Chapter 1

Approximations and error

Contents:

OPENING PROBLEM

Three friends are looking at the Eureka Tower in Melbourne, Australia. They are trying to work out how tall it is, but they keep on arguing.

- Andreas tries to carefully count the floors. He thinks there are 92. He then estimates the average height of a floor to be 3 metres. So, his estimate for the height is $92 \times 3 = 276$ m.
- Bernadette tries to carefully count the floors too, but she is less sure of the exact number. She thinks it is somewhere from 89 to 93, so she decides to use 90 as her estimate. She is sure that each floor is more than 3 m but less than 4 m. She therefore estimates 3.5 m for each floor. Her estimate for the height is $90 \times 3.5 = 315$ m.
- Carlos knows most about buildings. He is confident the height of a floor is about 3.3 m. He is too lazy to try to count each individual floor, so he counts them in lots of about 10. He is fairly sure 90 is the best guess, so his estimate for the height is $90 \times 3.3 = 297$ m.

Things to think about:

a Who do you think had the best *method*?

b Who was wisest in the way they rounded numbers?

c The Eureka Tower is in fact about 297.3 m tall.

  i Whose estimate was the most accurate?

  ii What was the percentage error in Andreas' estimate?

A ROUNDING NUMBERS

There are many occasions when it is sensible to give an **approximate** answer.

For example, it is unreasonable to give the exact distance between the Earth and the Sun, because it is continually changing. The distance varies from its *perihelion*, about 146 million km, to its *aphelion*, about 152 million km.

We use the symbol $\approx$ or sometimes $\doteqdot$ to show that an answer has been approximated.

RULES FOR ROUNDING OFF

- If the digit after the one being rounded off is **less than 5** (0, 1, 2, 3, or 4) we round **down**.
- If the digit after the one being rounded off is **5 or more** (5, 6, 7, 8, 9) we round **up**.

Example 1 Self Tutor

Round off:

a 436 to the nearest 10 **b** 716 to the nearest 100

c 1050 to the nearest 100 **d** 19 628 to the nearest 1000.

a $436 \approx 440$ {round up, as 6 is greater than 5}

b $716 \approx 700$ {round down, as 1 is less than 5}

c $1050 \approx 1100$ {5 is rounded up}

d $19\,628 \approx 20\,000$ {round up, as 6 is greater than 5}

EXERCISE 1A.1

1 Round off to the nearest 10:

a 86 **b** 81 **c** 85 **d** 128 **e** 162
f 104 **g** 635 **h** 1822 **i** 699 **j** 3045

2 Round off to the nearest 100:

a 215 **b** 264 **c** 3750 **d** 3950 **e** 26 341

3 Round off to the nearest 1000:

a 8365 **b** 3500 **c** 19 210 **d** 19 650 **e** 114 823

4 Round off to the accuracy given:

a The height of Mt Everest is 8848 m. (to the nearest 10 m)

b The surface area of Lake Baikal in Russia is 31 722 km^2. (to the nearest 1000 km^2)

c The population of New Zealand in 2018 was 4 749 598. (to the nearest 1000)

d The attendance at an English football match was 85 512 people. (to the nearest 100 people)

e The area of Australia is 7 692 000 km^2. (to the nearest 100 000 km^2)

f An average weight of an adult African elephant is 5443 kg. (to the nearest 100 kg)

g The distance between Paris and Sydney is 16 950 km. (to the nearest 100 km)

h The average distance from Earth to the Moon is 384 400 km. (to the nearest 100 000 km)

i The population of South America in 2018 was 428 240 515. (to the nearest 1 000 000)

ROUNDING DECIMAL NUMBERS

A survey found that a total of 6428 flights were taken by 825 people last year. However, it is not sensible to give the average number of flights per person as 7.791 515 152. An approximate answer of 7.8 is more appropriate.

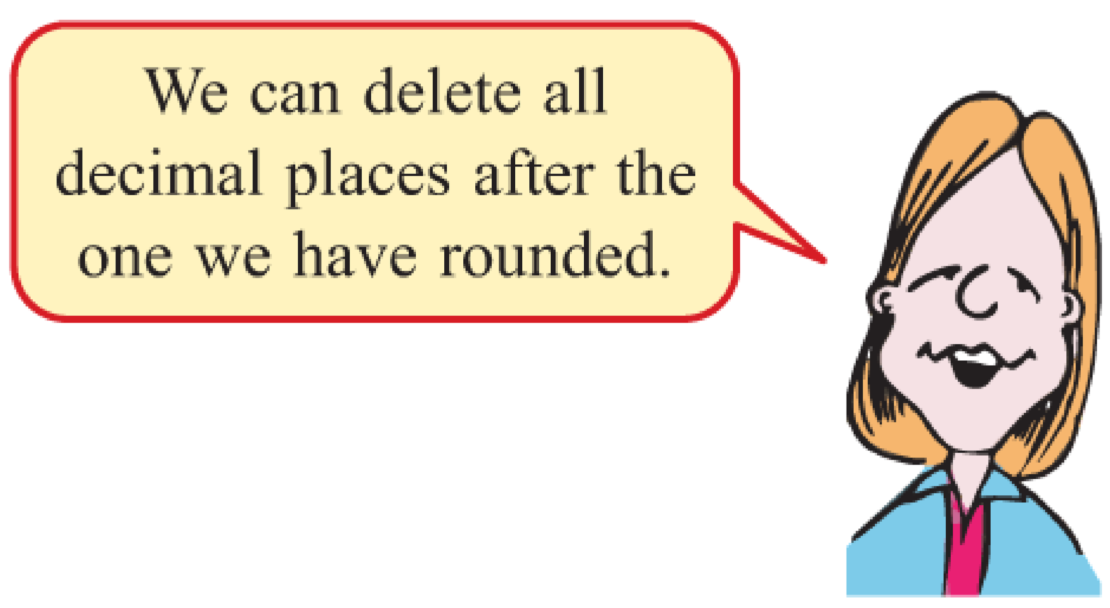
Example 2 Self Tutor

Round: **a** 8.43 to one decimal place

b 3.5169 to two decimal places.

a $8.43 \approx 8.4$ {round down, as $3 < 5$}

b $3.5169 \approx 3.52$ {round up, as $6 > 5$}



EXERCISE 1A.2

1 Round the following to the number of decimal places stated in brackets.

a 6.181 [1] **b** 6.181 [2] **c** 3.25 [1] **d** 17.403 [2]

e 2.131 58 [3] **f** 0.1940 [1] **g** 0.0972 [2] **h** 102.382 [2]

2 In 2009 Usain Bolt ran 100 m in 9.58 seconds. Round this time to 1 decimal place.

3 The average height of children in a class is 1.435 m. Round this height to 2 decimal places.

4 The thickness of a sheet of paper is 0.012 cm. Round this thickness to 2 decimal places.

5 The number π is a mathematical constant. The first six digits of π are 3.141 59.
Round π to:

a 1 decimal place **b** 3 decimal places **c** 4 decimal places.

6 The fraction $\frac{5}{19} \approx 0.263\,157\,895$. Round this number to:

a 1 decimal place **b** 2 decimal places **c** 6 decimal places.

7 Evaluate, giving your answer to 3 decimal places.

a $\sqrt{2}$ **b** $\sqrt{5}$ **c** $\sqrt{23}$ **d** $\sqrt[3]{4}$ **e** $\sqrt[3]{-15}$ **f** $\sqrt[3]{450}$

8 Calculate the following, rounding your answers to 2 decimal places:

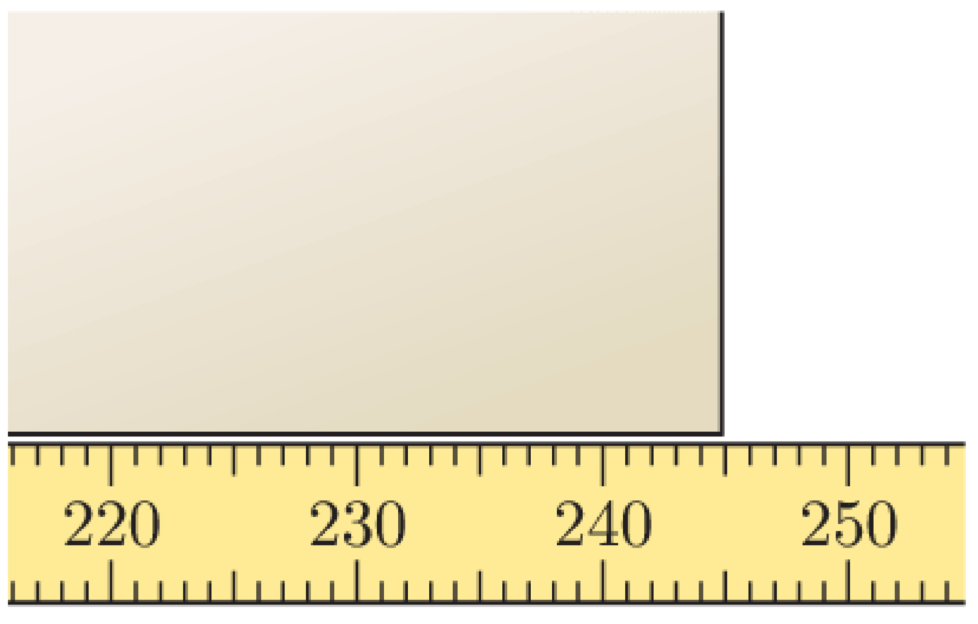
a $(16.8 + 12.4) \times 17.1$ **b** $16.8 + 12.4 \times 17.1$ **c** $127 \div 9 - 5$

d $127 \div (9 - 5)$ **e** $37.4 - 16.1 \div (4.2 - 2.7)$ **f** $\dfrac{16.84}{7.9 + 11.2}$

g $\dfrac{27.4}{3.2} - \dfrac{18.6}{16.1}$ **h** $\dfrac{27.9 - 17.3}{8.6} + 4.7$ **i** $\dfrac{0.0768 + 7.1}{18.69 - 3.824}$

9 Over a 23 game water polo season, Kerry scored 40 goals for her team. Find Kerry's average number of goals per game, correct to 2 decimal places.

10 Wang used a tape measure to check the length of his kitchen bench. After viewing the tape, he recorded the length as 2.45 m, which he rounded up to 2.5 m.
When his mother asked him about the length, he said it was about 3 m.
Explain what Wang has done wrong, and discuss why we need to be careful when we make approximations.



DISCUSSION

$\sqrt{61}$ is approximately 7.810 249 676 ....

Mateo rounded this number to 2 decimal places, giving $\sqrt{61} \approx 7.81$

Hiba rounded this number to 3 decimal places, giving $\sqrt{61} \approx 7.810$

Would it be fair to say that Hiba's estimate is "more accurate", even though both estimates have the same value?

ROUNDING OFF TO SIGNIFICANT FIGURES

The first **significant figure** of a decimal number is the first (left-most) non-zero digit.

For example:
- the first significant figure of 4567 is 4
- the first significant figure of 0.012 34 is 1.

Every digit to the right of the first significant figure is regarded as another significant figure.

To round off to a number of significant figures:

> Count off the specified number of significant figures then look at the next digit.
> - If the digit is less than 5, round **down**.
> - If the digit is 5 or more, round **up**.
>
> Delete all figures following the significant figures, replacing with 0s where necessary.

Example 3 — Self Tutor

Round: **a** 3.461 to 2 significant figures **b** 0.007 24 to 2 significant figures **c** 708 to 1 significant figure **d** 20.158 to 3 significant figures.

a 3.461 $\approx$ 3.5 {2 significant figures}

↑

This is the 2nd significant figure, so we look at the next digit which is 6.
The 6 tells us to round the 4 up to a 5 and delete the remaining digits.

b 0.007 24 $\approx$ 0.0072 {2 significant figures}

↑

These zeros at the front are place holders and so must stay.
The first significant figure is 7, and the second significant figure is 2.
The 4 tells us to leave the 2 as it is and delete the remaining digits.

c 708 $\approx$ 700 {1 significant figure}

↑

7 is the first significant figure so it has to be rounded.
The 0 tells us to keep the original 7 in the hundreds place, so we convert the 08 into 00.
These two zeros are place holders. They are not significant figures, but they need to be there to make sure the 7 has value 700.

d 20.158 $\approx$ 20.2 {3 significant figures}

↑

This 0 is significant as it lies between two non-zero digits.
The third significant figure is 1.
The 5 tells us to round the 1 up to a 2 and delete the remaining digits.

In IB examinations you are expected to give answers to 3 significant figures unless otherwise specified in the question.

EXERCISE 1A.3

1 Round to 2 significant figures:

a 128 **b** 8342 **c** 2.568 **d** 0.0134
e 163 870 **f** 1.086 **g** 3958 **h** 6.611

2 Round to 3 significant figures:

a 83 064 **b** 10 044 **c** 0.105 26 **d** 31.695
e 70.707 **f** 4.0007 **g** 0.036 71 **h** 19.989

3 Round to 4 significant figures:

a 16.382 **b** 438.207 **c** 6 873 681 **d** 0.028 885

4 The exact crowd size at a rock concert was 96 257 people. Round the crowd size to:

a 1 significant figure **b** 2 significant figures **c** 3 significant figures.

5 Evaluate the following, giving your answers to 3 significant figures:

a $\sqrt{7}$ **b** 2π **c** $36 \div 17$ **d** 517×3802
e $(0.986)^5$ **f** $\dfrac{16.3 - 2.68}{3.1}$ **g** $\sqrt{5.4 - 2.18}$ **h** $\dfrac{9.58}{\sqrt{2.8}}$

6 A theatre has 32 rows with 28 seats in each. Find the total number of seats, rounding your answer to 2 significant figures.

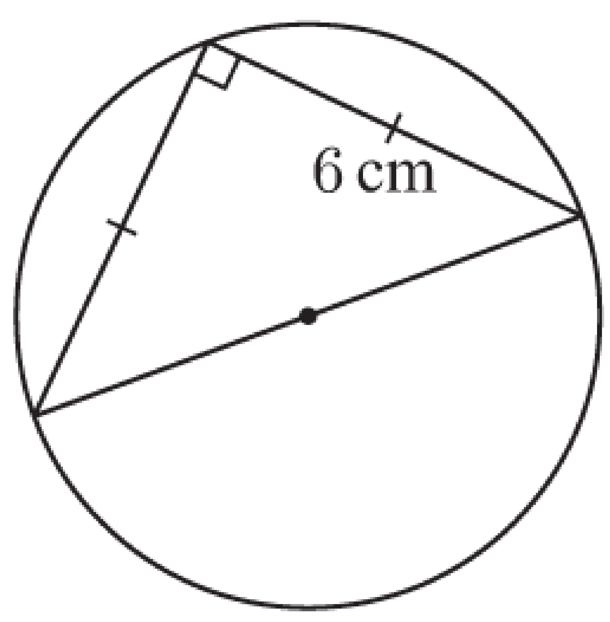
7 A ballroom has dimensions 30.1 m $\times$ 8.5 m. Find its area, rounding your answer to 3 significant figures.

8 The proceeds of a garage sale were $752.25, and this was shared equally between 4 people. Calculate the amount each person received:

a to 3 significant figures **b** to 5 significant figures.

9 The speed of sound in dry air at 20°C is 343 m s^{-1}. Calculate how many metres sound travels in one hour, giving your answer to two significant figures.

10 To calculate the area of this circle, Eric performed the following calculations:



Diameter of circle $= \sqrt{6^2 + 6^2} = \sqrt{72} \approx 8.49$ cm {3 significant figures}
$\therefore$ radius of circle $= 8.49 \div 2 \approx 4.25$ cm {3 significant figures}
$\therefore$ area of circle $= \pi \times 4.25^2 \approx 56.7$ cm^2 {3 significant figures}

a Explain why Eric's final answer of 56.7 cm^2 may not be accurate to 3 significant figures.

b Find the correct area of the circle, rounded to 3 significant figures.

B APPROXIMATIONS

A fast way of estimating a calculation is to perform a **one figure approximation**:

- Leave single digit numbers as they are.
- Round all other numbers to one significant figure.
- Perform the calculation.

Example 4 **Self Tutor**

Estimate:

a 872×52 **b** $61\,812 \div 384$ **c** 4.37×0.482

a 872×52
$\approx 900 \times 50$
$\approx 45\,000$

b $61\,812 \div 384$
$\approx 60\,000 \div 400$
$\approx 600 \div 4$
≈ 150

c 4.37×0.482
$\approx 4 \times 0.5$
≈ 2

EXERCISE 1B

1 Estimate using a one figure approximation:

a 32×6 **b** 58×7 **c** 81×30
d 207×3 **e** 487×50 **f** 6117×4
g 48×23 **h** 61×42 **i** 103×47
j 3125×18 **k** 422×307 **l** 3818×27
m 2.7×1.15 **n** 5.36×0.68 **o** 28.37×6.13

2 Estimate using a one figure approximation:

a $86 \div 3$ **b** $64 \div 5$ **c** $512 \div 21$
d $610 \div 43$ **e** $4182 \div 19$ **f** $78\,638 \div 82$
g $318 \div 62$ **h** $47\,320 \div 193$ **i** $0.628 \div 3$
j $46.1 \div 5.2$ **k** $631.7 \div 0.29$ **l** $18.7 \div 3.86$

3 Estimate, using one figure approximations, the cost of:

a 8 kg of apples at €2.80/kg **b** 3 airline tickets at \$213 each
c 7 theatre tickets at \$87.30 each **d** 55 L of fuel at £1.49/L.

4 Estimate using one figure approximations:

a the distance travelled if Brodie drives for 4.2 hours at 63 $\text{km}\,\text{h}^{-1}$
b the number of days in 14 years
c the average weight carried per truck if 423 tonnes of cargo is divided equally between 18 trucks
d the total pay for a part-time worker who works on average 18.2 hours per week for 12 weeks, and earns €21.50 per hour.

DISCUSSION

Suppose there are 19 biscuits in a packet, and 32 packets are packed in a carton. Using a one figure approximation, we estimate there are $20 \times 30 = 600$ biscuits in the carton. This is close to the actual number of 608 biscuits.

Will one figure approximations always be this close to the actual value? What is it about these particular numbers that makes the estimate close? By comparison, you might consider 24 packets each containing 14 biscuits.

ACTIVITY 1 ESTIMATION

An **estimation** is a value which has been found by judgement or prediction instead of carrying out a more accurate measurement.

We often estimate when it is difficult for us to measure. Whenever we do this it is important to round numbers to sensible accuracy.

In order to make reasonable estimations we often appeal to our previous experience.

Work in pairs for this Activity.

What to do:

PRINTABLE WORKSHEET

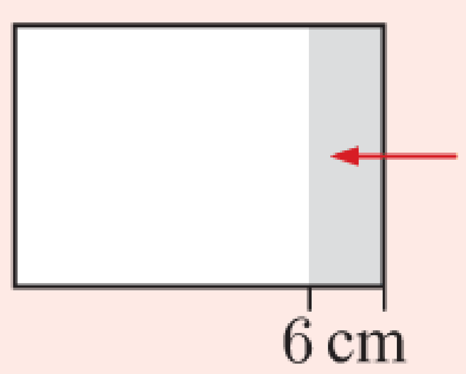
1 For each of the following objects:

- **i** Each person should *estimate* the quantity without measuring.
- **ii** As a pair, take measurements and hence find a good approximation for the quantity.
- **iii** Discuss the errors in your estimates.

a the height of your classroom doorway
b the length of your classroom
c the temperature in your classroom
d the perimeter of the basketball court
e the area of the basketball court
f the capacity of a large container.

2 **a** *Estimate* the area of the school car park without measuring.
b Discuss how you could determine an upper bound for what the area could be.
c Discuss how you could obtain a more accurate approximation for the area.

3 **a** Measure in mm the length and width of a sheet of 80 gsm A4 photocopying paper.
b What is its area in m^2 and how many sheets make up 1 m^2?
c 80 gsm means 80 grams per square metre. What is the mass of one sheet of A4 paper?
d What is the approximate mass of this part of the sheet?



e Crumple the 6 cm strip into your hand and feel how heavy it is.

C ERRORS IN MEASUREMENT

When we take measurements, we are usually reading some sort of scale.

The scale of a ruler may have centimetres marked on it, but when we measure the length of an object, it is likely to fall between two divisions. We **approximate** the length of the object by recording the value at the nearest centimetre mark. In doing so our answer may be inaccurate by up to a half a centimetre.

A measurement is accurate to $\pm\frac{1}{2}$ of the smallest division on the scale.

ACTIVITY 2 MEASURING DEVICES

Examine a variety of measuring instruments at school and at home. Make a list of the names of these instruments, what they measure, what their units are, and the degree of accuracy to which they can measure.

For example:

A ruler measures length. In the Metric System it measures in centimetres and millimetres, and can measure to the nearest millimetre. Its accuracy is $\pm\frac{1}{2}$ mm.

Example 5 Self Tutor

Ling uses a ruler to measure the length l of her pencil case. She records the length as 18.7 cm.

Find the range of values in which the length may lie.

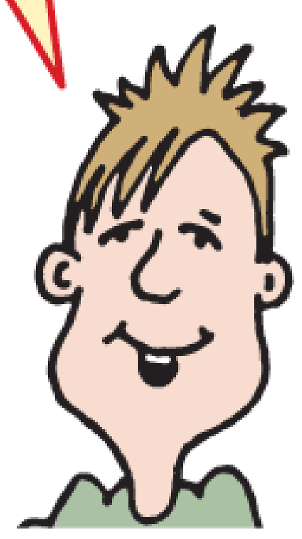
18.7 cm is 187 mm, so the measuring device must be accurate to the nearest half mm.

$\therefore$ the range of values is $187 \pm \frac{1}{2}$ mm

The actual length is in the range $186\frac{1}{2}$ mm to $187\frac{1}{2}$ mm.

$\therefore$ $18.65 \text{ cm} < l < 18.75 \text{ cm}$.

We do not include the endpoints of the interval because the length can never be *exactly* these values.



EXERCISE 1C

1 State the accuracy of the following measuring devices:

- **a** a tape measure marked in cm
- **b** a measuring cylinder with 1 mL graduations
- **c** a beaker with 100 mL graduations
- **d** a set of scales with marks every 500 g
- **e** a thermometer with marks every 0.1°C.

2 Roni checks his weight every week using scales with 1 kg graduations. This morning he recorded a weight of 68 kg. In what range of values does Roni's actual weight w lie?

3 Find the range of possible values corresponding to the following measurements:

- **a** 27 mm
- **b** 38.3 cm
- **c** 4.8 m
- **d** 1.5 kg
- **e** 25 g
- **f** 3.75 kg

4 Tom's digital thermometer said his temperature was 36.4°C. In what range of values did Tom's actual temperature T lie?

5 Joanne's exercise watch displays the distance she has run to 3 significant figures. State the *least* distance Joanne could have run, if the watch displays:

- **a** 1.06 km
- **b** 9.72 km
- **c** 10.1 km

Comment on the accuracy of the watch.

6 Four students measured the width of their classroom using the same tape measure. The measurements were 6.1 m, 6.4 m, 6.0 m, 6.1 m.

- **a** Which measurement is likely to be incorrect? Explain your answer.
- **b** What answer would you give for the width of the classroom? Explain your answer.
- **c** What graduations do you think were on the tape measure?

7 Hasan has many lengths of rope. He has measured each length to be 2.4 m.

- **a** In what range of values does the actual length of a rope l lie?
- **b** If Hasan carefully places n of his ropes end to end, in what range of values will the total length of rope L lie?

8 In the 800 m race at the sports carnival, the times recorded for Jiao and Liang were 2 min 8 s and 2 min 13 s respectively.
Find the range of possible values for the time t by which Jiao beat Liang.

Example 6 **Self Tutor**

A rectangular board was measured as 78 cm by 24 cm. Find the boundary values for its perimeter.

The length of the board could be from $77\frac{1}{2}$ cm to $78\frac{1}{2}$ cm.

The width of the board could be from $23\frac{1}{2}$ cm to $24\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the perimeter is $2 \times 77\frac{1}{2} + 2 \times 23\frac{1}{2} = 202$ cm

and the upper boundary of the perimeter is $2 \times 78\frac{1}{2} + 2 \times 24\frac{1}{2} = 206$ cm

The perimeter is between 202 cm and 206 cm, which is 204 ± 2 cm.

The **boundary values** are the smallest and largest values that the actual value could be.

9 A rectangular bath mat was measured as 86 cm by 38 cm. Find the boundary values of its perimeter.

10 A rectangular garden bed is measured as 252 cm by 143 cm. Find the range of possible values for the total length of edging l required to border the garden bed.

Example 7 **Self Tutor**

A paving brick is measured as $18 \text{ cm} \times 10 \text{ cm}$. What are the boundary values for its actual area?

The length of the paving brick could be from $17\frac{1}{2}$ cm to $18\frac{1}{2}$ cm.

The width of the paving brick could be from $9\frac{1}{2}$ cm to $10\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $17\frac{1}{2} \times 9\frac{1}{2} = 166.25 \text{ cm}^2$

and the upper boundary of the area is $18\frac{1}{2} \times 10\frac{1}{2} = 194.25 \text{ cm}^2$.

The area is between 166.25 cm^2 and 194.25 cm^2.

This could also be represented as $\dfrac{166.25 + 194.25}{2} \pm \dfrac{194.25 - 166.25}{2} \text{ cm}^2$
which is $180.25 \pm 14 \text{ cm}^2$.

11 A rectangle is measured to be 6 cm by 8 cm. Find:

a the largest area it could have

b the smallest area it could have.

12 Find the boundary values for the actual area of a glass window measured as 42 cm by 26 cm.

13 The base of a triangle is measured as 9 cm and its height is measured as 8 cm. In what range of values does its actual area A lie?

14 Find the boundary values for the actual volume of a box measuring 4 cm by 8 cm by 6 cm.

15 Find the range of values in which the actual volume V of a house brick measuring 21.3 cm by 9.8 cm by 7.3 cm must lie.

16 A cylinder is measured to have radius 5 cm and height 15 cm. Find the boundary values for the cylinder's volume.

17 A cone is measured to have radius 8.4 cm and height 4.6 cm. Find the boundary values for the cone's volume.

18 Eko measures the diameter of a ball to be 18.2 cm. Do you expect the rounding in Eko's measurement to have more effect on a calculation of the ball's surface area, or a calculation of its volume? Explain your answer.

19 Rachel measures the base side lengths of a square-based pyramid to be 4.6 cm, and its height to be 5.2 cm. Find the boundary values for the pyramid's:

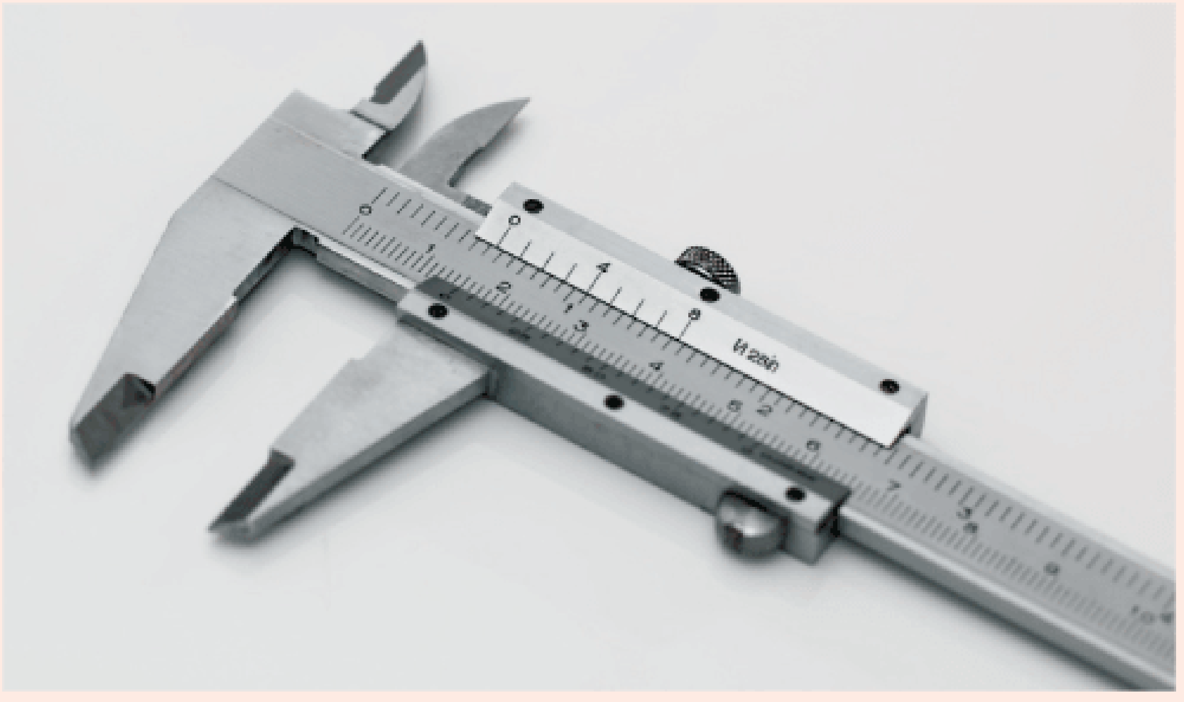
a volume

b surface area.

RESEARCH — VERNIER SCALES

A **vernier scale** is used to measure the length of objects with a high degree of accuracy.

Research how vernier scales work.



D ABSOLUTE AND PERCENTAGE ERROR

Whenever we measure a quantity there is almost always a difference between our measurement and the actual value. We call this difference the **error**.

The *size* or *magnitude* of the error, whether the measured or estimated value is too high or too low, is called the **absolute error**.

If the actual or exact value is V_E and the approximate value is V_A then the

$$\textbf{absolute error} = |V_A - V_E|$$

Error is often expressed as a percentage of the exact value:

$$\textbf{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

Example 8 **Self Tutor**

Suppose you estimate a fence's length to be 70 m, whereas its true length is 78.3 m. Find, correct to one decimal place, the:

a absolute error **b** percentage error.

a absolute error $= |V_A - V_E|$
$= |70 - 78.3|$ m
$= |-8.3|$ m
$= 8.3$ m

b percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{8.3}{78.3} \times 100\%$
$\approx 10.6\%$

EXERCISE 1D

1 Find the absolute error and percentage error in rounding:
  a the yearly profit of €1 367 540 made by a company to €1.37 million
  b a population of 31 467 people to 31 000 people
  c a retail sales figure of \$458 110 to \$460 000
  d the number of new cars sold in a year from 2811 to 3000.

2 Find the absolute error and percentage error if you estimate:
  a the mass of a brick to be 5 kg when its actual mass is 6.238 kg
  b the perimeter of a property to be 100 m when its actual perimeter is 97.6 m
  c the capacity of a container to be 20 L when its actual capacity is 23.8 L
  d the time to write a computer program to be 50 hours when it actually takes 72 hours.

Example 9 **Self Tutor**

Alan wants to lay carpet on his 4.2 m by 5.1 m lounge room floor. He estimates the area of the lounge room by rounding each measurement to the nearest metre.

a Find Alan's estimate of the lounge room area.
b The carpet costs \$39 per square metre. Find the cost of the carpet using Alan's estimate of the area.
c Find the actual area of Alan's lounge room.
d Find the percentage error in Alan's estimation.
e Will Alan have enough carpet to cover his lounge room? How should he have rounded the measurements?

a Area ≈ 4 m $\times$ 5 m
≈ 20 m^2

b Cost $= 20 \times \$39$
$= \$780$

c Actual area $= 4.2 \times 5.1$
$= 21.42$ m^2

d Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{|20 - 21.42|}{21.42} \times 100\%$
$\approx 6.63\%$

e Alan only has 20 m^2 of carpet, so he will not have enough to cover his lounge room. He should have rounded the measurements *up* to make sure he had enough carpet.

3 Jon's apartment is a 10.3 m by 9.7 m rectangle.

a Find the actual area of the apartment.

b Estimate the floor area by rounding each length to the nearest metre.

c Find the absolute error and percentage error in your estimate.

4 The cost of freight for a parcel is dependent on its volume. Justine lists the dimensions of a parcel as 24 cm by 15 cm by 9 cm on the consignment note. The actual dimensions are 23.9 cm $\times$ 14.8 cm $\times$ 9.2 cm.

a Calculate the actual volume of the parcel.

b Estimate the volume using the dimensions given on the consignment note.

c Find the absolute error and percentage error in the calculation using the consignment note.

5 A hotel wants to cover an 8.2 m by 9.4 m rectangular courtyard with synthetic grass. The manager estimates the area by rounding each measurement to the nearest metre.

a Find the manager's estimate of the area.

b The synthetic grass costs \$85 per square metre. Find its cost using the manager's estimate.

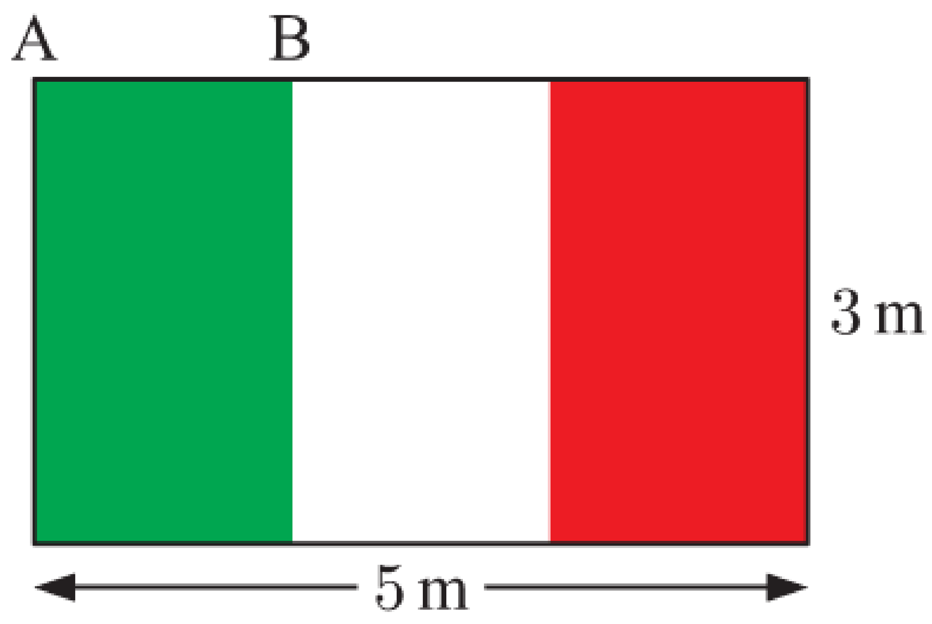
c Find the actual area of the rectangle.

d Calculate the percentage error in the manager's estimate.

e Will the hotel have enough grass to cover the courtyard?

f Find the cost of the grass if the manager had rounded each measurement *up* to the next metre.

6 The Italian flag has three different regions of equal size. Consider the flag alongside.



a Find the area of the green section exactly.

b Find the length AB correct to 1 decimal place.

c Use your rounded value in **b** to estimate the area of the green section.

d Find the percentage error in your estimate.

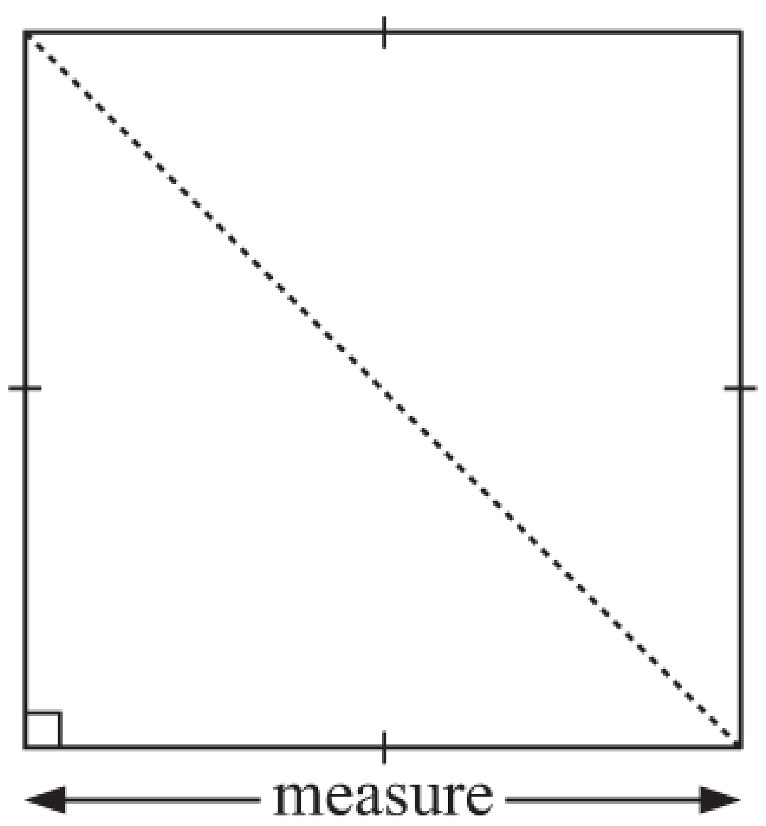
7 Hemi estimates that she can drive at an average speed of 70 km h^{-1} between her house and the beach, 87 km away. One particular journey took her 1 hour and 20 minutes.

a Calculate Hemi's average speed for this journey.

b Find the absolute error and percentage error in her estimate.

8 The sentence below is translated from an ancient Indian text, the *Śulba sūtra*:

> *The measure is to be increased by its third and this (third) again by its own fourth less the thirtyfourth part (of that fourth); this is (the value of) the diagonal of a square (whose side is the measure).*



a Use Pythagoras' theorem to show that the diagonal of a square is $\sqrt{2}$ times the measure of its side.

b Hence show that the text estimates the value of $\sqrt{2}$ as $\frac{577}{408}$.

c Find the percentage error for this estimate, giving your answer in scientific notation.

Example 10 — Self Tutor

The side length of a square is measured as 22 cm, rounded to the nearest centimetre.

a Use this measurement to estimate the area of the square.

b Find the boundary values for the area of the square.

c Hence find the maximum percentage error in the estimate.

a Area $\approx$ 22 cm $\times$ 22 cm $\approx$ 484 cm^2

b The side length of the square could be from $21\frac{1}{2}$ cm to $22\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $21\frac{1}{2} \times 21\frac{1}{2} = 462.25$ cm^2

and the upper boundary of the area is $22\frac{1}{2} \times 22\frac{1}{2} = 506.25$ cm^2.

c If the exact area V_E was 462.25 cm^2, the

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|484 - 462.25|}{462.25} \times 100\% \approx 4.71\%$$

If the exact area V_E was 506.25 cm^2, the

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|484 - 506.25|}{506.25} \times 100\% \approx 4.40\%$$

$\therefore$ the maximum percentage error in the estimate $\approx$ 4.71%.

9 The side lengths of a rectangle are measured as 2.3 m and 1.4 m, rounded to one decimal place.

a Use these measurements to estimate the area of the rectangle.

b Find the boundary values for the area of the rectangle.

c Hence find the maximum percentage error in the estimate.

10 Jasper measured the dimensions of his cylindrical can of tuna. He found the radius was 4 cm and the height was 5 cm, rounded to the nearest centimetre.

a Use these measurements to estimate the volume of the can.

b Find the boundary values for the volume of the can.

c Hence find the maximum percentage error in the estimate.

11 Carolina completed a 250 km car trip (rounded to the nearest km). The GPS in her car displays the average speed for the trip as 56.8 km h^{-1} (rounded to 1 decimal place).

a Estimate the time it took Carolina to complete the trip.

b Find the maximum possible:

i absolute error

ii percentage error in the estimate.

DISCUSSION

Why is it important to understand errors?

What things can go wrong if people measure inaccurately or round off incorrectly?

You may wish to consider the examples in the previous Exercise, and also cases such as:

- If a pilot flies off-course by 0.1° for 1000 km, how far away from his target will he be?
- What happens to a patient if a doctor injects 2 mg of a drug instead of 2 μg?

REVIEW SET 1A

1 Round off to the nearest 100:

a 7423 **b** 32 191 **c** 10 543 **d** 408 961

2 The mathematical constant $e \approx 2.718\,281\,828\,459\ldots$. Round this value to:

a 2 decimal places **b** 5 decimal places **c** 8 decimal places.

3 Evaluate the following, rounding your answers to 3 significant figures:

a $\sqrt{27}$ **b** $\dfrac{2.3 \times 9.4}{1.3}$ **c** 0.307^3

4 Estimate using a one figure approximation:

a 47×7 **b** 89×16 **c** $267 \div 48$

5 **a** Estimate $5877 \div 32$ using a one figure approximation.

b Do you think the exact value of $5877 \div 32$ is greater or less than your estimate? Explain your answer.

6 A triangular garden is measured to have sides of length 8 m, 12 m, and 14 m, rounded to the nearest metre. Find the range of possible values for the perimeter P of the garden.

7 **a** How accurate is a tape measure marked in cm?

b Find the range of possible values for a measurement of 36 cm.

c If the sides of a square are measured to be 36 cm, in what range of values must the actual area A lie?

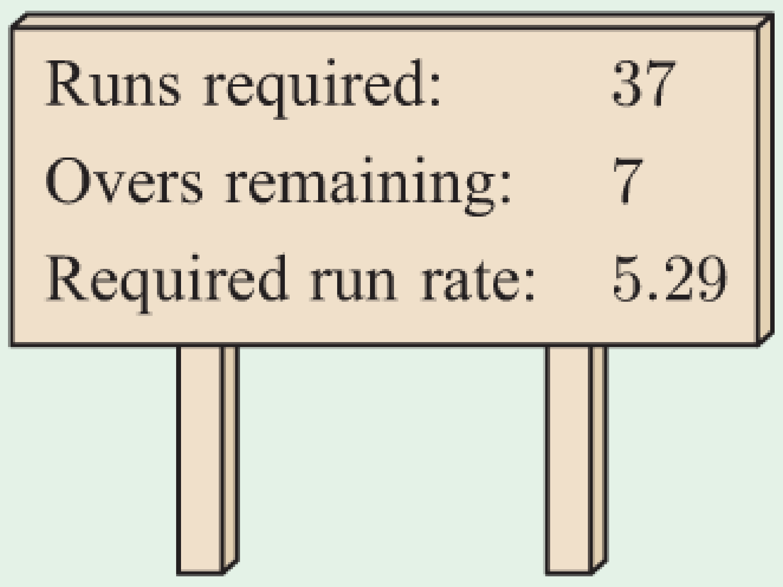
8 Find the absolute error and percentage error if you:

a estimate your credit card balance to be \$2000 when it is \$2590

b round 26.109 cm to 26 cm

c estimate the number of people at a music festival to be 4000 when there are 4386 in attendance.

9 In limited overs cricket, teams must score as many runs as possible within a particular number of overs. For the team batting second, the *required run rate* is found by dividing the number of *runs required* by the number of *overs remaining*.
On this scoreboard, the required run rate has been rounded to 2 decimal places. Find the percentage error in the calculation.



10 A circular gazebo has diameter 2.8 m. In calculating the area of the gazebo, the diameter is rounded to the nearest metre.

a Find the actual area of the gazebo.

b Find the calculated area of the gazebo.

c Find the absolute error and percentage error in the calculation.

REVIEW SET 1B

1 Round 74 815 to:

a the nearest 10 **b** the nearest 100 **c** the nearest 1000.

2 A self-service car wash received 248 customers in the last 13 days. Calculate the average number of customers per day, rounded to 1 decimal place.

3 Evaluate $\sqrt{5.4 \times 7.6}$, rounding your answer to:

a 2 decimal places **b** 4 significant figures.

4 Estimate using a one figure approximation:

a 28×74 **b** 5.84×8.09 **c** $57.9 \div 23.5$

5 Lucinda and Daniel bought 88 wedding invitations costing £4.90 each.

a Find the exact total cost of the invitations.

b Use a one figure approximation to estimate the total cost of the invitations.

c Find the percentage error in your estimate.

6 Dafne competed in the 100 m sprint at her school sports carnival. Her time for the race, rounded to 1 decimal place, was recorded as 14.9 seconds.
Find the range of possible values for:

a her time t seconds **b** her average speed s m s^{-1}.

7 A box has dimensions 5 cm by 7 cm by 10 cm, rounded to the nearest centimetre. Find the boundary values for the surface area A of the box.

8 Edward measures the width and height of a television screen to the nearest 10 cm. The width is approximately 150 cm and the height is approximately 90 cm.

a Use Edward's measurements to estimate the length of the diagonal of the screen.

b Given that the diagonal has length 177.8 cm, find the absolute error and percentage error of the estimate.

9 An architect designs a support beam to be $\sqrt{5}$ metres long. The builder working from the architect's plans converts this length to a decimal number.

a Write down the length of the support beam correct to the nearest:

i metre **ii** centimetre **iii** millimetre.

b For each answer in **a**, write down how many significant figures were specified.

c The architect insists that there be no more than 1% error. Which of the approximations in **a**, if any, will satisfy this?

10 In order to quickly estimate areas and volumes, people use various approximations for π.
Find the percentage error if π is approximated by:

a 3 **b** 3.1 **c** 3.14 **d** $\frac{22}{7}$ **e** $\frac{355}{113}$

11 The radius of a circle is measured as 3.5 cm, rounded to 1 decimal place.

a Use this measurement to estimate the area of the circle.

b Find the boundary values for the area of the circle.

c Find the maximum percentage error in the estimate.

EXERCISE 1A.1

1 **a** 90 **b** 80 **c** 90 **d** 130 **e** 160 **f** 100 **g** 640 **h** 1820 **i** 700 **j** 3050

2 **a** 200 **b** 300 **c** 3800 **d** 4000 **e** 26 300

3 **a** 8000 **b** 4000 **c** 19 000 **d** 20 000 **e** 115 000

4 **a** 8850 m **b** 32 000 km^2 **c** 4 750 000 people **d** 85 500 people **e** 7 700 000 km^2 **f** 5400 kg **g** 17 000 km **h** 400 000 km **i** 428 000 000 people

EXERCISE 1A.2

1 **a** 6.2 **b** 6.18 **c** 3.3 **d** 17.40 **e** 2.132 **f** 0.2 **g** 0.10 **h** 102.38

2 9.6 s **3** 1.44 m **4** 0.01 cm

5 **a** 3.1 **b** 3.142 **c** 3.1416

6 **a** 0.3 **b** 0.26 **c** 0.263 158

7 **a** ≈ 1.414 **b** ≈ 2.236 **c** ≈ 4.796 **d** ≈ 1.587 **e** ≈ -2.466 **f** ≈ 7.663

8 **a** 499.32 **b** 228.84 **c** 9.11 **d** 31.75 **e** 26.67 **f** 0.88 **g** 7.41 **h** 5.93 **i** 0.48

9 1.74 goals per game

10 While 2.45 m $\approx$ 2.5 m is correct, Wang should have used the original value of 2.45 m to round to the nearest integer, so 2.45 m $\approx$ 2 m.

EXERCISE 1A.3

1 **a** 130 **b** 8300 **c** 2.6 **d** 0.013 **e** 160 000 **f** 1.1 **g** 4000 **h** 6.6

2 **a** 83 100 **b** 10 000 **c** 0.105 **d** 31.7 **e** 70.7 **f** 4.00 **g** 0.0367 **h** 20.0

3 **a** 16.38 **b** 438.2 **c** 6 874 000 **d** 0.028 89

4 **a** 100 000 people **b** 96 000 people **c** 96 300 people

5 **a** ≈ 2.65 **b** ≈ 6.28 **c** ≈ 2.12 **d** $\approx 1\,970\,000$ **e** ≈ 0.932 **f** ≈ 4.39 **g** ≈ 1.79 **h** ≈ 5.73

6 900 seats **7** 256 m^2 **8** **a** \$188 **b** \$188.06

9 1 200 000 m

10 **a** Eric has rounded each answer to 3 significant figures before using it in the next calculation, rather than using exact values.
b ≈ 56.5 cm^2

EXERCISE 1B

1 **a** 180 **b** 420 **c** 2400 **d** 600 **e** 25 000 **f** 24 000 **g** 1000 **h** 2400 **i** 5000 **j** 60 000 **k** 120 000 **l** 120 000 **m** 3 **n** 3.5 **o** 180

2 **a** 30 **b** 12 **c** 25 **d** 15 **e** 200 **f** 1000 **g** 5 **h** 250 **i** 0.2 **j** 10 **k** 2000 **l** 5

3 **a** €24 **b** \$600 **c** \$630 **d** £60

4 **a** 240 km **b** 4000 days **c** 20 tonnes **d** €4000

EXERCISE 1C

1 **a** $\pm\frac{1}{2}$ cm **b** $\pm\frac{1}{2}$ mL **c** ± 50 mL **d** ± 250 g **e** ± 0.05°C

2 67.5 kg $< w <$ 68.5 kg

3 **a** 26.5 mm to 27.5 mm **b** 38.25 cm to 38.35 cm **c** 4.75 m to 4.85 m **d** 1.45 kg to 1.55 kg **e** 24.5 g to 25.5 g **f** 3.745 kg to 3.755 kg

4 36.35°C $< T <$ 36.45°C

5 **a** 1.055 km **b** 9.715 km **c** 10.05 km
The watch displays the correct distance accurate to ± 0.005 km.

6 **a** 6.4 m, as it is the most different from the other measurements.
b 6.05 m, as it is in the range of values for a measurement of 6.0 m or 6.1 m.
c 10 cm

7 **a** 2.35 m $< l <$ 2.45 m **b** $2.35n$ m $< L < 2.45n$ m

8 4 s $< t <$ 6 s **9** 248 ± 2 cm **10** 788 cm $< l <$ 792 cm

11 **a** 55.25 cm^2 **b** 41.25 cm^2

12 1092.25 ± 34 cm^2 **13** 36.125 ± 4.25 cm^2

14 196.5 ± 52.125 cm^3 **15** $\approx 1523.90 \pm 21.79$ cm^3

16 $\approx 1197.73 \pm 275.28$ cm^3 **17** 339.95 ± 7.74 cm^3

18 volume $= \frac{4}{3}\pi r^3$, surface area $= 4\pi r^2$
The rounding will have more effect on the volume, as the error is multiplied through 3 times rather than twice.

19 **a** 36.69 ± 1.15 cm^3 **b** 73.48 ± 1.54 cm^2

EXERCISE 1D

1 **a** €2460, $\approx 0.180\%$ **b** 467 people, $\approx 1.48\%$ **c** \$1890, $\approx 0.413\%$ **d** 189 cars, $\approx 6.72\%$

2 **a** 1.238 kg, $\approx 19.8\%$ **b** 2.4 m, $\approx 2.46\%$ **c** 3.8 L, $\approx 16.0\%$ **d** 22 hours, $\approx 30.6\%$

3 **a** 99.91 m^2 **b** 100 m^2 **c** 0.09 m^2, $\approx 0.0901\%$

4 **a** 3254.224 cm^3 **b** 3240 cm^3 **c** 14.224 cm^3, $\approx 0.437\%$

5 **a** 72 m^2 **b** \$6120 **c** 77.08 m^2 **d** $\approx 6.59\%$ **e** no **f** \$7650

6 **a** 5 m^2 **b** 1.7 m **c** 5.1 m^2 **d** 2%

7 **a** 65.25 km h^{-1} **b** 4.75 km h^{-1}, $\approx 7.28\%$

8 **b** **Hint:** Let the measure have the value 1.
c $\approx 1.50 \times 10^{-4}$ %

9 **a** ≈ 3.22 m^2 **b** $3.0375 < A < 3.4075$ **c** $\approx 6.01\%$

10 **a** ≈ 251 cm^3 **b** $173 < V < 350$ **c** $\approx 45.1\%$

11 **a** ≈ 4.40 hours ($\approx$ 4 h 24 min 5 s)
b **i** ≈ 45.7 s **ii** $\approx 0.287\%$

REVIEW SET 1A

1 **a** 7400 **b** 32 200 **c** 10 500 **d** 409 000

2 **a** 2.72 **b** 2.718 28 **c** 2.718 281 83

3 **a** ≈ 5.20 **b** ≈ 16.6 **c** ≈ 0.0289

4 **a** 350 **b** 1800 **c** 6

5 **a** 200
b Less, as in our one figure approximation we increased the numerator and decreased the denominator.

6 32.5 m $< P <$ 35.5 m

7 **a** accurate to $\pm\frac{1}{2}$ cm **b** 35.5 cm to 36.5 cm
c 1260.25 cm^2 $< A <$ 1332.25 cm^2

8 **a** \$590, $\approx 22.8\%$ **b** 0.109 cm, $\approx 0.417\%$
c 386 people, $\approx 8.80\%$

9 $\approx 0.0811\%$

10 **a** $\pi \times 1.4^2 \approx 6.16$ m^2 **b** $\pi \times 1.5^2 \approx 7.07$ m^2
c ≈ 0.911 m^2, $\approx 14.8\%$

REVIEW SET 1B

1 **a** 74 820 **b** 74 800 **c** 75 000

2 19.1 customers per day **3** **a** 6.41 **b** 6.406

4 **a** 2100 **b** 48 **c** 3

5 a £431.20 b £450 c $\approx 4.36\%$
6 a $14.85 \text{ s} < t < 14.95 \text{ s}$ b $6.66 \text{ m s}^{-1} < s < 6.77 \text{ m s}^{-1}$
7 $267.5 \text{ cm}^2 < A < 355.5 \text{ cm}^2$
8 a ≈ 175 cm b ≈ 2.87 cm, $\approx 1.61\%$
9 a i 2 m ii 2.24 m iii 2.236 m
b i 1 ii 3 iii 4 c 2.24 m and 2.236 m
10 a $\approx 4.51\%$ b $\approx 1.32\%$ c $\approx 0.0507\%$
d $\approx 0.0402\%$ e $\approx 8.49 \times 10^{-6}\,\%$
11 a $\approx 38.5 \text{ cm}^2$ b $37.4 < A < 39.6 \text{ cm}^2$ c $\approx 2.92\%$

EXERCISE 2A

1 a $232 b $13 920 c $1920
2 a £282.60 b £10 173.60 c £673.60
3 a $1036.80 b $2610 4 €1903.28
5 a Balance Bank b Cash Credit Union
c If Becky is able to afford the larger repayments, she should choose Cash Credit Union as she will pay less interest.
6 b $27 509.01
c i $200 ii $186.21
The balance of the loan is less in month 6 which means the interest paid will also be less.
d i $4.02 ii $6497.40 ($6498 using technology)
e The monthly repayment was rounded up, so every month the payments have reduced the balance by a little extra.
7 a $148.64 b $729.28
8 a £490.61 b £15 598.73
9 a $8500 b i $518.58 ii $1871.60 iii $5492.57
10 a 6.30% p.a. b $395.65
11 a i $789.19 ii $512.92 iii $395.92
b The 3 year loan charges the least interest of $3410.84 as more is paid off each month and therefore less interest is charged.
12 a €386.90 b €5214 c €10 169.13
d Ally pays more interest in the first $2\frac{1}{2}$ years than in the second $2\frac{1}{2}$ years.
13 a $1827.33 b $188 559.20 c $162 745.03
d i $3165.28 ii $159 196.40 iii $29 362.80

EXERCISE 2B

1 a 25 years 4 months b $3693.84
2 a €3163.24 b €413 500.41
3 a 12 years 7 months b 3 years 1 month longer
4 a £5614.06
b No, he can only afford to spend £5614.06 per month. Otherwise his money will run out before he turns 84.
5 a $1 094 748.09
b i $8600.27 ii 11 years 11 months
6 a £11 512.29 b £394 007.62 c £1312.64
7 a $\$4500 \times 12 \times 20 = \$1\,080\,000$
b Maggie will earn interest on the money in the annuity account as she makes her regular withdrawals.
c $618 117.53
8 The money will last forever.
9 a 7.19% b i 2 years 10 months ii €679.24
10 a $5121.03 b $322 605.07 c $6708.44

REVIEW SET 2A

1 a $455.43 b $27 325.80 c $4325.80
2 a €157.24 b €1086.93
3 a $2884.74
b Total repayments $= \$2884.74 \times 12 \times 25 = \$865\,422$
Total interest charged $= \$865\,422 - \$410\,000 = \$455\,422$
4 a 8 years 7 months b $2996.23
5 a €7861.43 b 14 years 3 months c €727 698.90
6 a $799 813.28 b $314 877.35

REVIEW SET 2B

1 a $279.08 b $1395.84
2 a $17 500 b $1260.97 c $2675.52 d $9347.67
3 a i 11 742.52 pesos ii 8286.45 pesos
b The 4 year loan charges the least interest of 63 640.96 pesos as more is paid off each month and therefore less interest is charged.
4 a An annuity fund is an investment where an individual makes a lump-sum deposit, and then makes regular *withdrawals* from the account. We have previously considered compound interest investments that make regular *deposits* into an account.
b Diane is technically correct, but she will be able to withdraw more than £2000 per month since the money in the fund will earn interest.
c £3167.02
5 a €2467.29 b €448.52
6 a 4.90% p.a. b 6 years 7 months

EXERCISE 3A

1 a, d, and e are functions, since in each case, no two different ordered pairs have the same x-coordinate.
2 a Is a function, since for any value of x there is at most one value of y.
b Is a function, since for any value of x there is at most one value of y.
c Is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2$, $y = \pm\sqrt{5}$.
3 a function b function c function
d not a function e not a function f function
g function h not a function
4 Not a function as a 2 year old child could pay $0 or $20.
5 No, because a vertical line (the y-axis) would cut the relation more than once.
6 No. A vertical line is not a function. It will not pass the "vertical line" test.
7 a $y^2 = x$ is a relation but not a function.
$y = x^2$ is a function (and a relation).
$y^2 = x$ has a horizontal axis of symmetry (the x-axis).
$y = x^2$ has a vertical axis of symmetry (the y-axis).
Both $y^2 = x$ and $y = x^2$ have vertex $(0, 0)$.
$y^2 = x$ is a rotation of $y = x^2$ clockwise through 90° about the origin *or* $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.
b i The part of $y^2 = x$ in the first quadrant.
ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.
8 a Both curves are functions since any vertical line will cut each curve at most once.
b $y = \sqrt[3]{x}$

EXERCISE 3B

1 a 2 b 8 c -1 d -13 e 1
2 a 2 b 2 c -16 d -68 e $\frac{17}{4}$