Solution

The two fixed points could be taken as $(2, 3, 1)$ and $(4, 2, 0)$ while the vectors are $\mathbf{v}_1 = (1, 2, -3)$ and $\mathbf{v}_2 = (3, -1, 1)$.

$$d = \left| \frac{(2, -1, -1)(-1, -10, -7)}{|(-1, -10, -7)|} \right|$$

$$= \left| \frac{-2 + 10 + 7}{\sqrt{1 + 100 + 49}} \right| = \frac{15}{\sqrt{150}} = \frac{\sqrt{6}}{2}$$

Note: The minimum distance could be found using other methods too. One of them would be to consider the line going from any point on L_1 to any point on L_2. This will give a parametric equation in s and t. Then considering that this line will be perpendicular to both L_1 and L_2, i.e. $\mathbf{u} \cdot \mathbf{v}_1 = 0$, $\mathbf{u} \cdot \mathbf{v}_2 = 0$, enables us to set up a system of two equations that could be solved for s and t. Lastly, we get the distance between the points corresponding to the specific values we just established.

Exercise 14.4

1 Find a vector equation, a set of parametric equations and a set of Cartesian equations of the line containing the point A and parallel to the vector $\mathbf{u}$.

a) $A(-1, 0, 2)$, $\mathbf{u} = (1, 5, -4)$

b) $A(3, -1, 2)$, $\mathbf{u} = (2, 5, -1)$

c) $A(1, -2, 6)$, $\mathbf{u} = (3, 5, -11)$

2 Find all three forms of the equation of the line that passes through the points A and B.

a) $A(-1, 4, 2)$, $B(7, 5, 0)$

b) $A(4, 2, -3)$, $B(0, -2, 1)$

c) $A(1, 3, -3)$, $B(5, 1, 2)$

3 a) Write the equation of the line through the points $(3, -2)$ and $(5, 1)$ in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

b) Write the equation of the line through the points $(0, -2)$ and $(5, 0)$ in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

4 The equation of a line in 2-space is given by $\mathbf{r} = (2, 1) + t(3, -2)$. Write the equation in the form $ax + by = c$.

5 Find the equation of a line through $(2, -3)$ that is parallel to the line with equation $\mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$.

6 Find the equation of a line through $(-2, 1, 4)$ and parallel to the vector $3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

7 In each of the following, find the point of intersection of the two given lines, and if they do not intersect, explain why.

a) L_1: $\mathbf{r} = (2, 2, 3) + t(1, 3, 1)$
L_2: $\mathbf{r} = (2, 3, 4) + t(1, 4, 2)$

b) L_1: $\mathbf{r} = (-1, 3, 1) + t(4, 1, 0)$
L_2: $\mathbf{r} = (-13, 1, 2) + t(12, 6, 3)$

c) L_1: $\mathbf{r} = (1, 3, 5) + t(7, 1, -3)$
L_2: $\mathbf{r} = (4, 6, 7) + t(-1, 0, 2)$

d) L_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

L_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$

8 Find the vector and parametric equations of each line:

a) through the points (2, −1) and (3, 2)

b) through the point (2, −1) and parallel to the vector $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

c) through the point (2, −1) and perpendicular to the vector $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

d) with y-intercept (0, 2) and in the direction of $2\mathbf{i} - 4\mathbf{j}$

9 Consider the line with equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

a) For what value of t does this line pass through the point $\left(0, \frac{11}{2}, \frac{9}{2}\right)$?

b) Does the point (−1, 4, 6) lie on this line?

c) For what value of m does the point $\left(\frac{1-2m}{2}, 2m, 3\right)$ lie on the given line?

10 Consider the following equations representing the paths of cars after starting time $t \geqslant 0$, where distances are measured in km and time in hours. For each car, determine

(i) starting position
(ii) the velocity vector
(iii) the speed.

a) $\mathbf{r} = (3, -4) + t\begin{pmatrix} 7 \\ 24 \end{pmatrix}$

b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -12 \end{pmatrix}$

c) $(x, y) = (5, -2) + t(24, -7)$

11 Find the velocity vector of each of the following racing cars taking part in the Paris–Dakar rally:

a) direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ with a speed of 160 km/h

b) direction $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ with a speed of 170 km/h

12 After leaving an intersection of roads located at 3 km east and 2 km north of a city, a car is moving towards a traffic light 7 km east and 5 km north of the city at a speed of 30 km/h. (Consider the city as the origin for an appropriate coordinate system.)

a) What is the velocity vector of the car?

b) Write down the equation of the position of the car after t hours.

c) When will the car reach the traffic light?

13 Consider the vectors $\mathbf{u} = (1, a, b)$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

a) Find a and b so that $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

b) If O is the origin, P a point whose position vector is $\mathbf{v}$ and Q is with position vector $\mathbf{w}$, find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

c) Hence, find the sine of the angle and use it to find the area of the triangle OPQ.

14 The triangle ABC has vertices at the points $A(-1, 2, 3)$, $B(-1, 3, 5)$ and $C(0, -1, 1)$.

a) Find the size of the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence, or otherwise, find the area of triangle ABC.

Let L_1 be the line parallel to $\overrightarrow{AB}$ which passes through $D(2, -1, 0)$, and L_2 be the line parallel to $\overrightarrow{AC}$ which passes through $E(-1, 1, 1)$.

c) (i) Find the equations of the lines L_1 and L_2.

(ii) Hence, show that L_1 and L_2 do not intersect.

15 Consider the points $A(1, 3, -17)$ and $B(6, -7, 8)$ which lie on the line l.

a) Find an equation of line l, giving the answer in parametric form.

b) The point P is on l such that $\overrightarrow{OP}$ is perpendicular to l. Find the coordinates of P.

16 a) Starting with the equation of a line in the form $mx + ny = p$, find a vector equation of the line.

b) (i) Starting with a vector equation of a line where $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, with $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, find an equation of the line in the form $mx + ny = p$.

(ii) What is the relationship between the components of the direction vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and the slope of the line?

17 Find a parametrization for the line segment between points A and B in each of the following questions.

(i) $A(0, 0, 0)$, $B(1, 1, 3)$

(ii) $A(-1, 0, 1)$, $B(1, 1, -2)$

(iii) $A(1, 0, -1)$, $B(0, 3, 0)$

18 Find a vector equation and a set of parametric equations of the line through the point $(0, 2, 3)$ and parallel to the line $\mathbf{r} = (\mathbf{i} - 2\mathbf{j}) + 2t\mathbf{k}$.

19 Find a vector equation and a set of parametric equations of the line through the point $(1, 2, -1)$ and parallel to the line $\mathbf{r} = t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

20 Find a vector equation and a set of parametric equations of the line through the origin and the point $A(x_0, y_0, z_0)$.

21 Find a vector equation and a set of parametric equations of the line through $(3, 2, -3)$ and perpendicular to

a) the xz-plane

b) the yz-plane.

22 Write a set of symmetric equations for the line through the origin and the point $A(x_0, y_0, z_0)$, $x_0, y_0, z_0 \neq 0$.

In questions 23–29, determine whether the lines l_1 and l_2 are parallel, skew or intersecting. If they intersect, find the coordinates of the point of intersection.

23 $l_1: x - 3 = 1 - y = \dfrac{z-5}{2}$, $l_2: \mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{j} + \mathbf{k})$

24 $l_1: \begin{cases} x = -1 + s \\ y = 2 - 3s \\ z = 1 + 2s \end{cases}$ $\quad l_2: \begin{cases} x = 2 - 2m \\ y = -1 + 6m \\ z = -4m \end{cases}$

25 $l_1: \dfrac{x-3}{2} = \dfrac{1+y}{4} = 2 - z$, $l_2: \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

26 $l_1: x - 1 = \dfrac{y-1}{3} = \dfrac{z+4}{2}$, $l_2: 1 - x = -1 - y = \dfrac{z}{2}$

27 l_1: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(-6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$, l_2: $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + m(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

28 $\frac{x-2}{5} = y - 1 = \frac{z-2}{3}$ and $\frac{x+4}{3} = \frac{7-y}{3} = \frac{10-z}{4}$

29 $x = 1 + t, y = 2 - 2t, z = t + 5$ and $x = 2 + 2t, y = 5 - 9t, z = 2 + 6t$

30 Find the point on the line

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(-3\mathbf{i} + \mathbf{j} + \mathbf{k})$$

that is closest to the origin. (Hint: use the parametric form and the distance formula and minimize the distance using derivatives!)

31 Find the point on the line

$$\mathbf{r} = 4\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

that is closest to the origin.

32 Find the point on the line

$$\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

that is closest to the point $(-1, 4, 1)$.

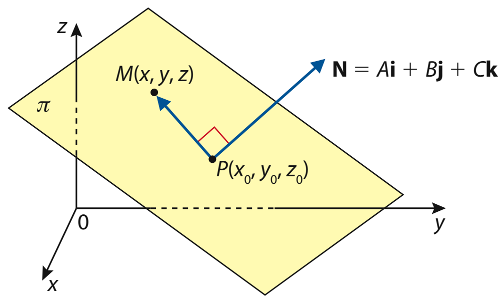
14.5 Planes

To define/specify a plane is to identify it in a way that makes it unique. One way is to set up an equation in a frame that will identify every point that belongs to the plane. There are several ways of specifying a plane but we will only mention four of them here. The rest will be cases that we address in some problems later. For more helpful geometric concepts please refer to the book's website.

A plane can be defined

- by three non-collinear points
- by two intersecting straight lines
- to be perpendicular to a certain direction and at a specific distance from the origin (for example)
- by being drawn through a given point and perpendicular to a given direction.

A direction, for our purposes, can be defined by a vector. In the case of a plane, the vector determining the direction is perpendicular to the plane and is said to be **normal to the plane**.



Equations of a plane

From the many ways of defining a plane above, the last two are mostly appropriate for deriving equations of a plane.

Cartesian (scalar) equation of a plane

Consider a plane π and a fixed point $P(x_0, y_0, z_0)$ on that plane. A vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, called the normal vector to the plane, is a vector perpendicular to the plane.