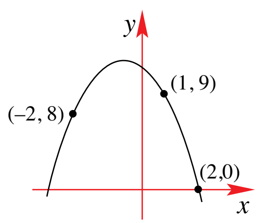
(d)


9. Find the equation of the quadratic function with graph

(a)


(b)


(c)


(d)


2.4.3 QUADRATIC INEQUALITIES

Quadratic inequations arise from replacing the '=' sign in a quadratic by an inequality sign. Solving inequations can be carried out in two ways, either algebraically or graphically.

Method 1 Algebraic Method

This method relies on factorising the quadratic and then using the fact that when two terms, a and b are multiplied, the following rules apply:

1. $ab > 0 \Leftrightarrow a > 0$ and $b > 0$ or $a < 0$ and $b < 0$
2. $ab < 0 \Leftrightarrow a > 0$ and $b < 0$ or $a < 0$ and $b > 0$

The same rules apply if we replace ' > ' with ' ≥ ' and ' < ' with ' ≤ '

EXAMPLE 2.30

Find

(a) $\{x|\ x^2 - 6x + 8 > 0\}$ (b) $\{x|\ 2x^2 + 5x - 3 \le 0\}$ (c) $\{x|\ x^2 - 3 < 2x\}$

Solution

(a) We start by factorising the quadratic: $x^2 - 6x + 8 = (x-2)(x-4)$

Then, $x^2 - 6x + 8 > 0 \Leftrightarrow (x-2)(x-4) > 0$

Which means either $x - 2 > 0$ **and** $x - 4 > 0$ i.e., $x > 2$ **and** $x > 4 \Rightarrow x > 4$ – (1)

or $x - 2 < 0$ **and** $x - 4 < 0$ i.e., $x < 2$ **and** $x < 4 \Rightarrow x < 2$ – (2)

Then, combining (1) and (2) we have $\{x|\ x^2 - 6x + 8 > 0\} = \{x|\ x < 2\} \cup \{x|\ x > 4\}$.

(b) Now, $2x^2 + 5x - 3 \le 0 \Leftrightarrow (2x-1)(x+3) \le 0$.

Meaning that either $2x - 1 \le 0$ **and** $x + 3 \ge 0$ i.e., $x \le \frac{1}{2}$ **and** $x \ge -3$ – (1)

or $2x - 1 \ge 0$ **and** $x + 3 \le 0$ i.e., $x \ge \frac{1}{2}$ **and** $x \le -3$ – (2)

From result (1) we have that $-3 \le x \le \frac{1}{2}$.

However, the inequalities in result (2) are inconsistent, i.e., we cannot have that x is both greater than or equal to $\frac{1}{2}$ **and** less than or equal to –3 simultaneously. Therefore we discard this inequality. Therefore, $\{x|\ 2x^2 + 5x - 3 \le 0\} = \left\{x|\ -3 \le x \le \frac{1}{2}\right\}$

(c) This time we need some rearranging:

$$x^2 - 3 < 2x \Leftrightarrow x^2 - 2x - 3 < 0 \Leftrightarrow (x+1)(x-3) < 0$$

Then, we must have that $x + 1 < 0$ **and** $x + 3 > 0$ i.e., $x < -1$ **and** $x > 3$ – (1)

$x + 1 > 0$ **and** $x + 3 < 0$ i.e., $x > -1$ **and** $x < 3$ – (2)

This time (1) is inconsistent, so we discard it and from (2) we have $-1 < x < 3$.

Therefore, $\{x|\ x^2 - 3 < 2x\} = \{x|\ -1 < x < 3\}$.

Method 2 Graphical Method

This method relies on examining the graph of the corresponding quadratic function and then

1. quoting the x–values that produce y–values that lie above (or on) the x–axis (i.e., $y > 0$ or $y \geq 0$)

or 2. quoting x–values that produce y–values that lie below (or on) the x–axis (i.e., $y < 0$ or $y \leq 0$)

We consider inequations from Example 2.30

EXAMPLE 2.31

Use a graphical method to find

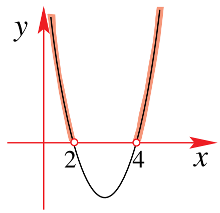
(a) $\{x | x^2 - 6x + 8 > 0\}$ (b) $\{x | 2x^2 + 5x - 3 \leq 0\}$ (c) $\{x | x^2 - 3 < 2x\}$

Solution

(a) The corresponding function in this case is $f(x) = x^2 - 6x + 8$. That part of the graph corresponding to $f(x) > 0$ is highlighted in red. The values of x that correspond to these parts are

$$x < 2 \text{ as well as } x > 4$$

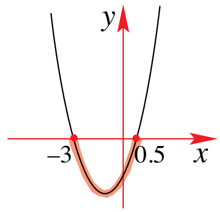
Therefore, s.s. = $\{x | x < 2\} \cup \{x | x > 4\}$



(b) The corresponding function in this case is $f(x) = 2x^2 + 5x - 3$. That part of the graph corresponding to $f(x) \leq 0$ is highlighted in red. The values of x that correspond to these parts are

$$-3 \leq x \leq \frac{1}{2}$$

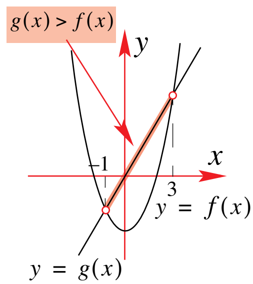
Therefore, s.s. = $\left\{x | -3 \leq x \leq \frac{1}{2}\right\}$



(c) This time we have two functions, that of $f(x) = x^2 - 3$ and $g(x) = 2x$, and we want to find those values of x where $f(x) < g(x)$.

We do this by sketching both graphs on the same set of axes and then finding those values of x for which $f(x) < g(x)$ i.e., where the graph of $y = g(x)$ lies above that of $y = f(x)$.

Once we have found the point of intersection, i.e., once we have solved $f(x) = g(x)$, we refer to the graph.



Now, $f(x) = g(x) \Leftrightarrow x^2 - 3 = 2x \Leftrightarrow x^2 - 2x - 3 = 0$

$$\Leftrightarrow (x-3)(x+1) = 0$$

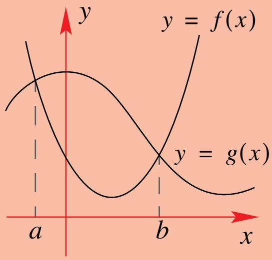
$$\Leftrightarrow x = 3 \text{ or } x = -1$$

Then, $f(x) < g(x)$ for $-1 < x < 3$.

i.e., $\{x | x^2 - 3 < 2x\} = \{x | -1 < x < 3\}$.

Part (c) in Example 2.31 leads us to a process that deals with any expression of the form $f(x) < g(x)$, $f(x) > g(x)$, $f(x) \leq g(x)$ or $f(x) \geq g(x)$. Basically, we have the following:

To solve an inequality between two functions, $f(x)$ and $g(x)$,
i.e., to solve for $f(x) < g(x)$, $f(x) > g(x)$, $f(x) \le g(x)$ or $f(x) \ge g(x)$
We proceed as follows:



1. Sketch the corresponding graphs of both $f(x)$ and $g(x)$ on the same set of axes.

2. Find where the two graphs intersect, i.e., solve $f(x) = g(x)$ so that we find $x = a$ and $x = b$.

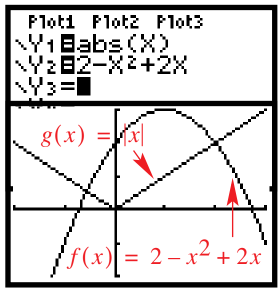
3. Identify where one function lies above or below the other. Then, depending on the inequality, quote the values of x that correspond to that region (or those regions).
e.g., $f(x) > g(x) \Leftrightarrow x \in\ ]-\infty, a] \cup\ ]b, \infty[$ and $f(x) \le g(x) \Leftrightarrow x \in [a, b]$

EXAMPLE 2.32

Use a graphical method to find $\{x |\ |x| < 2 - x^2 + 2x\}$

Solution

We make use of the TI–83 to solve this problem.

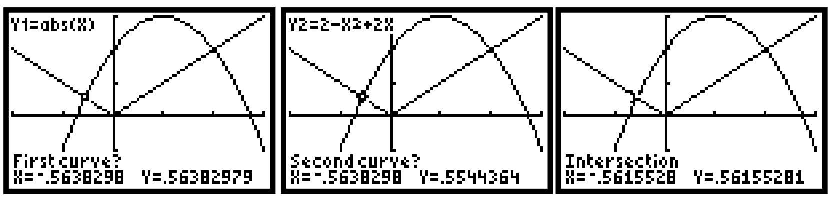


Let $g(x) = |x|$ and $f(x) = 2 - x^2 + 2x$, we sketch these graphs on the same set of axes.

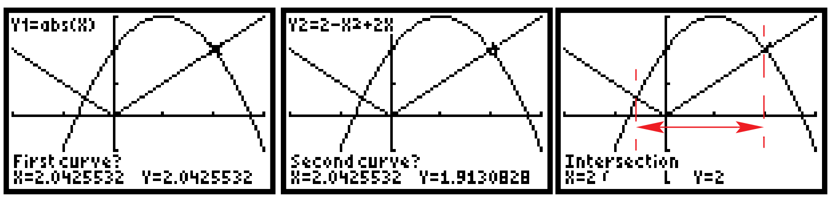
Next we need to find where the two graphs intersect. That is we need to solve the equation $g(x) = f(x)$. So, we have:

$$|x| = 2 - x^2 + 2x$$

Using the **intersect** option from the **CALC** menu we have:



Having found one point of intersection, we now find the other:



Therefore, from our results we have that $\{x |\ |x| < 2 - x^2 + 2x\} = \{x | -0.5616 < x < 2\}$.
Notice once again, the the graphics calculator could only provide an approximate answer for one of the points of intersection.

EXERCISES 2.4.3

1. Find the solution set for each of the following inequalities.

(a) $(x-1)(x+2) > 0$ (b) $(x+3)(x-2) \le 0$
(c) $x(4-x) \le 0$ (d) $(1-3x)(x-3) > 0$
(e) $(3+2x)(x+1) \ge 0$ (f) $(5-2x)(3-4x) < 0$

2. Find the solution set for each of the following inequalities.

(a) $x^2+3x+2>0$ (b) $x^2-x-6<0$ (c) $2x^2-5x-3 \ge 0$
(d) $x^2-4 \le 0$ (e) $x^2+x-5<0$ (f) $-x^2+x+6 \le 0$
(g) $-x^2+x+1 \ge 0$ (h) $-2x^2-3x+5 \ge 0$ (i) $2x^2+5x-3>0$
(j) $x^2-4x+3<0$ (k) $2x^2+x-1<0$ (l) $x^2+3<0$
(m) $-x^2-2>0$ (n) $2x^2-7x \le 15$ (o) $3x^2+5x>2$

3. (a) For what value(s) of k is the inequation $x^2+2kx-k>0$ true for all values of x?
(b) For what value(s) of k is the inequation $x^2-kx+2 \ge 0$ true for all values of x?
(c) For what value(s) of n is the inequation $x^2+2x \ge 2n$ true for all values of x?

4. By sketching on the same set of axes, the graphs of the functions $f(x)$ and $g(x)$, solve the inequalities

i. $f(x) < g(x)$ ii. $f(x) \ge g(x)$

(a) $f(x) = x+2$, $g(x) = x^2$ (b) $f(x) = x-1$, $g(x) = x^2-4x+5$
(c) $f(x) = x^2+2$, $g(x) = 4x-1$ (d) $f(x) = 3x^2-1$, $g(x) = x+1$
(e) $f(x) = 5-x^2$, $g(x) = x^2-3$ (f) $f(x) = x^2-3x-3$, $g(x) = x-4$

5. On the same set of axes sketch the graphs of $f(x) = |x-1|$ and $g(x) = 1-x^2$.
Hence find $\{x : |x-1| < 1-x^2\}$.

6. Given that $f(x) = x^2+3x+2$ and $g(x) = 4-x^2$, find $\{x|\ f(x) \le g(x)\}$.

7. (a) Find $\{x : |x^2-4x| < k\}$ for i. $k=2$ ii. $k=4$ iii. $k=8$

(b) Find i. $\{x : |2x-3| \le 3x-x^2\}$ ii. $\left\{x : |3-|x|| \le \left|3-\frac{1}{3}x^2\right|\right\}$

8. Find (a) $\left\{x : \frac{x-2}{x+3} > 0\right\}$ (b) $\left\{x : \frac{4-x}{x+1} > 0\right\}$

(c) $\left\{x : \frac{1-2x}{x^2+1} > k\right\}$ where i. $k=0$ ii. $k=1$

9. (a) Find $\{x : 2|x|+1 < 3-2(x-k)^2\}$ for i. $k = \pm 1$ ii. $k=2$
(b) Find $\{k : 2|x|+1 < 3-2(x-k)^2, k>0\} = \varnothing$.

2.4.4 SIMULTANEOUS EQUATIONS INVOLVING LINEAR – QUADRATIC EQUATIONS

In part (c) of Examples 2.30 and 2.31 we have already found the need to solve simultaneous equations where one equation was a quadratic, i.e., $f(x) = x^2 - 3$ and the other was linear, i.e., $g(x) = 2x$. In this instance, we equated the two functions, $f(x) = g(x)$ so that $x^2 - 3 = 2x$, then transposed to get a new quadratic, i.e., $x^2 - 2x - 3 = 0$ which could readily be solved for x. In this section we formalise the process of solving simultaneous equations involving a quadratic and linear expression or two quadratic expressions.

To solve a **linear–quadratic** system of equations it is often the case that the method of substitution is most appropriate. The process is as follows:

Step 1: Arrange the equations so that they are both in the form $y = \ldots$ i.e., y is expressed **explicitly** in terms of x.

Step 2: Label the two equations
$$y = ax^2 + bx + c \quad -(1)$$
$$y = mx + k \quad -(2)$$

Step 3: Equate (1) and (2): $mx + k = ax^2 + bx + c$

Step 4: Transpose to obtain a new quadratic:
$$ax^2 + (b - m)x + (c - k) = 0$$

Step 5: Solve for x and then find y by substituting into (1) or (2).

EXAMPLE 2.33

Solve the simultaneous sytem of equations

(a) $y = x^2 + 3x - 6$, $y = 2x - 4$ (b) $y = -2x^2 + 4x + 9$, $x + 2y = 7$

Solution

(a) We label the equations as follows;
$$y = x^2 + 3x - 6 \quad -(1)$$
$$y = 2x - 4 \quad -(2)$$

Equating (1) to (2) gives: $2x - 4 = x^2 + 3x - 6$

Solving we have: $0 = x^2 + x - 2$

$\Leftrightarrow \quad = (x + 2)(x - 1)$

$\Leftrightarrow x = -2, 1$

Substituting $x = -2$ into (2): $y = 2(-2) - 4 = -8$

$x = 1$ into (2): $y = 2(1) - 4 = -2$

The solution can be expressed as two coordinate pairs: (–2,–8), (1,–2).

(b) The first step in this case is to make y the subject of the second equation and then to substitute this into the first equation.

i.e., $x + 2y = 7 \Leftrightarrow 2y = 7 - x \Leftrightarrow y = \frac{1}{2}(7 - x)$.

Therefore we have, $y = -2x^2 + 4x + 9$ – (1)

$y = \frac{1}{2}(7 - x)$ – (2)

Substituting [or equating] (2) into (1) gives:

$$\frac{7-x}{2} = -2x^2 + 4x + 9$$
$$\Leftrightarrow 7 - x = -4x^2 + 8x + 18$$
$$\Leftrightarrow 4x^2 - 9x - 11 = 0$$
$$\therefore x = \frac{9 \pm \sqrt{(-9)^2 - 4 \times 4 \times (-11)}}{2 \times 4}$$
$$= \frac{9 \pm \sqrt{257}}{8}$$
$$\approx -0.88, 3.13$$

Substituting into (2): $y = \frac{1}{2}\left(7 - \frac{9 \pm \sqrt{257}}{8}\right) = \frac{47}{16} \pm \frac{\sqrt{257}}{16}$

$\approx 3.94, 1.94$

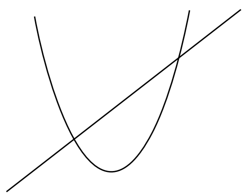
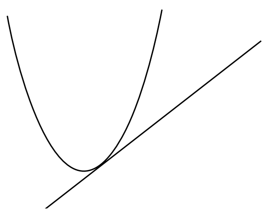
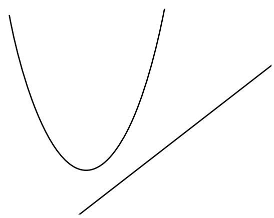
The approximate coordinate pairs for the solution to this problem are (–0.88,3.94), (3.13,1.94).

Again, we see that the discriminant can be used to determine the geometrical relationship between the parabola and the straight line.

When solving the simultaneous system of equations

$$y = px^2 + qx + r \quad – (1)$$
$$y = mx + k \quad – (2)$$

which results in solving the quadratic $ax^2 + bx + c = 0$ (after equating (1) to (2)) we have three possible outcomes:

Case 1	Case 2	Case 3
		
i.e., $\Delta = b^2 - 4ac > 0$ The straight line cuts the parabola twice.	i.e., $\Delta = b^2 - 4ac = 0$ The straight line touches the parabola at one point. We say the straight line is a **tangent** to the parabola.	i.e., $\Delta = b^2 - 4ac < 0$ The straight line neither touches nor cuts the parabola anywhere.

EXAMPLE 2.34

Find the value(s) of m for which the straight line with equation $y = mx - 2$ is a tangent to the parabola with equation $y = x^2 - 3x + 7$.

Solution

We start by solving the system of equations as we have done previously:

$$y = x^2 - 3x + 7 \quad -(1)$$
$$y = mx - 2 \quad -(2)$$

equating (1) to (2): $x^2 - 3x + 7 = mx - 2$

$$\Leftrightarrow x^2 - (m+3)x + 9 = 0$$

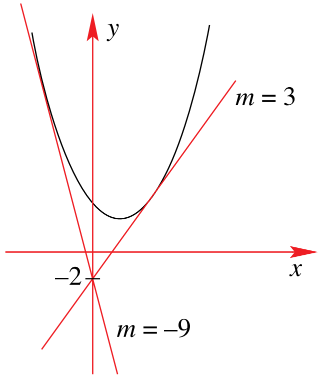
Then, for the straight line to be a tangent, it means that the line and the parabola touch. This in turn implies that the discriminant is zero.

That is, $\Delta = [-(m+3)]^2 - 4 \times 1 \times 9 = (m+3)^2 - 36$

$$= (m+3+6)(m+3-6) \text{ [using the diff. of two squares]}$$
$$= (m+9)(m-3)$$

Then, setting $\Delta = 0$ we have $(m+9)(m-3) = 0 \Leftrightarrow m = -9$ or $m = 3$.

Geometrically we have:



To solve a **quadratic–quadratic** system of equations we use the same method of substitution that was used for the linear–quadratic set up.

EXAMPLE 2.35

Solve simultaneously the system of equations $y = 2x^2 + 3x + 1$, $y = 2x - x^2 + 3$

Solution

Set up the equations as follows:

$$y = 2x^2 + 3x + 1 \quad -(1)$$
$$y = 2x - x^2 + 3 \quad -(2)$$

Equating (1) to (2): $2x^2 + 3x + 1 = 2x - x^2 + 3$

$$\Leftrightarrow 3x^2 + x - 2 = 0$$
$$\Leftrightarrow (3x-2)(x+1) = 0$$
$$\therefore x = \frac{2}{3} \text{ or } x = -1$$

Using (2): When $x = \frac{2}{3}$, $y = 2\left(\frac{2}{3}\right) - \left(\frac{4}{9}\right) + 3 = \frac{35}{9}$ and when $x = -1$, $y = -2 - 1 + 3 = 0$

Therefore, the pairs that satisfy this system of equations are $\left(\frac{2}{3}, \frac{35}{9}\right)$ and $(-1, 0)$.

EXERCISES 2.4.4

1. Solve the following pairs of simultaneous equations.

(a) $y = 2x + 1$
$y = x^2 + 2x - 3$

(b) $y = x + 1$
$y = x^2 + 2x - 1$

(c) $y = 3x - 1$
$y = 3x^2 - 2x - 3$

(d) $3x - 2y = 3$
$y = x^2 + 2x - 3$

(e) $x - 2y = 5$
$y = x^2 + 4x - 7$

(f) $x + 2y = 0$
$x^2 + 3y = 4$

(g) $y = 2x$
$y = x^2 + x - 3$

(h) $x + 2y = 3$
$y = -x^2 + 2x - 3$

(i) $y = 3x + 1$
$y = x^2 + 2x - 3$

(j) $x - y = 1$
$y = x^2 + x - 5$

(k) $3x - 2y = 3$
$y = -x^2 + 2x - 3$

2. Solve the following simultaneous equations.

(a) $y = x^2 - 4x + 7$
$y = 2x^2 + 2x$

(b) $y = 4x^2 - 16x + 8$
$y = 9x - 8x^2 - 4$

(c) $y = 3x^2 - 2x + 2$
$y = 2x^2 + x + 2$

(d) $y = 4x^2 + 3ax - 2a^2$
$y = 2x^2 + 2ax - a^2$ where a is a real number

(e) $y = x^2 - 2x + 4$
$y = 2 - 2x - x^2$

(f) $y = x^2 + 3x - 2$
$y = 2x^2 - x + 2$

(g) $y = x^2 + 4x + 3$
$y = 4x - x^2$

(h) $y = 5x^2 + x + 4$
$y = x^2 + 5x + 3$

3. For what value(s) of m will the straight line with equation $y = mx - 6$
(a) touch (b) intersect (c) never meet
the parabola with equation $y = x^2$.

4. Find the distance between the points of intersection of the line with equation $y = 2x + 1$ and the parabola with equation $y = x^2 - 4x + 6$.

5. Find the value of a such that the line $y = 2x + a$ has exactly one intersection point with the parabola with equation $y = x^2 + 3x + 2$.

6. If the graph of the function $f(x) = 4x^2 - 20x - 4$ intersects the graph of the function $g(x) = 1 + 4x - x^2$ at the point (m, k), show that the other point of intersection occurs where $x = -\frac{1}{m}$.

7. For what value(s) of k is the line $2x = 3y + k$ a tangent to the parabola $y = x^2 - 3x + 4$?

8. Show that the equations $\begin{cases} y = kx - 1 \\ y = x^2 + (k-1)x + k^2 \end{cases}$ have no real solution for all values of k.

9. For what values of k do $\begin{cases} y = kx \\ y = kx^2 + 3x + k \end{cases}$ have no solution?

10. For what values of q do the equations $\begin{cases} y = (1-q)x + 2 \\ y = -\frac{1}{qx} \end{cases}$ have only 1 real solution?

11. Find the value for c, for some fixed value of m, so that the straight line $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$, $a \neq 0$.

12. The parabola with equation $y = ax^2 + x + ab$ meets the straight line with equation $y = a^2bx + 2ab$ where a and b are real constants at the points where $x = x_1$ and $x = x_2$. Show that $x_1x_2 + b = 0$.

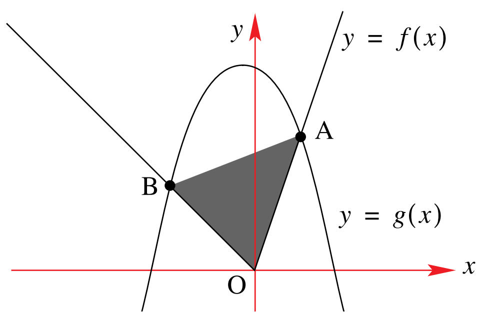
13. (a) Solve the simultaneous equations

i. $y = 3x^2$, $y = 14 - 11x$ ii. $y = 3x^2$, $y = x + 14$

(b) The functions $f(x) = x + |2x|$ and $g(x) = -x^2 - \frac{2}{3}x + \frac{14}{3}$ intersect at the points A(a, b) and B(c, d) shown in the diagram below.

i. Show that $3a^2 + 11a - 14 = 0$.

ii. Show that $3c^2 - c - 14 = 0$.



(c) i. Find the equation of the straight line passing through the points A and B

ii. Find the area of the region enclosed by the triangle OAB.

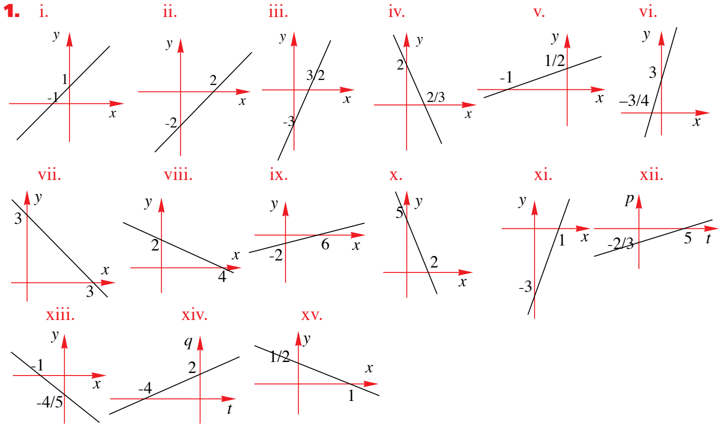
(h) $x < -\frac{5}{12} \cup x > \frac{1}{12}$ (i) $x \le -4 \cup x \ge 16$ **6.** $p < 3$ **7.** (a) $-\frac{2}{3} < x < 2$ (b) $-3 \le x \le 1$

(c) $0 < x < 2$ **8.** (a) $\frac{a}{1+a} < x < \frac{a}{1-a}$ (b) $\frac{-1}{1+a} < x < \frac{1}{1-a}$ (c) $\left]-\infty, \frac{-a^2}{a+1}\right] \cup \left[\frac{a^2}{a-1}, \infty\right[$

9. (a) $-\frac{4}{3} < x < \frac{4}{3}$ (b) $-\frac{3}{2} < x < \frac{3}{4}$

EXERCISE 2.3.1

1.

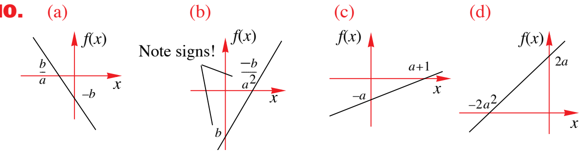


2. (a) 2 (b) 3 (c) $\frac{5}{3}$ **3.** (a) $y = 2x - 1$ (b) $y = 3x + 9$ (c) $y = -x - 1$ **4.** (a) $-\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{3}{2}$

(d) $-\frac{4}{5}$ **5.** $y = 2x$ **6.** $y = -x + 1$ **7.** $y = \frac{x+2}{2}$ **8.** 2 **9.** (a) $y = \frac{5}{2}x$ (b) $y = -\frac{3}{2}x + 3$

(c) $y = \frac{5}{6}x - \frac{1}{2}$ (d) $y = -2x + 1$

10. (a) (b) (c) (d)



EXERCISE 2.3.2

1. (i) $x = 1, y = 2$ (ii) $x = 3, y = 5$ (iii) $x = -1, y = 2$ (iv) $x = 0, y = 1$ (v) $x = -2, y = -3$ (vi) $x = -5, y = 1$ **2.** (i) $x = \frac{13}{11}, y = \frac{17}{11}$ (ii) $x = \frac{9}{14}, y = \frac{3}{14}$ (iii) $x = 0, y = 0$

(iv) $x = \frac{4}{17}, y = -\frac{22}{17}$ (v) $x = -\frac{16}{7}, y = \frac{78}{7}$ (vi) $x = \frac{5}{42}, y = -\frac{3}{28}$ **3.** (i) -3 (ii) -5 (iii) -1.5

4. (i) $m = 2, a = 8$ (ii) $m = 10, a = 24$ (iii) $m = -6, a = 9$.

5. (a) $x = 1, y = a - b$ (b) $x = -1, y = a + b$ (c) $x = \frac{1}{a}, y = 0$ (d) $x = b, y = 0$

(e) $x = \frac{a-b}{a+b}, y = \frac{a-b}{a+b}$ (f) $x = a, y = b - a^2$

EXERCISE 2.3.3

1. (a) $x = 4, y = -5, z = 1$ (b) $x = 0, y = 4, z = -2$ (c) $x = 10, y = -7, z = 2$ (d) $x = 1, y = 2, z = -2$ (e) $\varnothing$ (f) $x = 2t - 1, y = t, z = t$ (g) $x = 2, y = -1, z = 0$ (h) $\varnothing$

EXERCISE 2.4.1

1. (a) -5 (b) 4, 6 (c) -3, 0 (d) 1, 3 (e) -6, 3 (f) $-2, \frac{5}{3}$ (g) 2 (h) -3, 6 (i) -6, 1 (j) $0, \frac{3}{2}$

2. (a) -1 (b) -7, 5 (c) $-\frac{2}{5}$, 3 (d) -2, 1 (e) -3, 1 (f) 4, 5

3. (a) $-1 \pm \sqrt{6}$ (b) $3 \pm \sqrt{5}$ (c) $1 \pm \sqrt{5}$ (d) $\frac{-1 \pm \sqrt{33}}{8}$ (e) $\frac{9 \pm \sqrt{73}}{4}$ (f) $\frac{1 \pm \sqrt{85}}{6}$

4. (a) $\frac{3 \pm \sqrt{37}}{2}$ (b) $\frac{5 \pm \sqrt{33}}{2}$ (c) $\frac{3 \pm \sqrt{33}}{2}$ (d) $\frac{7 \pm \sqrt{57}}{2}$ (e) $\frac{-7 \pm \sqrt{65}}{2}$ (f) -4, 2 (g) $-1 \pm 2\sqrt{2}$

(h) $\frac{-5 \pm \sqrt{53}}{2}$ (i) $\frac{3 \pm \sqrt{37}}{2}$ (j) no real solutions (k) $4 \pm \sqrt{7}$ (l) no real solutions (m) $\frac{2 \pm \sqrt{13}}{2}$

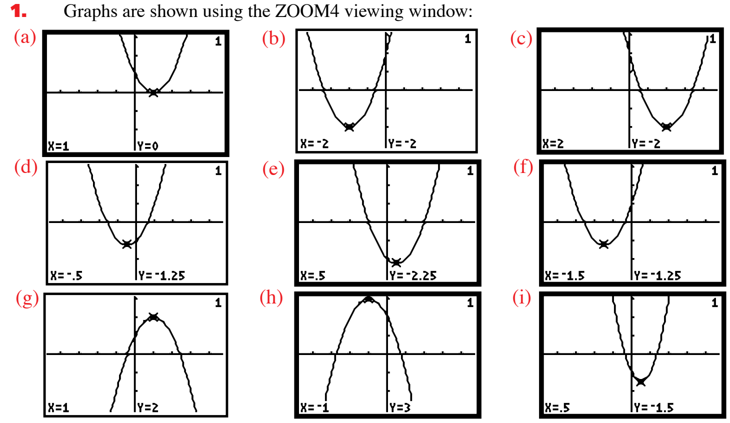
(n) $\frac{3 \pm 2\sqrt{11}}{5}$ (o) $\frac{6 \pm \sqrt{31}}{5}$ (p) $\frac{6 \pm \sqrt{29}}{7}$ **5.** (a) $-2 < p < 2$ (b) $p = \pm 2$ (c) $p < -2$ or $p > 2$

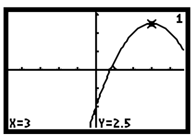
6. (a) $m = 1$ (b) $m < 1$ (c) $m > 1$ **7.** (a) $m = \pm 2\sqrt{2}$ (b) $]-\infty, -2\sqrt{2}[\cup]2\sqrt{2}, \infty[$

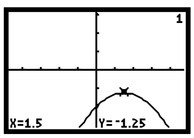
(c) $]-2\sqrt{2}, 2\sqrt{2}[$ **8.** (a) $k = \pm 6\sqrt{2}$ (b) $]-\infty, -6\sqrt{2}[\cup]6\sqrt{2}, \infty[$ (c) $]-6\sqrt{2}, 6\sqrt{2}[$ **10.** 4

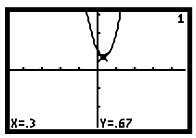
EXERCISE 2.4.2

1. Graphs are shown using the ZOOM4 viewing window:

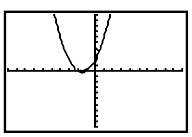


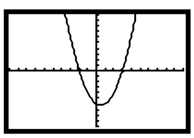
(j) 

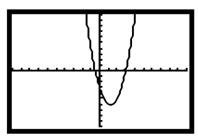
(k) 

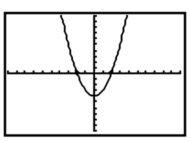
(l) 

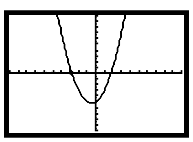
2. Graphs are shown using the ZOOM6 viewing window:

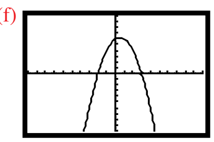
(a) 
(–2,0),(–1,0),(0,2)

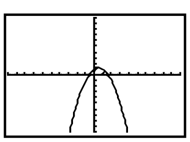
(b) 
(–2,0),(3,0),(0,–6)

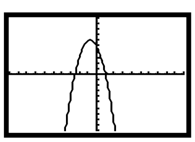
(c) 
(–0.5,0),(3,0),(0,–3)

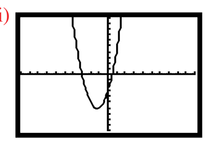
(d) 
(–2,0),(2,0),(0,–4)

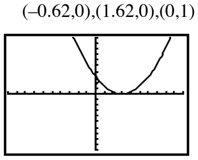
(e) 
(–2.79,0),(1.79,0),(0,–5)

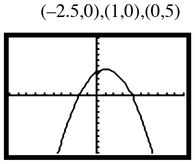
(f) 
(–2,0),(3,0),(0,6)

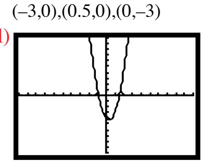
(g) 
(–0.62,0),(1.62,0),(0,1)

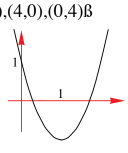
(h) 
(–2.5,0),(1,0),(0,5)

(i) 
(–3,0),(0.5,0),(0,–3)

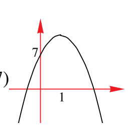
(j) 
(3,0),(0,3)

(k) 
(–2,0),(4,0),(0,4)ß

(l) 
(–0.87,0),(1.54,0),(0,–4)

3. (a) $x = 1$ (b) (1,–1) (c) i. $\left(\frac{2 \pm \sqrt{2}}{2}, 0\right)$ ii. (0,1) 

4. (a) $x = 1$ (b) (1,9)

(c) i. $\left(\frac{2 \pm 3\sqrt{2}}{2}, 0\right)$ ii. (0,7) 

5. (a) $k = \frac{9}{4}$ (b) $k < \frac{9}{4}$ (c) $k > \frac{9}{4}$ **6.** (a) $k = \frac{25}{8}$

(b) $k < \frac{25}{8}$ (c) $k > \frac{25}{8}$ **7.** (a) $k = \pm 1$ (b) $-1 < k < 1$ (c) $k < -1 \cup k > 1$

8. (a) $y = \frac{5}{12}(x-2)(x-6)$ (b) $y = -\frac{3}{8}(x+4)^2$ (c) $y = \frac{3}{4}(x-2)^2 + 1$ (d) $y = 3x^2 - 6x + 7$

9. (a) $y = -\frac{2}{5}x(x-6)$ (b) $y = \frac{3}{4}(x-3)^2$ (c) $y = \frac{7}{9}(x+2)^2 + 3$ (d) $y = -\frac{7}{3}x^2 - 2x + \frac{40}{3}$

EXERCISE 2.4.3

1. (a) $]-\infty,-2[\cup]1,\infty[$ (b) $[-3,2]$ (c) $]-\infty,0]\cup[4,\infty[$ (d) $]\frac{1}{3},3[$ (e) $]-\infty,-1.5]\cup[-1,\infty[$ (f) $]0.75,2.5[$

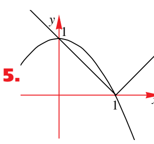
2. (a) $]-\infty,-2[\cup]-1,\infty[$ (b) $]-2,3[$ (c) $]-\infty,-0.5]\cup[3,\infty[$ (d) $[-2,2]$ (e) $]\frac{-1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}[$

(f) $]-\infty,-2]\cup[3,\infty[$ (g) $[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}]$ (h) $[-2.5,1]$ (i) $]-\infty,-3[\cup]0.5,\infty[$ (j) $]1,3[$ (k) $]-1,0.5[$ (l) Ø

(m) Ø (n) $[-1.5,5]$ (o) $]-\infty,-2[\cup]\frac{1}{3},\infty[$ **3.** (a) $-1 < k < 0$ (b) $-2\sqrt{2} < k < 2\sqrt{2}$ (c) $n \leq -0.5$

4. (a) i. $]-\infty,-1[\cup]2,\infty[$ ii. $[-1,2]$ (b) i. $]-\infty,2[\cup]3,\infty[$ ii. $[2,3]$ (c) i. $]1,3[$ ii. $]-\infty,1]\cup[3,\infty[$

(d) i. $]-\frac{2}{3},1[$ ii. $]-\infty,-\frac{2}{3}]\cup[1,\infty[$ (e) i. $]-\infty,-2[\cup]2,\infty[$ ii. $[-2,2]$ (f) i. $]2-\sqrt{3}, 2+\sqrt{3}[$

ii. $]-\infty, 2-\sqrt{3}]\cup[2+\sqrt{3},\infty[$ **5.** 

$]0,1[$ **6.** $[-2,0.5]$

7. (a) i. $]2-\sqrt{6}, 2-\sqrt{2}[\cup]2+\sqrt{2}, 2+\sqrt{6}[$ ii. $]2(1-\sqrt{2}), 2(1+\sqrt{2})\backslash\{2\}$ iii. $]2(1-\sqrt{3}), 2(1+\sqrt{3})$

(b) i. $[\frac{5-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2}]$ ii. all real values **8.** (a) $\{x: x < -3\}\cup\{x: x > 2\}$ (b) $\{x: -1 < x < 4\}$

(c) i. $\{x: x < 0.5\}$ ii. $\{x: -2 < x < 0\}$ **9.** (a) i. $]0,1[$ $(k = 1)$; $]-1,0[$ $(k = -1)$ ii. Ø (b) $k > 1.25$

EXERCISE 2.4.4

1. (a) (–2,–3) (2,5) (b) (–2,–1) (1,2) (c) $\left(-\frac{1}{3}, -2\right)$, (2, 5) (d) $\left(-\frac{3}{2}, -\frac{15}{4}\right)$, (1, 0)

(e) $\left(-\frac{9}{2}, -\frac{19}{4}\right)$, (1, –2) (f) $\left(\frac{3+\sqrt{73}}{4}, \frac{-3-\sqrt{73}}{8}\right)$, $\left(\frac{3-\sqrt{73}}{4}, \frac{-3+\sqrt{73}}{8}\right)$

(g) $\left(\frac{1-\sqrt{13}}{2}, 1-\sqrt{13}\right)$, $\left(\frac{1+\sqrt{13}}{2}, 1+\sqrt{13}\right)$ (h) no real solutions

(i) $\left(\frac{1-\sqrt{17}}{2}, \frac{5-3\sqrt{17}}{2}\right)$, $\left(\frac{1+\sqrt{17}}{2}, \frac{5+3\sqrt{17}}{2}\right)$ (j) (–2,–3), (2,1) (k) no real solutions

2. (a) (1,4), (–7,84) (b) $\left(\frac{4}{3}, -\frac{56}{9}\right)$, $\left(\frac{3}{4}, -\frac{7}{4}\right)$ (c) (0, 2), (3, 23) (d) $(-a, -a^2)$, $\left(\frac{a}{2}, \frac{a^2}{2}\right)$ (e) Ø

(f) (2,8) (g) Ø (h) $\left(\frac{1}{2}, \frac{23}{4}\right)$ **3.** (a) $\pm 2\sqrt{6}$ (b) $m < -2\sqrt{6}$, $m > 2\sqrt{6}$ (c) $-2\sqrt{6} < m < 2\sqrt{6}$

4. $\sqrt{80}$ **5.** 1.75 **7.** $-\frac{23}{12}$ **9.** $(-\infty, -3) \cup (1, \infty)$ **10.** 0.5 **11.** $c = \frac{a}{m}$

13. (a) i. (1,3), $\left(-\frac{14}{3}, \frac{196}{3}\right)$ ii. (–2, 12), $\left(\frac{7}{3}, \frac{49}{3}\right)$ (c) i. A(1,3), B(–2,2) ii. 4 sq. units