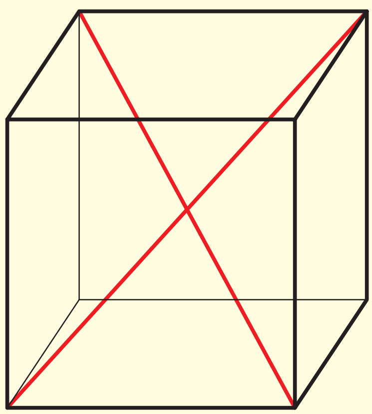
13 Vectors

In this chapter you will learn:

- to use vectors to represent displacements and positions in two and three dimensions
- to perform algebraic operations with vectors, and understand their geometric interpretation
- to calculate the distance between two points
- to use vectors to calculate the angle between two lines
- to use vectors to find areas of parallelograms and triangles
- to use two new operations on vectors, called scalar product and vector product.

Introductory problem

What is the angle between the diagonals of a cube?



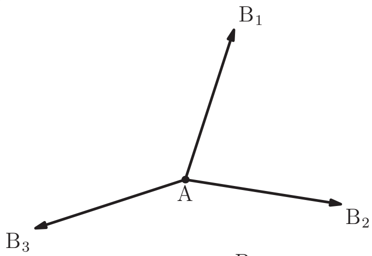
Solving problems in three dimensions can be difficult, as two-dimensional diagrams cannot always make clear what is happening. Vectors are a useful tool to describe geometrical properties using equations, which can often be analysed more easily. In this chapter we will develop techniques to calculate angles, distances and areas in two and three dimensions. We will apply those techniques to further geometrical problems in chapter 14.

13A Positions and displacements

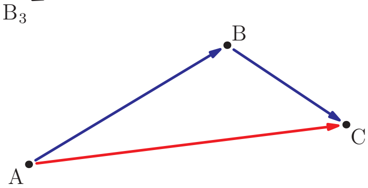
You may know from physics that **vectors** are used to represent quantities which have both magnitude (size) and direction, such as force or velocity. Vector quantities are different from **scalar** quantities which are fully described by a single number. In pure mathematics, vectors are used to represent displacements from one point to another, and so to describe geometrical figures.

If there is a fixed point A and a point B is 10 cm away from it, this information alone does not tell you where point B is.

Vectors are an example of abstraction in mathematics: a single concept that can be applied to many different situations. Force, velocity and displacements appear to have very little in common, yet they can all be described and manipulated using the rules of vectors. In the words of the French mathematician and physicist Henry Poincaré (1854–1912): 'Mathematics is the art of giving the same name to different things'.

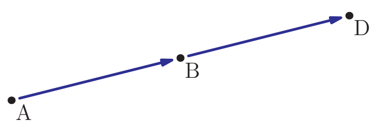


The position of B relative to A can be represented by the **vector displacement** $\overrightarrow{AB}$. The vector contains both distance and direction information; it describes a way of getting from A to B.

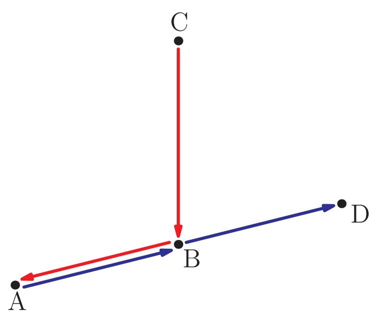


If we now add a third point, C, then there are two ways of getting from A to C: either directly, or via B. To express the second possibility using vectors, we use the addition sign to represent moving from A to B followed by moving from B to C: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Always remember that a vector represents a way of getting from one point to another, but it does not tell us anything about the actual position of the starting and the end points.



If getting from B to D involves moving the same distance and in the same direction as getting from A to B, then $\overrightarrow{BD} = \overrightarrow{AB}$.



To return from the end point to the starting point, we use the minus sign and so $\overrightarrow{BA} = -\overrightarrow{AB}$.

We can also use the subtraction sign with vectors: $\overrightarrow{CB} - \overrightarrow{AB} = \overrightarrow{CB} + \overrightarrow{BA}$

To get from A to D we need to move in the same direction, but twice as far, as in getting from A to B. We can express this by writing $\overrightarrow{AD} = 2\overrightarrow{AB}$ or, equivalently, $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AD}$.

It is convenient to give vectors letters, as we do with variables in algebra. To emphasise that something is a vector, rather than a scalar (number) we use either bold type or an arrow on top. When writing by hand, we use underlining instead of bold type. For example, we can denote vector $\overrightarrow{AB}$ by $\boldsymbol{a}$ (or $\vec{a}$). Then in the diagrams above, $\overrightarrow{BD} = \boldsymbol{a}$, $\overrightarrow{BA} = -\boldsymbol{a}$ and $\overrightarrow{AD} = 2\boldsymbol{a}$.

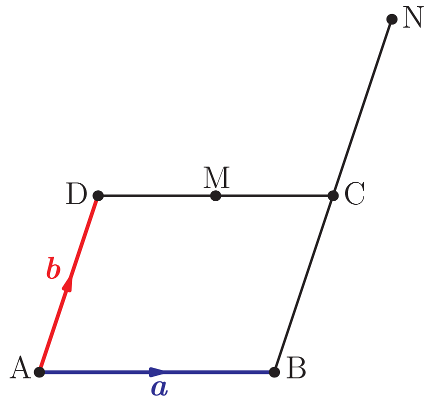
EXAM HINT

Fractions of a vector are usually written as multiples: $\frac{1}{2}\overrightarrow{AD}$, not $\frac{\overrightarrow{AD}}{2}$.

Worked example 13.1

The diagram shows a parallelogram *ABCD*.

Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. *M* is the midpoint of *CD* and *N* is the point on *BC* such that *CN* = *BC*.



Express vectors $\overrightarrow{CM}$, $\overrightarrow{BN}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

We can think of $\overrightarrow{CM}$ as describing a way of getting from *C* to *M* moving only along the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$

Going from *C* to *M* is the same as going half way from *B* to *A*, and $\overrightarrow{BA} = -\overrightarrow{AB}$

$$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA} = -\frac{1}{2}\underline{a}$$

Going from *B* to *N* is twice the distance and in the same direction as from *B* to *C*, and $\overrightarrow{BC} = \overrightarrow{AD}$

$$\overrightarrow{BN} = 2\overrightarrow{BC} = 2\underline{b}$$

To get from *M* to *N* we can go from *M* to *C* and then from *C* to *M*

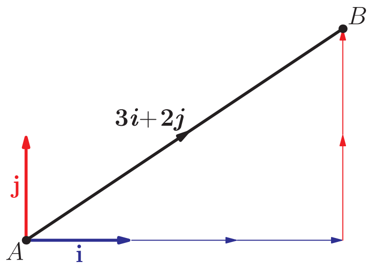
$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN}$$

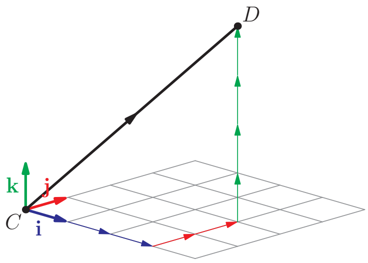
$\overrightarrow{MC} = -\overrightarrow{CM}$ and $\overrightarrow{CN} = \overrightarrow{BC}$

$$= -\overrightarrow{CM} + \overrightarrow{BC}$$

$$= \frac{1}{2}\underline{a} + \underline{b}$$

To do further calculations with vectors we also need to describe them with numbers, not just diagrams. You are already familiar with coordinates, which are used to represent positions of points. A similar idea can be used to represent vectors.





$\vec{CD} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

Start in the two-dimensional plane and select two directions perpendicular to each other, and define vectors of length 1 in those two directions by ***i*** and ***j***. Then any vector in the plane can be expressed in terms of ***i*** and ***j***, as shown in the diagram. ***i*** and ***j*** are called **base or unit vectors**.

To represent displacements in three-dimensional space, we need three base vectors, all perpendicular to each other. They are conventionally called ***i, j*** and ***k***.

An alternative notation is to use **column vectors**. In this notation, displacements shown above are written as:

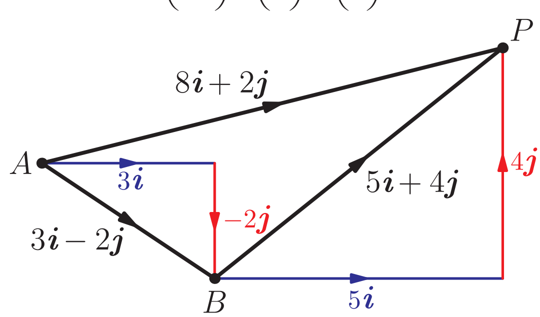
$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \ \overrightarrow{CD} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The numbers in the column are called the **components** of a vector.

Using components in column vectors makes it easy to add displacements.

To get from A to B in the diagram below, we need to move 3 units in the ***i*** direction, and to get from B to P we need to move 5 units in the ***i*** direction; thus getting from A to P requires moving 8 units in the ***i*** direction. Similarly in the ***j*** direction we move –2 units from A to B and 4 units from B to P, making the total displacement from A to P equal to 2 units. As the total displacement from A to P is $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$, we can write it in component form as: $(3\boldsymbol{i} - 2\boldsymbol{j}) + (5\boldsymbol{i} + 4\boldsymbol{j}) = 8\boldsymbol{i} + 2\boldsymbol{j}$ or using the column vector notation as:

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$



Reversing the direction of a vector is also simple: to get from B to A we need to move –3 units in the ***i*** direction and 2 units in the ***j*** direction and so $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

These rules for adding and subtracting vectors also apply in three dimensions.

EXAM HINT

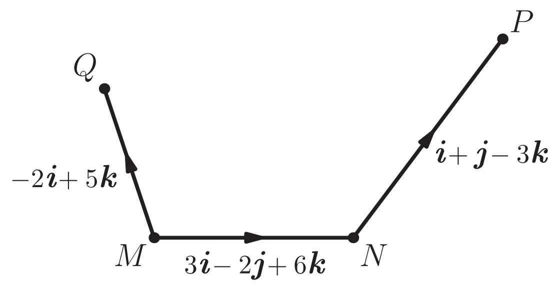
You must be familiar with both base vector and column vector notation, as both will be used in questions. When you write your answers, you can use whichever notation you prefer.

EXAM HINT

Vector diagrams do not have to be accurate or to scale to be useful. A two-dimensional sketch of a 3D situation is often enough to show you what is going on.

Worked example 13.2

The diagram shows points M, N, P and Q such that $\overrightarrow{MN} = 3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$, $\overrightarrow{NP} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{MQ} = -2\boldsymbol{j} + 5\boldsymbol{k}$.



Write the following vectors in component form:

(a) $\overrightarrow{MP}$ (b) $\overrightarrow{PM}$ (c) $\overrightarrow{PQ}$

We can get from M to P via N

(a) $\overrightarrow{MP} = \overrightarrow{MN} + \overrightarrow{NP}$
$= (3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}) + (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k})$
$= 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

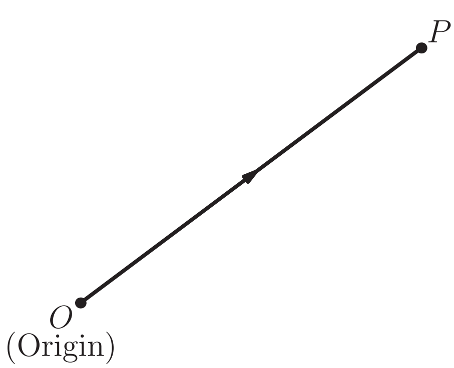
We have already found $\overrightarrow{MP}$

(b) $\overrightarrow{PM} = -\overrightarrow{MP} = -4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$

We can get from P to Q via M using the answers to (a) and (b)

(c) $\overrightarrow{PQ} = \overrightarrow{PM} + \overrightarrow{MQ}$
$= (-4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + (-2\boldsymbol{j} + 5\boldsymbol{k})$
$= -4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

Vectors represent displacements, but they can also represent the positions of points. If we use one fixed point, called the **origin**, then the position of a point can be described by its displacement from the origin. For example, the position of point P in the diagram can be represented by its **position vector**, $\overrightarrow{OP}$.



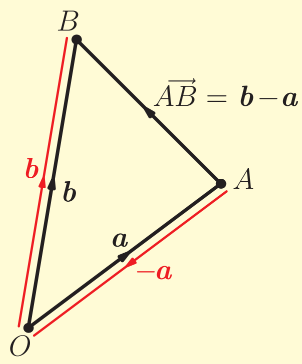
> **EXAM HINT**
> The position vector of point A is usually denoted by $\boldsymbol{a}$.

If we know position vectors of two points A and B we can find the displacement $\overrightarrow{AB}$ as shown in the diagram below: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

KEY POINT 13.1

If points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ then

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.



Position vectors are closely related to coordinates. If the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ have directions set along the coordinate axes, then the components of any position vector are simply the coordinates of the point.

Worked example 13.3

Points A and B have coordinates (3, −1, 2) and (5, 0, 3) respectively. Write as column vectors:

(a) the position vectors of A and B

(b) the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the point	(a) $\underline{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$
Use relationship $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$	(b) $\overrightarrow{AB} = \underline{b} - \underline{a}$ $= \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Worked example 13.4

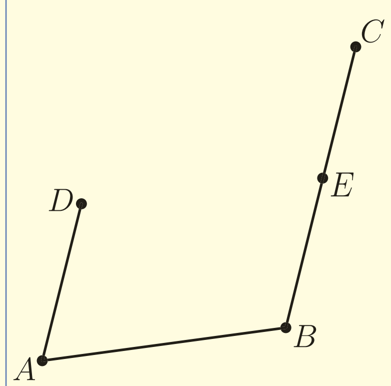
Points A, B, C and D have position vectors $\boldsymbol{a}=\begin{pmatrix}3\\-1\\1\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}5\\0\\3\end{pmatrix}$, $\boldsymbol{c}=\begin{pmatrix}7\\8\\-3\end{pmatrix}$, $\boldsymbol{d}=\begin{pmatrix}4\\3\\-2\end{pmatrix}$

Point E is the midpoint of the line BC.

(a) Find the position vector of E.

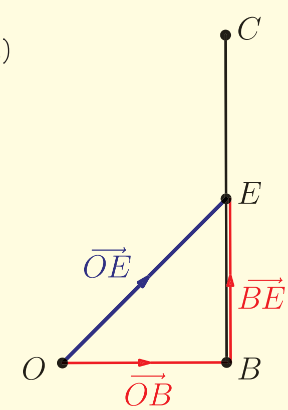
(b) Show that $ABED$ is a parallelogram.

Draw a diagram to show what is going on



For this part, we only need to look at points B, C and E. It may help to show the origin on the diagram

(a)



$$\overrightarrow{OE}=\overrightarrow{OB}+\overrightarrow{BE}$$
$$=\overrightarrow{OB}+\frac{1}{2}\overrightarrow{BC}$$

Use relationship $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$

$$=\underline{b}+\frac{1}{2}(\underline{c}-\underline{b})$$
$$=\frac{1}{2}\underline{b}+\frac{1}{2}\underline{c}$$
$$=\begin{pmatrix}2.5\\0\\1.5\end{pmatrix}+\begin{pmatrix}3.5\\4\\-1.5\end{pmatrix}=\begin{pmatrix}6\\4\\0\end{pmatrix}$$

continued . . .

In a parallelogram, opposite sides are equal length and parallel, which means that the vectors corresponding to those sides are equal

We need to show that $\overrightarrow{AD} = \overrightarrow{BE}$

(b) $\overrightarrow{AD} = \underline{d} - \underline{a}$

$$= \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{BE} = \underline{e} - \underline{b}$

$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{AD} = \overrightarrow{BE}$

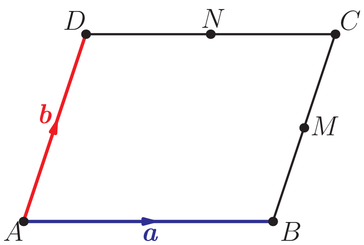
$ABED$ is a parallelogram.

In Worked example 13.4a we derived a general formula for finding the position vector of a midpoint of a line segment.

KEY POINT 13.2

The position vector of the midpoint of $[AB]$ is $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.

Exercise 13A

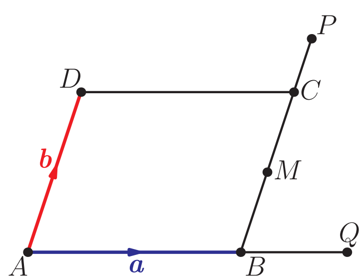


1. The diagram shows a parallelogram $ABCD$ with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of BC and N is the midpoint of CD. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) (i) $\overrightarrow{BC}$ (ii) $\overrightarrow{AC}$

(b) (i) $\overrightarrow{CD}$ (ii) $\overrightarrow{ND}$

(c) (i) $\overrightarrow{AM}$ (ii) $\overrightarrow{MN}$



2. In the parallelogram $ABCD$, $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of BC, Q is the point on AB such that $BQ = \frac{1}{2}AB$ and P is the point on the extended line BC such that $BC : CP = 3 : 1$, as shown on the diagram.

Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) (i) $\overrightarrow{AP}$ (ii) $\overrightarrow{AM}$

(b) (i) $\overrightarrow{QD}$ (ii) $\overrightarrow{MQ}$

(c) (i) $\overrightarrow{DQ}$ (ii) $\overrightarrow{PQ}$

3. Write the following vectors in column vector notation (in three dimensions):

(a) (i) $4\boldsymbol{i}$ (ii) $-5\boldsymbol{j}$

(b) (i) $3\boldsymbol{i}+\boldsymbol{k}$ (ii) $2\boldsymbol{j}-\boldsymbol{k}$

4. Three points O, A and B are given. Let $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$.

(a) Express $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) C is the midpoint of AB. Express $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Point D lies on the line (AB) on the same side of B as A, so that $AD=3AB$. Express $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. *[5 marks]*

EXAM HINT

Remember that (AB) represents the infinite line through A and B, while $[AB]$ is the line segment (the part of the line between A and B).

5. Points A and B lie in a plane and have coordinates (3, 0) and (4, 2) respectively. C is the midpoint of $[AB]$.

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors.

(b) Point D is such that $\overrightarrow{AD}=\begin{pmatrix}7\\-2\end{pmatrix}$.

Find the coordinates of D. *[5 marks]*

6. Points A and B have position vectors $\overrightarrow{OA}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}4\\-2\\5\end{pmatrix}$.

(a) Write $\overrightarrow{AB}$ as a column vector.

(b) Find the position vector of the midpoint of $[AB]$. *[5 marks]*

7. Point A has position vector $\boldsymbol{a}=2\boldsymbol{i}-3\boldsymbol{j}$ and point D is such that $\overrightarrow{AD}=\boldsymbol{i}-\boldsymbol{j}$. Find the position vector of point D. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

Point C lies on (AB) so that $AC:BC=2:3$. Find the position vector of C. *[5 marks]*

9. Points P and Q have position vectors $\boldsymbol{p}=2\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k}$ and $\boldsymbol{q}=\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k}$.

(a) Find the position vector of the midpoint M of $[PQ]$.

(b) Point R lies on the line (PQ) such that $QR=QM$. Find the coordinates of R $(R\neq M)$. *[6 marks]*

10. Points A, B and C have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find the position vector of point D such that $ABCD$ is a parallelogram. *[5 marks]*

13B Vector algebra

Although the idea of representing forces by directed line segments dates back to antiquity, the algebra of vectors was first developed in the 19th Century and was originally used to study complex numbers, which you will meet in chapter 15.

In the previous section we used vectors to describe positions and displacements of points in space, but vectors can represent quantities other than displacements; for example velocities or forces. Whatever the vectors represent, they always follow the same algebraic rules. In this section we will summarise those rules, which can be expressed using either diagrams or equations.

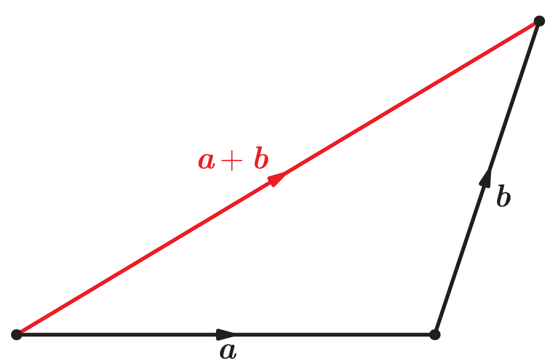
EXAM HINT

The ability to switch between diagrams and equations is essential for solving harder vector problems.

Vector addition can be done on a diagram by joining the starting point of the second vector to the end point of the first. In component form, it is carried out by adding corresponding components. When vectors represent displacements, vector addition represents one displacement followed by another.

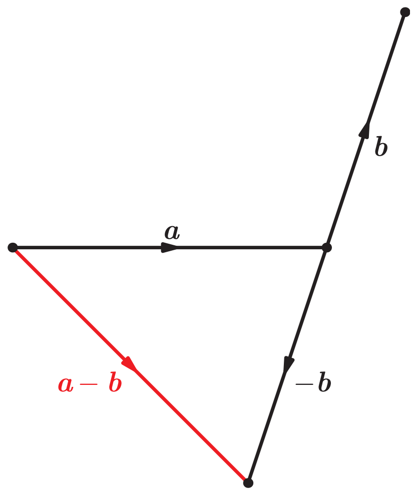
EXAM HINT

Remember that vectors only show relative positions of two points, they don't have a fixed starting point. So we are free to 'move' the second vector to the end point of the first.



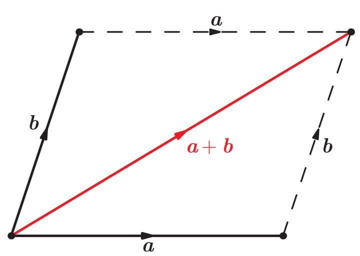
$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

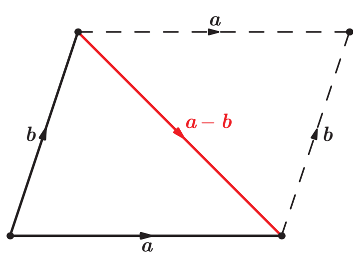
Vector subtraction is the same as adding a negative vector. ($-\boldsymbol{a}$ is the same length but the opposite direction to $\boldsymbol{a}$). In component form you simply subtract corresponding components.



$$\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

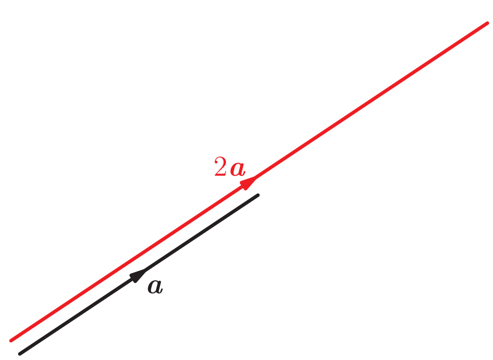
You can also consider vector addition as the diagonal of the parallelogram formed by the two vectors. The difference of two vectors can be represented by the other diagonal of the parallelogram formed by the two vectors.



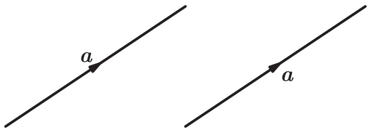


Scalar multiplication changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar.

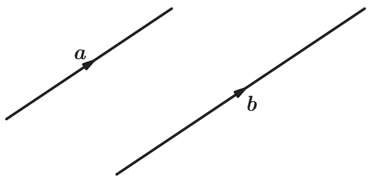
For any vector $\boldsymbol{a}$, $k\boldsymbol{a}$ represents a displacement in the same direction but with distance multiplied by k.



$$2\begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$



Two vectors are **equal** if they have the same magnitude and direction. All their components are equal. They represent the same displacements but may have different start and end points.



If two vectors are in the same direction then they are **parallel**. Parallel vectors are scalar multiples of each other. This is because multiplying a vector by a scalar does not change its direction.

$$\begin{pmatrix}2\\-3\\1\end{pmatrix} \text{ is parallel to } \begin{pmatrix}6\\-9\\3\end{pmatrix} \text{ because } \begin{pmatrix}6\\-9\\3\end{pmatrix} = 3\begin{pmatrix}2\\-3\\1\end{pmatrix}.$$

KEY POINT 13.3

> If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel we can write $\boldsymbol{b} = t\boldsymbol{a}$ for some scalar t.

The next example illustrates the vector operations we have just described.

Worked example 13.5

Given vectors $\boldsymbol{a} = \begin{pmatrix}1\\2\\7\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}-3\\4\\2\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}-2\\p\\q\end{pmatrix}$:

(a) Find $2\boldsymbol{a} - 3\boldsymbol{b}$.
(b) Find the values of p and q such that $\boldsymbol{c}$ is parallel to $\boldsymbol{a}$.
(c) Find the value of scalar k such that $\boldsymbol{a} + k\boldsymbol{b}$ is parallel to vector $\begin{pmatrix}0\\10\\23\end{pmatrix}$.

(a) $2\underline{a} - 3\underline{b} = 2\begin{pmatrix}1\\2\\7\end{pmatrix} - 3\begin{pmatrix}-3\\4\\2\end{pmatrix}$

$$= \begin{pmatrix}2\\4\\14\end{pmatrix} - \begin{pmatrix}-9\\12\\6\end{pmatrix} = \begin{pmatrix}11\\-8\\8\end{pmatrix}$$

continued . . .

If vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are parallel we can write $\mathbf{v_2} = t\mathbf{v_1}$

(b) Write $\underline{c} = t\underline{a}$ for some scalar t

Then: $\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$

If two vectors are equal then all their components are equal

$$\Rightarrow \begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$\therefore p = -4,\ q = -14$

Write vector $\mathbf{a} + k\mathbf{b}$ in terms of k

(c) $\underline{a} + k\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -3k \\ 4k \\ 2k \end{pmatrix} = \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix}$

Then use $\mathbf{a} + k\mathbf{b} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

Parallel to $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

$$\Rightarrow \begin{cases} 1-3k = 0 & (1) \\ 2+4k = 10t & (2) \\ 7+2k = 23t & (3) \end{cases}$$

Find k from the first equation, but check that all three equations are satisfied

(1) $1-3k = 0 \Rightarrow k = \frac{1}{3}$

(2) $2+4\left(\frac{1}{3}\right) = 10t \Rightarrow t = \frac{1}{3}$

(3) $7+2\left(\frac{1}{3}\right) = 23\left(\frac{1}{3}\right)$ (correct)

$\therefore k = \frac{1}{3}$

Exercise 13B

1. Let $\boldsymbol{a}=\begin{pmatrix}7\\1\\12\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}5\\-2\\3\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}1\\1\\2\end{pmatrix}$. Find the following vectors:

   (a) (i) $3\boldsymbol{a}$ (ii) $4\boldsymbol{b}$

   (b) (i) $\boldsymbol{a}-\boldsymbol{b}$ (ii) $\boldsymbol{b}+\boldsymbol{c}$

   (c) (i) $2\boldsymbol{b}+\boldsymbol{c}$ (ii) $\boldsymbol{a}-2\boldsymbol{b}$

   (d) (i) $\boldsymbol{a}+\boldsymbol{b}-2\boldsymbol{c}$ (ii) $3\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}$

2. Let $\boldsymbol{a}=\boldsymbol{i}+2\boldsymbol{j}$, $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{k}$ and $\boldsymbol{c}=2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$. Find the following vectors:

   (a) (i) $-5\boldsymbol{b}$ (ii) $4\boldsymbol{a}$

   (b) (i) $\boldsymbol{c}-\boldsymbol{a}$ (ii) $\boldsymbol{a}-\boldsymbol{b}$

   (c) (i) $\boldsymbol{a}-\boldsymbol{b}+2\boldsymbol{c}$ (ii) $4\boldsymbol{c}-3\boldsymbol{b}$

3. Given that $\boldsymbol{a}=4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$, find the vector $\boldsymbol{b}$ such that:

   (a) $\boldsymbol{a}+\boldsymbol{b}$ is the zero vector (b) $2\boldsymbol{a}+3\boldsymbol{b}$ is the zero vector

   (c) $\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{j}$ (d) $\boldsymbol{a}+2\boldsymbol{b}=3\boldsymbol{i}$

4. Given that $\boldsymbol{a}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}5\\3\\3\end{pmatrix}$ find vector $\boldsymbol{x}$ such that $3\boldsymbol{a}+4\boldsymbol{x}=\boldsymbol{b}$. *[4 marks]*

5. Given that $\boldsymbol{a}=3\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}$, $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c}=\boldsymbol{i}+\boldsymbol{k}$, find the value of the scalar t such that $\boldsymbol{a}+t\boldsymbol{b}=\boldsymbol{c}$. *[4 marks]*

6. Given that $\boldsymbol{a}=\begin{pmatrix}2\\0\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}3\\1\\3\end{pmatrix}$ find the value of the scalar p such that $\boldsymbol{a}+p\boldsymbol{b}$ is parallel to the vector $\begin{pmatrix}3\\2\\3\end{pmatrix}$. *[5 marks]*

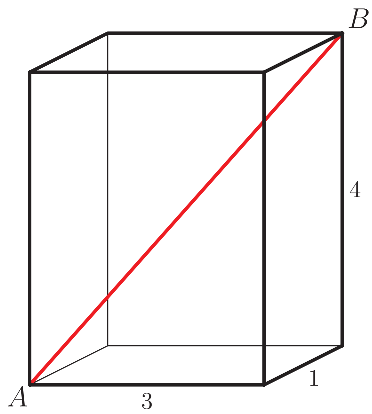
7. Given that $\boldsymbol{x}=2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{y}=4\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ find the value of the scalar λ such that $\lambda x+y$ is parallel to vector $\mathbf{j}$. *[5 marks]*

8. Given that $\boldsymbol{a}=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{b}=2q\boldsymbol{i}+\boldsymbol{j}+q\boldsymbol{k}$ find the values of scalars p and q such that $p\boldsymbol{a}+\boldsymbol{b}$ is parallel to vector $\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$. *[6 marks]*

13C Distances

Geometry problems often involve finding distances between points. In this section we will see how to use vectors to do this.

Consider two points, A and B such that the displacement $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. The distance AB can be found by using Pythagoras' theorem in three dimensions: $AB = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$. This quantity is called the **magnitude** of $\overrightarrow{AB}$, and written as $\left|\overrightarrow{AB}\right|$.



To find the distance between A and B, using their position vectors, we first need to find the displacement vector $\overrightarrow{AB}$ and then calculate its magnitude.

Worked example 13.6

Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$. Find the exact distance AB.

The distance is the magnitude of the displacement vector, so find $\overrightarrow{AB}$ first

$$\overrightarrow{AB} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$$
$$= \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

Now use the formula for the magnitude

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2 + 2^2} = \sqrt{22}$$

EXAM HINT

Don't forget that squaring a negative number gives a positive value.

KEY POINT 13.4

The magnitude of a vector, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The distance between points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{b} - \boldsymbol{a}|$.

The symbol $\gtreqless$ means 'greater than, equal to or less than'. This may appear to be a useless symbol, but it highlights an important idea in vectors – they cannot be put into order. So while it is correct to say that $|v| \gtreqless |u|$ it is not possible to say the same about the vectors themselves.

We saw in Section 13B that multiplying a vector by a scalar produces a vector in the same direction but of different magnitude. In more advanced applications of vectors it is useful to be able to use vectors of length 1, called **unit vectors**. The base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are examples of unit vectors.

Worked example 13.7

(a) Find the unit vector in the same direction as $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 5 parallel to $\boldsymbol{a}$.

(a) Call the required unit vector $\hat{a}$.

To produce a vector in the same direction but of different magnitude as ***a***, we need to multiply ***a*** by a scalar. We need to find the value of the scalar

Then $\hat{a} = k\hat{a}$ and $\hat{a} = 1$

$|k\underline{a}| = k|\underline{a}| = 1$

$\Rightarrow k = \frac{1}{|\underline{a}|}$

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\therefore k = \frac{1}{3}$

Find the vector ***â***

The unit vector is

$\hat{a} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$

To get a vector of magnitude 5 we need to multiply the unit vector by 5

(b) Let $\underline{b}$ be parallel to $\underline{a}$ and $|\underline{b}| = 5$

Then $\underline{b} = 5\hat{a}$

$\therefore \underline{b} = \begin{pmatrix} \frac{10}{3} \\ \frac{-10}{3} \\ \frac{5}{3} \end{pmatrix}$

EXAM HINT

Note that part (b) has two possible answers, as ***b*** could be in the opposite direction. To get the second answer we would take the scalar to be –5 instead of 5.

The last example showed the general method for finding the unit vector in a given direction.

KEY POINT 13.5

The unit vector in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Exercise 13C

1. Find the magnitude of the following vectors in two dimensions.

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j}$ $\quad \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j}$

2. Find the magnitude of the following vectors in three dimensions.

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ $\quad \boldsymbol{d} = \boldsymbol{j} - \boldsymbol{k}$

3. Find the distance between the following pairs of points in the plane.

(a) (i) $A\,(1, 2)$ and $B\,(3, 7)$ (ii) $C\,(2, 1)$ and $D\,(1, 2)$

(b) (i) $P\,(-1, -5)$ and $Q\,(-4, 2)$ (ii) $M\,(1, 0)$ and $N\,(0, -2)$

4. Find the distance between the following pairs of points in three dimensions.

(a) (i) $A\,(1, 0, 2)$ and $B\,(2, 3, 5)$

(ii) $C\,(2, 1, 7)$ and $D\,(1, 2, 1)$

(b) (i) $P\,(3, -1, -5)$ and $Q\,(-1, -4, 2)$

(ii) $M\,(0, 0, 2)$ and $N\,(0, -3, 0)$

5. Find the distance between the points with the given position vectors.

(a) $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$

(b) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

(c) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

(d) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{j} - \boldsymbol{k}$

6. (a) (i) Find a unit vector parallel to $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

(ii) Find a unit vector parallel to $6\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.

(b) (i) Find a unit vector in the same direction as $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

(ii) Find a unit vector in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

7. Find the possible values of the constant c such that the vector $\begin{pmatrix} 2c \\ c \\ -c \end{pmatrix}$ has magnitude 12. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

C is the midpoint of $[AB]$. Find the exact distance AC. *[4 marks]*

9. Let $\boldsymbol{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a} + \lambda\mathbf{b}| = 5\sqrt{2}$. *[6 marks]*

10. (a) Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 3 in the same direction as $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. *[6 marks]*

11. Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and

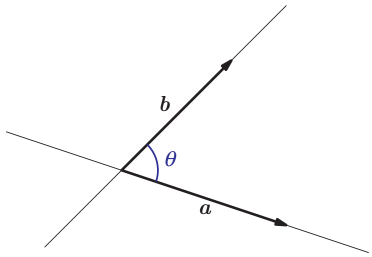
$\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ where O is the origin.

Find the possible values of t such that $AB = 3$. *[5 marks]*

12. Points P and Q have position vectors $\boldsymbol{p} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{q} = (2+t)\boldsymbol{i} + (1-t)\boldsymbol{j} + (1+t)\boldsymbol{k}$. Find the value of t for which the distance PQ is the minimum possible and find this minimum distance. *[6 marks]*

13D Angles

In geometry problems you are often asked to find angles between two lines. The diagram shows two lines with angle θ between them. $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in the directions of the two lines. Note that both arrows are pointing *away* from the intersection point. It turns out that $\cos\theta$ can be expressed in terms of the components of the two vectors. This result can be derived using the cosine rule. See Fill-in proof sheet 12 'Deriving scalar products' on the CD-ROM.



KEY POINT 13.6

If θ is the angle between vectors, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}.$$

The expression in the numerator of the above fraction has many important uses, and is called the **scalar product**.

KEY POINT 13.7

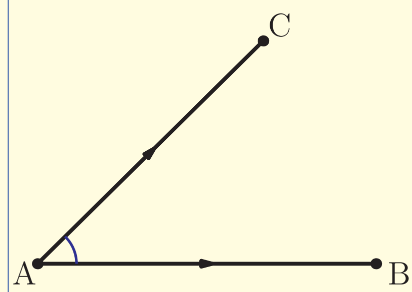
Scalar product

The quantity $a_1b_1 + a_2b_2 + a_3b_3$ is called the scalar product (**inner product**, or **dot product**) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a} \bullet \boldsymbol{b}$.

Worked example 13.8

Given points A (3, −5, 2), B (4, 1, 1) and C (−1, 1, 2) find the size of the angle $B\hat{A}C$ in degrees.

It's always a good idea to draw a diagram to be sure which vectors the angle lies between



The required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

Let $\theta = B\hat{A}C$

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

We need the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix}4\\1\\1\end{pmatrix} - \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}1\\6\\-1\end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}-4\\6\\0\end{pmatrix}$$

$$\cos\theta = \frac{1\times(-4)+6\times6+(-1)\times0}{\sqrt{1^2+6^2+1^2}\sqrt{4^2+6^2+0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}} = 0.7199$$

$$\therefore \theta = \arccos(0.7199) = 44.0°$$

It is very straightforward to check whether two vectors are perpendicular. If $\theta = 90°$ then $\cos\theta = 0$, so the top of the fraction in the formula for $\cos\theta$ must be zero. We do not even have to calculate the magnitudes of the two vectors.

KEY POINT 13.8

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}\bullet\boldsymbol{b} = 0$.

Worked example 13.9

If $\boldsymbol{p} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ find the value of the scalar t such that $\boldsymbol{p} + t\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

Two vectors are perpendicular if their dot product equals 0

$$(\underline{p} + t\underline{q}) \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Find the components of $\boldsymbol{p} + t\boldsymbol{q}$ in terms of t

$$\underline{p} + t\underline{q} = \begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix}$$

So

$$\begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Form and solve the equation

$$\Leftrightarrow 3(4+2t) + 5(-1+t) + 1(2+t) = 0$$
$$\Leftrightarrow 9 + 12t = 0$$
$$\Leftrightarrow \quad t = -\frac{3}{4}$$

Exercise 13D

1. Calculate the angle between the following pairs of vectors, giving your answers in radians.

   (a) (i) $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

   (b) (i) $2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

   (ii) $3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{i} - 2\boldsymbol{k}$

   (c) (i) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ (ii) $\boldsymbol{i} - \boldsymbol{j}$ and $2\boldsymbol{i} + 3\boldsymbol{j}$

2. The angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is θ. Find the exact value of $\cos\theta$ in the following cases:

   (a) (i) $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

   (ii) $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

(b) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(c) (i) $\boldsymbol{a} = -2\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i}$ (ii) $\boldsymbol{a} = 5\boldsymbol{i}$ and $\boldsymbol{b} = 3\boldsymbol{j}$

3. (a) The vertices of a triangle have position vectors

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{c} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}.$$

Find, in degrees, the angles of the triangle.

(b) Find, in degrees, the angles of the triangle with vertices (2, 1, 2), (4, –1, 5) and (7, 1, –2).

4. Which of the following pairs of vectors are perpendicular?

(a) (i) $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

(b) (i) $5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$

(ii) $\boldsymbol{i} - 3\boldsymbol{k}$ and $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

5. Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$.

Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$. *[5 marks]*

6. Four points are given with coordinates A (2, –1, 3), B (1, 1, 2), C (6, –1, 2) and D (7, –3, 3).
Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$. *[5 marks]*

7. Four points have coordinates A (2, 4, 1), B (k, 4, $2k$), C ($k + 4$, $2k + 4$, $2k + 2$) and D (6, $2k + 4$, 3).

(a) Show that $ABCD$ is a parallelogram for all values of k.

(b) When $k = 1$ find the angles of the parallelogram.

(c) Find the value of k for which $ABCD$ is a rectangle.

[8 marks]

8. Vertices of a triangle have position vectors $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{c} = 5\boldsymbol{i}$.

(a) Show that the triangle is right-angled.

(b) Calculate the other two angles of the triangle.

(c) Find the area of the triangle. *[8 marks]*

13E Properties of the scalar product

In the last section we defined the scalar product of vectors

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as

$$\boldsymbol{a} \bullet \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and saw that if θ is the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ then:

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

In this section we look at various properties of the scalar product in more detail; in particular, its algebraic rules. The scalar product has many properties similar to multiplication of numbers. These properties can be proved by using components of the vectors.

KEY POINT 13.9

Algebraic properties of the scalar product

$$\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$$

$$(-\boldsymbol{a}) \bullet \boldsymbol{b} = -(\boldsymbol{a} \bullet \boldsymbol{b})$$

$$(k\boldsymbol{a}) \bullet \boldsymbol{b} = k(\boldsymbol{a} \bullet \boldsymbol{b})$$

$$\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{b}) + (\boldsymbol{a} \bullet \boldsymbol{c})$$

But there are some properties of multiplication of numbers which do *not* apply to scalar product. For example, it is not possible to calculate the scalar product of three vectors: the expression $(\boldsymbol{a} \bullet \boldsymbol{b}) \bullet \boldsymbol{c}$ has no meaning, as $\boldsymbol{a} \bullet \boldsymbol{b}$ is a scalar (and so has no direction), and scalar product involves multiplying two vectors.

Two important properties of scalar product concern perpendicular and parallel vectors. These are very useful when solving geometry problems.

All the operations with vectors work in both two and three dimensions. If there were a fourth dimension, the position of each point could be described using four numbers. We could use analogous rules to calculate 'distances' and 'angles'. Does this mean that we can acquire knowledge about a four-dimensional world which we can't see, or even imagine?

EXAM HINT

These are not in the formula booklet!

KEY POINT 13.10

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$, in particular, $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.

The following two examples show you how you can use these rules.

Worked example 13.10

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors such that $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 3$, evaluate $(2\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} + 4\boldsymbol{b})$.

Multiply out the brackets as we would with numbers $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

$(2\underline{a} - \underline{b}) \bullet (\underline{a} + 4\underline{b}) = 2\underline{a} \bullet \underline{a} + 8\underline{a} \bullet \underline{b} - \underline{b} \bullet \underline{a} - 4\underline{b} \bullet \underline{b}$

$\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular so $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a} = 0$

$= 2\underline{a} \bullet \underline{a} - 4\underline{b} \bullet \underline{b}$

Use the fact that $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

$= 2|\underline{a}|^2 - 4|\underline{b}|^2$

$= 2 \times 5^2 - 4 \times 3^2$

$= 14$

Worked example 13.11

Points A, B and C have position vectors $\boldsymbol{a} = k\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$.

(a) Find $\overrightarrow{BC}$.

(b) Find $\overrightarrow{AB}$ in terms of k.

(c) Find the value of k for which (AB) is perpendicular to (BC).

Use $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

(a) $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

$$= \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

Use $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 3k \\ -k \\ k \end{pmatrix} = \begin{pmatrix} 3-3k \\ 4+k \\ -2-k \end{pmatrix}$$

continued . . .

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular then $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

(c) $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

$$\begin{pmatrix} 3-3k \\ 4+k \\ -2-2k \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = 0$$

$\Rightarrow -6+6k-12-3k-14-14k=0$

$\Rightarrow -11k = 32$

$k = -\frac{11}{32}$

Exercise 13E

1. Evaluate $\boldsymbol{a} \cdot \boldsymbol{b}$ in the following cases:

(a) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -12 \\ 4 \\ -8 \end{pmatrix}$

(b) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$

(c) (i) $\boldsymbol{a} = 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

(ii) $\boldsymbol{a} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

(d) (i) $\boldsymbol{a} = -3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{k}$

(ii) $\boldsymbol{a} = -3\boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{k}$

2. Given that θ is the angle between vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ find the exact value of $\cos\theta$.

(a) (i) $\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{p} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) (i) $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

3. (a) Given that $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 5$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 10$, find, in degrees, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Given that $|\boldsymbol{c}| = 9$, $|\boldsymbol{d}| = 12$ and $\boldsymbol{c} \cdot \boldsymbol{d} = -15$, find, in degrees, the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$.

4. (a) Given that $|\boldsymbol{a}| = 6$, $|\boldsymbol{b}| = 4$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $37°$, calculate $\boldsymbol{a}\bullet\boldsymbol{b}$.

(b) Given that $|\boldsymbol{a}| = 8$, $\boldsymbol{a}\bullet\boldsymbol{b} = 12$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$, find the exact value of $|\boldsymbol{b}|$.

5. Given that $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{c} = 5\boldsymbol{i} - 3\boldsymbol{k}$ and $\boldsymbol{d} = -2\boldsymbol{j} + \boldsymbol{k}$ verify that:

(a) $\boldsymbol{b}\bullet\boldsymbol{d} = \boldsymbol{d}\bullet\boldsymbol{b}$

(b) $\boldsymbol{a}\bullet(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\bullet\boldsymbol{b} + \boldsymbol{a}\bullet\boldsymbol{c}$

(c) $(\boldsymbol{c}-\boldsymbol{d})\bullet\boldsymbol{c} = |\boldsymbol{c}|^2 - \boldsymbol{c}\bullet\boldsymbol{d}$

(d) $(\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{a}+\boldsymbol{b}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{a}\bullet\boldsymbol{b}$

6. Find the values of t for which the following pairs of vectors are perpendicular.

(a) (i) $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} t+1 \\ 2t-1 \\ 2t \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

(b) (i) $5t\boldsymbol{i} - (2+t)\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - t\boldsymbol{k}$

(ii) $t\boldsymbol{i} - 3\boldsymbol{k}$ and $2t\boldsymbol{i} + \boldsymbol{j} + t\boldsymbol{k}$

We will meet equations of this type in the next chapter. See Worked example 14.17.

7. In this question, we will introduce a method using scalar product to find x and y such that $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2x \\ -3x \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 3y \\ y \end{pmatrix}$, and then use it to solve other similar equations.

(a) Use the usual method of simultaneous equations to find x and y.

(b) (i) Find the scalar product of both sides of the equation with $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and hence find x.

(ii) Find the scalar product of both sides of the equation with $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and hence find y.

(iii) Can you see why the vectors $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ were selected?

(c) (i) Find a vector perpendicular to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and a vector perpendicular to $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

(ii) Hence find x and y such that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + x\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + y\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

8. Given that $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$: calculate:

(a) $\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c})$

(b) $(\boldsymbol{b} - \boldsymbol{a}) \bullet (\boldsymbol{d} - \boldsymbol{c})$

(c) $(\boldsymbol{b} + \boldsymbol{d}) \bullet (2\boldsymbol{a})$ *[7 marks]*

9. (a) If $\boldsymbol{a}$ is a unit vector perpendicular to $\boldsymbol{b}$, find the value of $\boldsymbol{a} \bullet (2\boldsymbol{a} - 3\boldsymbol{b})$.

(b) If $\boldsymbol{p}$ is a unit vector making a 45° angle with vector $\boldsymbol{q}$ and $\boldsymbol{p} \bullet \boldsymbol{q} = 3\sqrt{2}$, find $|\boldsymbol{q}|$. *[6 marks]*

10. (a) $\boldsymbol{a}$ is a vector of magnitude 3 and $\boldsymbol{b}$ makes an angle of 60° with $\boldsymbol{a}$. Given that $\boldsymbol{a} \bullet (\boldsymbol{a} - \boldsymbol{b}) = \frac{1}{3}$, find the exact value of $|\boldsymbol{b}|$.

(b) Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors of equal magnitude such that $(3\boldsymbol{a} + \boldsymbol{b})$ is perpendicular to $(\boldsymbol{a} - 3\boldsymbol{b})$, prove that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. *[6 marks]*

11. Points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} - 19\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = 2\lambda\boldsymbol{i} + (\lambda + 2)\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = -6\boldsymbol{i} - 15\boldsymbol{j} + 7\boldsymbol{k}$.

(a) Find the value of λ for which BC is perpendicular to AC.

For the value of λ found above:

(b) find the angles of the triangle ABC

(c) find the area of the triangle ABC. *[8 marks]*

12. $ABCD$ is a parallelogram with AB parallel to DC. Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$.

(a) Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Simplify $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{b} - \boldsymbol{a})$.

(c) Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular. *[8 marks]*

13. Points A and B have position vectors $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2\lambda \\ \lambda \\ 4\lambda \end{pmatrix}$.

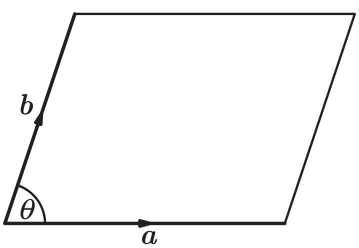
(a) Show that B lies on the line OA for all values of λ.

Point C has position vector $\begin{pmatrix} 12 \\ 2 \\ 4 \end{pmatrix}$.

(b) Find the value of λ for which $C\hat{B}A$ is a right angle.

(c) For the value of λ found above, calculate the exact distance from C to the line OA. *[8 marks]*

13F Areas



Given the coordinates of the four vertices of a parallelogram, how can we calculate its area? The area of the parallelogram is given by $ab\sin\theta$ where a and b are the lengths of the sides and θ is the angle between them. We could use the coordinates of the vertices to find the lengths of the sides, and then use the cosine rule to find angle θ. However, using vectors gives a quicker way to calculate the area.

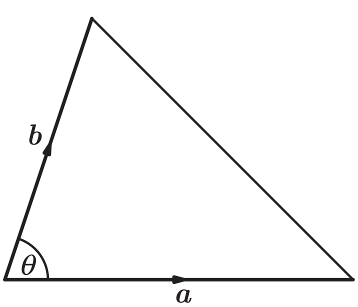
EXAM HINT
Notice that Notice that vectors a and b form two adjacent sides of the parallelogram. We can use any pair of adjacent sides.

KEY POINT 13.11

The area of the parallelogram with sides defined by vectors $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is equal to the magnitude of the vector $\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

This vector is called the **vector product** (or **cross product**) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a} \times \boldsymbol{b}$.

A parallelogram can be divided in half to form two triangles, so we can also use the vector product to calculate the area of a triangle.



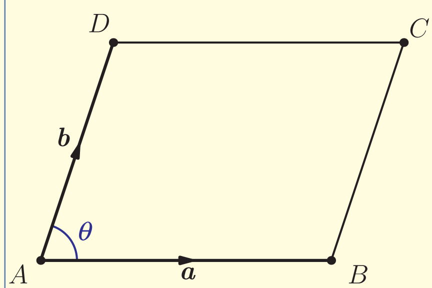
KEY POINT 13.12

The area of the triangle with two sides defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$.

Worked example 13.12

Find the area of the parallelogram with vertices $A\begin{pmatrix}1\\4\\2\end{pmatrix}, B\begin{pmatrix}3\\-3\\0\end{pmatrix}, C\begin{pmatrix}1\\1\\7\end{pmatrix}, D\begin{pmatrix}-1\\8\\9\end{pmatrix}$.

Draw a diagram to show which two vectors to use: we can choose any two adjacent sides of the parallelogram, e.g. *AB* and *AD*



$$\boldsymbol{a} = \overrightarrow{AB} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}$$

$$\boldsymbol{b} = \overrightarrow{AD} = \begin{pmatrix}-2\\4\\7\end{pmatrix}$$

Calculate $\boldsymbol{a}\times\boldsymbol{b}$ first, and then find its magnitude

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}\times\begin{pmatrix}-2\\4\\7\end{pmatrix}$$

$$= \begin{pmatrix}-49+8\\4-14\\8-14\end{pmatrix} = \begin{pmatrix}-41\\-10\\-6\end{pmatrix}$$

$$\text{Area} = |\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{41^2+10^2+6^2} = 55.4$$

Exercise 13F

1. Calculate $\boldsymbol{a}\times\boldsymbol{b}$ for the following pairs of vectors:

   (a) (i) $\boldsymbol{a} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}5\\2\\2\end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}-12\\4\\-8\end{pmatrix}$

   (b) (i) $\boldsymbol{a} = 4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

   (ii) $\boldsymbol{a} = -3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i}-4\boldsymbol{k}$

2. Calculate the area of the triangle with vertices:

(a) (i) (1, 3, 3), (−1, 1, 2) and (1, −2, 4)
(ii) (3, −5, 1), (−1, 1, 3) and (−1, 5, 2)

(b) (i) (−3, −5, 1), (4, 7, 2) and (−1, 2, 2)
(ii) (4, 0, 2), (4, 1, 5) and (4, −3, 2)

3. Given points A, B and C with coordinates (3, −5, 1), (7, 7, 2) and (−1, 1, 3).

(a) calculate $\boldsymbol{p} = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\boldsymbol{q} = \overrightarrow{BA} \times \overrightarrow{BC}$.

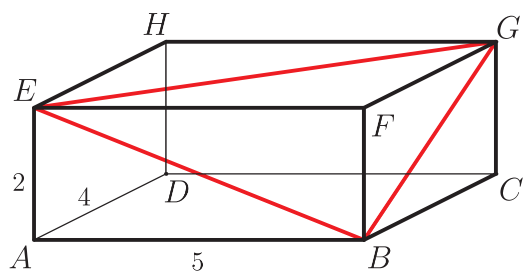
(b) What can you say about vectors $\boldsymbol{p}$ and $\boldsymbol{q}$?

4. The points A (3, 1, 2), B (−1, 1, 5) and C (7, 2, 3) are vertices of a parallelogram $ABCD$.

(a) Find the coordinates of D.

(b) Calculate the area of the parallelogram.

5. A cuboid $ABCDEFGH$ is shown in the diagram. The coordinates of four of the vertices are A (0, 0, 0), B (5, 0, 0), C (5, 4, 0) and E (0, 0, 2).



(a) Find the coordinates of the remaining four vertices.

Face diagonals BE, BG and EG are drawn as shown.

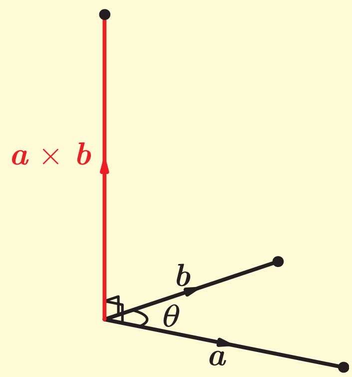
(b) Find the area of the triangle BEG.

13G Properties of the vector product

In this section we look at more properties of the vector product $\boldsymbol{a} \times \boldsymbol{b}$. We have already seen that the magnitude of this vector is equal to the area of the parallelogram defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

KEY POINT 13.13

The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a}\times\boldsymbol{b}$, has magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in the diagram.



In component form, $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}$.

If you study physics, you may have come across the 'right-hand rule' for the direction of the magnetic field. This is just one example of application of vector product; others include circular motion, fluid dynamics and Maxwell's theory of electromagnetism.

EXAM HINT

The Formula booklet gives you the equations, but not the diagrams.

Worked example 13.13

Find the exact value of the sine of the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that

$|\boldsymbol{a}|=3$, $|\boldsymbol{b}|=2$ and $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}3\\6\\1\end{pmatrix}$.

We are not given the components of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ so need to use the definition of the vector product involving magnitudes

$$|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$$
$$\sqrt{3^2+6^2+1^2}=(3\times 2)\sin\theta$$
$$\sqrt{46}=6\sin\theta$$
$$\therefore \sin\theta=\frac{\sqrt{46}}{6}$$

The fact that the vector product is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is very useful.

Worked example 13.14

Find a vector perpendicular to both $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$.

The vector product of two vectors is perpendicular to both of them

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix}$$

You may find the two different ways of multiplying vectors confusing. However, if you think about normal multiplication, you will realise that it can have at least two very different interpretations: It can be considered as repeated addition, taking two numbers and producing a third number as the answer; or the result can represent the area of a rectangle with given lengths of sides. The two 'types' of multiplication of numbers just happen to give the same numerical answer.

The vector product has many properties similar to multiplication of numbers, but the most important difference is that $\boldsymbol{a} \times \boldsymbol{b}$ is not the same as $\boldsymbol{b} \times \boldsymbol{a}$.

KEY POINT 13.14

Algebraic properties of vector product

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$
$$(k\boldsymbol{a}) \times \boldsymbol{b} = k(\boldsymbol{a} \times \boldsymbol{b})$$
$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$$

With the vector product it is possible to multiply three vectors together, but $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ is not the same as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

Again, there are special results concerning parallel and perpendicular vectors.

KEY POINT 13.15

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a} \times \boldsymbol{b} = 0$. In particular, $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$.

Worked example 13.15

Given that $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, evaluate $|(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b})|$.

Expand the brackets as we would with numbers $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$

$$(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b}) = 2\boldsymbol{a} \times \boldsymbol{a} + 6\boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a} - 3\boldsymbol{b} \times \boldsymbol{b}$$

continued . . .

$\boldsymbol{a} \times \boldsymbol{a} = 0$

$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$

For perpendicular vectors, $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$

$$= 6\boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a}$$

$$= 7\boldsymbol{a} \times \boldsymbol{b}$$

$$\therefore |(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b})| = 7|\boldsymbol{a} \times \boldsymbol{b}|$$
$$= 7|\boldsymbol{a}||\boldsymbol{b}| = 140$$

Exercise 13G

1. Find, in radians, the acute angle between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that:

   (a) $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 5$ and $|\boldsymbol{a} \times \boldsymbol{b}| = 7$

   (b) $|\boldsymbol{a}| = 12$, $|\boldsymbol{b}| = 3$ and $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

   (c) $|\boldsymbol{a}| = 7$, $|\boldsymbol{b}| = 1$ and $\boldsymbol{a} \times \boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$

   (d) $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 4$ and $|\boldsymbol{a} \times \boldsymbol{b}| = 0$

2. Find a vector perpendicular to the following pairs of vectors:

   (a) (i) $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$

   (b) (i) $\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

3. Find the unit vector perpendicular to the following pairs of vectors:

   (a) $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

4. Given that $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$ find the exact value of $|\boldsymbol{a} \times \boldsymbol{b}|$. *[4 marks]*

5. (a) Prove that $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b}) = 2\boldsymbol{a} \times \boldsymbol{b}$.

   (b) Simplify $(2\boldsymbol{a} - 3\boldsymbol{b}) \times (3\boldsymbol{a} + 2\boldsymbol{b})$. *[6 marks]*

6. (a) Explain why $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{a} = 0$.

(b) Evaluate $(\boldsymbol{a} \times \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{b})$. *[5 marks]*

7. Prove that for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$|\boldsymbol{a} \times \boldsymbol{b}|^2 + (\boldsymbol{a} \bullet \boldsymbol{b})^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2$. *[5 marks]*

Summary

- The position of B relative to A can be represented by the **vector displacement** $\overrightarrow{AB}$.
- Vectors can be expressed in terms of **base vectors** $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ or as **column vectors** using **components**. For example, $\overrightarrow{AB}$ can be represented by $(3\boldsymbol{i} + 2\boldsymbol{j})$ or $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. The numbers 3 and 2 are the components.
- Vectors can represent the position of points relative to the **origin**; the displacement of a point from the origin is the point's **position vector**.
- The position vector of the midpoint of $[AB]$ is $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.
- The displacement between points A and B with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{b} - \boldsymbol{a}$.
- The distance between the points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $|\boldsymbol{b} - \boldsymbol{a}|$.
- The **unit vector** in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel we can write $\boldsymbol{b} = t\boldsymbol{a}$ for some scalar t.
- The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- The angle θ, between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, is given by: $\cos\theta = \frac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ where $\boldsymbol{a} \bullet \boldsymbol{b}$ is the **scalar product** (dot product), given in terms of the components by: $\boldsymbol{a} \bullet \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors then $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$, in particular, $\boldsymbol{a} \bullet \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Algebraic properties of scalar product:

$$\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$$
$$(-\boldsymbol{a}) \bullet \boldsymbol{b} = -(\boldsymbol{a} \bullet \boldsymbol{b})$$
$$(k\boldsymbol{a}) \bullet \boldsymbol{b} = k(\boldsymbol{a} \bullet \boldsymbol{b})$$
$$\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{b}) + (\boldsymbol{a} \bullet \boldsymbol{c})$$

- The area of the parallelogram with sides defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the magnitude of the vector $\boldsymbol{a} \times \boldsymbol{b}$, which is called the **vector product** (or cross product). Its direction is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and it is given by:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ has magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Algebraic properties of vector product:

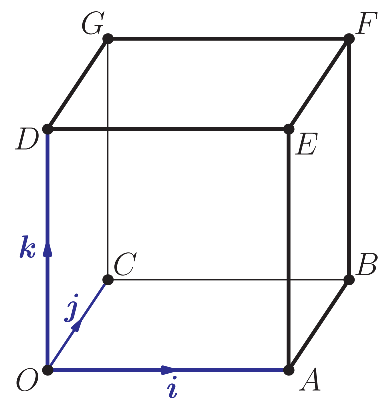
$$\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$$
$$(k\boldsymbol{a})\times\boldsymbol{b}=k\ (\boldsymbol{a}\times\boldsymbol{b})$$
$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})$$

- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$. In particular, $\boldsymbol{a}\times\boldsymbol{a}=0$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|$.

Introductory problem revisited

What is the angle between the diagonals of a cube?

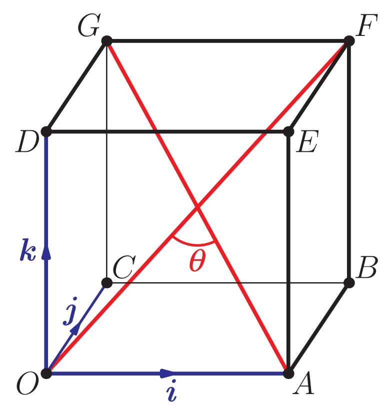
You can solve this problem by using the cosine rule in a triangle made by the diagonals and one side. However, using vectors gives a slightly faster solution, as we do not have to find the lengths of the sides of the triangle.



The angle between two lines can be found by using the vectors in the directions of two lines and the formula involving the scalar product. We do not know the actual positions of the vertices of the cube, or even the lengths of its sides, but as the answer does not depend on the size of the cube, we can look at the cube with side length 1, set with one vertex at the origin and sides parallel to the base vectors.

We want to find the angle between the diagonals OF and AG, so we need the coordinates of those four vertices. They are: $O(0, 0, 0)$, $A(1, 0, 0)$, $F(1, 1, 1)$, $G(0, 1, 1)$.

The required angle θ is between the lines OF and AG.



The corresponding vectors are: $\overrightarrow{OF}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad\overrightarrow{AG}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$

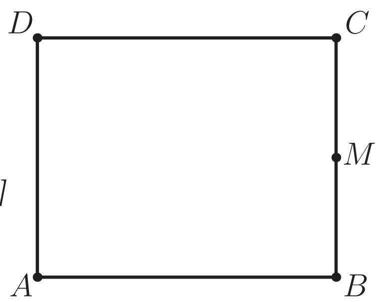
$$\cos\theta=\frac{\overrightarrow{OF}\bullet\overrightarrow{AG}}{|\overrightarrow{OF}||\overrightarrow{AG}|}$$
$$=\frac{-1+1+1}{\sqrt{3}\sqrt{3}}=\frac{1}{3}$$
$$\therefore\theta=70.5°$$

Mixed examination practice 13

Short questions

1. Given that $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$, find $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. *[6 marks]*

2. The diagram shows a rectangle $ABCD$. M is the midpoint of BC.



(a) Express MD in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

(b) Given that $AB = 6$ and $AD = 4$, show that $\overrightarrow{MD} \cdot \overrightarrow{MC} = 4$. *[5 marks]*

3. The position vectors of points N and L are:

$$\boldsymbol{n} = 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{l} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

(a) Find the vector product $\boldsymbol{n} \times \boldsymbol{l}$.

(b) Using your answer to part (a), or otherwise, find the area of the parallelogram with two sides $\overline{ON}$ and $\overline{OL}$. *[6 marks]*

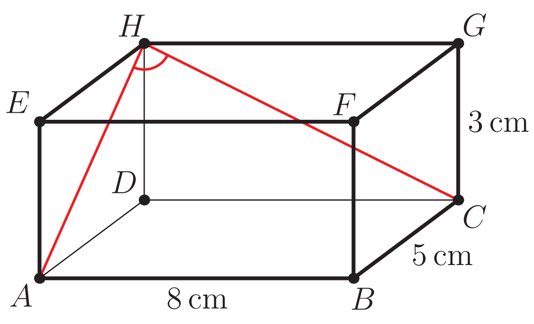
4. Let $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \\ p \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

(a) Find $\boldsymbol{a} \times \boldsymbol{b}$.

(b) Find the value of p, given that $\boldsymbol{a} \times \boldsymbol{b}$ is parallel to $\boldsymbol{c}$. *[6 marks]*

5. The rectangle box shown in the diagram has dimensions 8 cm × 5 cm × 3 cm.

Find, correct to the nearest one-tenth of a degree, the size of the angle $A\hat{H}C$. *[6 marks]*



6. Let α be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$ and $\boldsymbol{b} = (\sin\theta)\boldsymbol{i} + (\cos\theta)\boldsymbol{j}$ and $0 < \theta < \pi/4$. Express α in terms of θ.
(© *IB Organization 2000*) *[6 marks]*

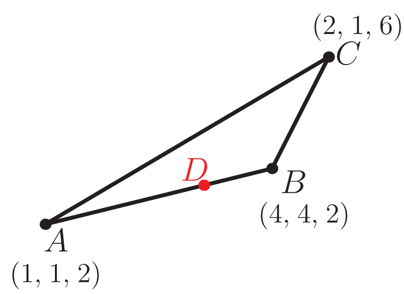
7. Given two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}-\boldsymbol{b}|$, find the value of $\boldsymbol{a}\bullet\boldsymbol{b}$.
(© *IB Organization 2002*) *[6 marks]*

8. (a) Show that $(\boldsymbol{b}-\boldsymbol{a})\bullet(\boldsymbol{b}-\boldsymbol{a}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{a}\bullet\boldsymbol{b}$.
(b) In triangle MNP, $M\hat{P}N = \theta$. Let $\overrightarrow{PM} = \boldsymbol{a}$ and $\overrightarrow{PN} = \boldsymbol{b}$. Use the result from part (a) to prove the cosine rule: $MN^2 = PM^2 + PN^2 - 2PM\,PN\cos\theta$. *[6 marks]*

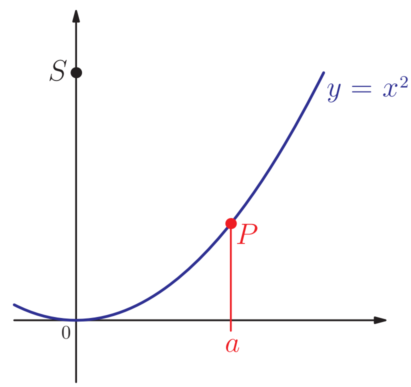
Long questions

1. Points A, B and D have coordinates (1, 1, 7), (−1, 6, 3) and (3, 1, k), respectively. AD is perpendicular to AB.
(a) Write down, in terms of $\boldsymbol{k}$, the vector $\overrightarrow{AD}$.
(b) Show that $\boldsymbol{k} = 6$.
Point C is such that $\overrightarrow{BC} = 2\overrightarrow{AD}$.
(c) Find the coordinates of C.
(d) Find the exact value of $\cos(A\hat{D}C)$. *[10 marks]*

2. A triangle has vertices $A(1, 1, 2)$, $B(4, 4, 2)$ and $C(2, 1, 6)$. Point D lies on the side AB and $AD:DB = 1:k$.
(a) Find $\overrightarrow{CD}$ in terms of k.
(b) Find the value of k such that CD is perpendicular to AB.
(c) For the above value of k, find the coordinates of D.
(d) Hence find the length of the altitude from vertex C. *[10 marks]*



3. Point P lies on the parabola $y = x^2$ and has x-coordinate $a\,(a > 0)$.



(a) Write down, in terms of a, the coordinates of P.

Point S has coordinates (0, 4) and O is the origin.

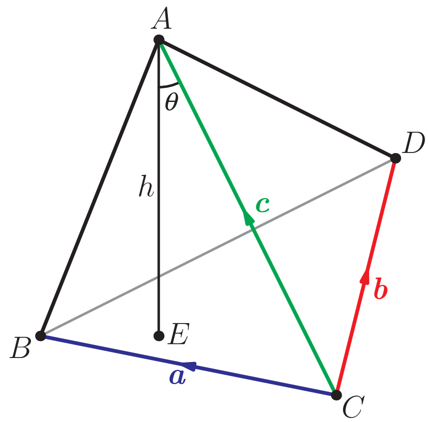
(b) Write down the vectors $\overrightarrow{PO}$ and $\overrightarrow{PS}$.

(c) Use scalar product to find the value of a for which OP is perpendicular to PS.

(d) For the value of a found above, calculate the exact area of the triangle OPS.

[10 marks]

4. Consider the tetrahedron shown in the diagram and define vectors $\boldsymbol{a} = \overrightarrow{CB}$, $\boldsymbol{b} = \overrightarrow{CD}$ and $\boldsymbol{c} = \overrightarrow{CA}$.



(a) Write down an expression for the area of the base in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ only.

(b) AE is the height of the tetrahedron, $|AE| = h$ and $C\hat{A}E = \theta$. Express h in terms of $\boldsymbol{c}$ and θ.

(c) Use the results of (a) and (b) to prove that the volume of the tetrahedron is given by $\frac{1}{6}|\boldsymbol{a} \times \boldsymbol{b}| \cdot \boldsymbol{c}$.

(d) Find the volume of the tetrahedron with vertices (0, 4, 0), (0, 6, 0) (1, 6, 1) and (3, −1, 2).

[14 marks]

14 Lines and planes in space

In this chapter you will learn:

- to describe all points in space which lie on the same straight line
- to describe all points in space which lie in the same plane
- to solve three-dimensional problems involving intersections, distances and angles between lines and planes
- the connection between solving systems of linear equations and finding intersections of planes.

Introductory problem

Which is more stable (less wobbly): a three-legged stool or a four-legged stool?

Answering the above question involves thinking about whether given points lie in the same plane (on the same flat surface). In this chapter we will use the work on vectors from the last chapter to solve problems involving points, lines and planes in three-dimensional space. Such calculations are used in many areas such as design, navigation and computer games.

14A Vector equation of a line

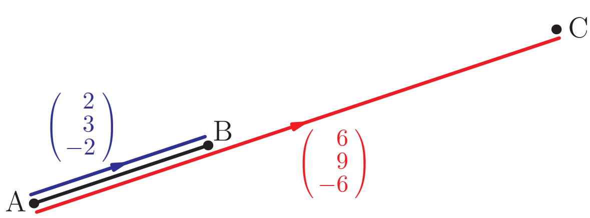
To solve problems involving lines in space we need a way of deciding whether a point lies on a given straight line.

Suppose we have two points, A (−1, 1, 4) and B (1, 4, 2). These two points determine a unique straight line (a 'straight line' extends indefinitely in both directions).

How can we check whether a third point lies on the same line?

We can use vectors to answer this question. For example, consider the point C (5, 10, −2).

Then $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 3\overrightarrow{AB}$.



EXAM HINT

Remember the IB notation for lines and line segments: (AB) means the (infinite) straight line through A and B, [AB] is the line segment (the part of the line between A and B), and AB is the length of [AB].

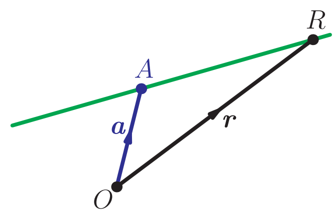
This means that (AC) is parallel to (AB). Since they both contain the point A, (AC) and (AB) must be the same straight line; in other words, C lies on the line (AB).

Next we ask how we can characterise all the points on the line (AB).

Using the above idea, we realise that a point R lies on (AB) if (AR) and (AB) are parallel. We know that this can be expressed using vectors by saying that $\overrightarrow{AR} = \lambda \overrightarrow{AB}$ for some value of the scalar λ (remember that a scalar means a number):

$$\overrightarrow{AR} = \begin{pmatrix} 2\lambda \\ 3\lambda \\ -2\lambda \end{pmatrix}.$$

We also know that $\overrightarrow{AR} = \boldsymbol{r} - \boldsymbol{a}$, where $\boldsymbol{r}$ and $\boldsymbol{a}$ are the position vectors of R and A, respectively.



See Section 13B for a reminder of vector algebra.

This means that $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 3\lambda \\ -2\lambda \end{pmatrix}$ is the position vector of a general point on the line (AB). In other words, R has coordinates $(-1+2\lambda, 1+3\lambda, 4-2\lambda)$ for some value of λ. Different values of λ correspond to different points on the line; for example, $\lambda = 0$ corresponds to point A, $\lambda = 1$ to point B and $\lambda = 3$ to point C.

You will see on page 416 that there is more than one possible vector equation of a line.

The line is parallel to the vector $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, so this vector determines the direction of the line. The expression for the position vector of $\boldsymbol{r}$ is usually written as $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, so it is easy to identify the direction vector.

KEY POINT 14.1

The expression $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$ is a **vector equation** of the line.

The vector $\boldsymbol{d}$ is the direction vector of the line and $\boldsymbol{a}$ is the position vector of one point on the line.

The vector $\boldsymbol{r}$ is the position vector of a general point on the line; different values of parameter λ give positions of different points on the line.

EXAM HINT

The formula booklet tells you the equation, but not what all the letters stand for.

Worked example 14.1

Write down a vector equation of the line passing through the point (−1, 1, 2) in the direction of the vector $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

The equation of the line is $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$, where $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{d}$ is the direction vector

$$\underline{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$$

You know that, in two dimensions, a straight line is determined by its gradient and one point. The gradient is a value which determines the direction of the line. For example, for a line with gradient 3, an increase of 1 unit in x produces an increase of 3 units in y and so the line is in the direction of the vector $\begin{pmatrix}1\\3\end{pmatrix}$.

In three dimensions, a straight line is still determined by its direction and one point, but we cannot use a single number for the gradient. The line in Worked example 14.1 had a direction vector $\begin{pmatrix}2\\2\\1\end{pmatrix}$, so an increase of 2 units in x produces *both* an increase of 2 units in y *and* an increase of 1 unit in z.

Two points determine a straight line. The next example shows how to find a vector equation when two points on the line are given.

Worked example 14.2

Find a vector equation of the line through points A (−1, 1, 2) and B (3, 5, 4).

For a vector equation of the line, we need one point and the direction vector

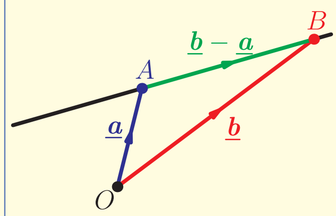
$$\underline{r} = \underline{a} + \lambda\underline{d}$$

The line passes through A (−1, 1, 2); we could use B (3, 5, 4) instead, but smaller values are preferable

$$\underline{a} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

continued . . .

The line is in the direction of $\overrightarrow{AB}$, as we can see by drawing a diagram.



$$\underline{d} = \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$\therefore \underline{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

What if we had used point B instead of A? Then we would have got the equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$. This equation represents the same line, but the value of the parameter λ corresponding to any particular point will be different. For example, with the first equation point A has $\lambda = 0$ and point B has $\lambda = 1$, while using the second equation point A has $\lambda = -1$ and point B has $\lambda = 0$.

Note that the direction vector is not unique either, since we are only interested in its direction and not its magnitude.

Hence $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}$ could also be used as direction vectors for the above line, as they are all in the same direction. So yet another form of the equation of the same line would be

$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}$ and with this equation, point A has $\lambda = 0$ and point B has $\lambda = -\frac{2}{3}$.

To simplify calculations, we usually choose the direction vector to be the one involving smallest possible integer values, although sometimes it is more convenient to use the corresponding unit vector.

See Key point 13.3 about parallel vectors.

Worked example 14.3

(a) Show that the equations $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}$ both represent the same straight line.

(b) Show that the equation $\boldsymbol{r} = \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix}$ represents a different straight line.

EXAM HINT

When a question asks for equations of several lines, then different letters should be used for the parameters. Commonly used letters are λ (lambda), μ (mu), *t* and *s*.

We need to show that the two lines have parallel direction vectors (they will then be parallel) and one common point (then they will be the same line)

Two vectors are parallel if one is a scalar multiple of the other

(a) Direction vectors are parallel, as

$$\begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

The point (5, 7, 5) will lie on the first line if we can find the value of λ which gives this position vector

Show that (5, 7, 5) lies on the first line:

Find the value of λ which gives the first coordinate

$$-1 + 2\lambda = 5$$
$$\therefore \lambda = 3$$

This value of λ must give the other two coordinates

$$\begin{cases} 1 + 3 \times 2 = 7 \\ 2 + 3 \times 1 = 5 \end{cases}$$

∴ (5, 7, 5) lies on the line.

Hence the two lines are the same.

Check whether the direction vectors are parallel

(b) $\begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

So the line is parallel to the other two.

Check whether (5, −3, 1) lies on the first line. Find the value of λ which gives the first coordinate

$$-1 + 2\lambda = -5$$
$$\therefore \lambda = -2$$

continued . . .

This value of λ must give the other two coordinates

$$\begin{cases} 1+(-2)\times 2=-3 \\ 2+(-2)\times 1=0\neq 1 \end{cases}$$

(5, −3, 1) does not lie on the line.

Hence the line is not the same as the first line.

In the above example we used the coordinates of the point to find the corresponding value of λ. However, sometimes we only know that a point lies on the line, but not its precise coordinates. The next example shows how we can work with a general point on the line (with an unknown value of λ).

Worked example 14.4

Point B (3, 5, 4) lies on the line with equation $r=\begin{pmatrix}-1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$. Find the possible positions of a point Q on the line such that $BQ = 15$.

We know that Q lies on the line, so it has the position vector $\begin{pmatrix}-1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ for some value of λ, and we need to find λ.

$$\underline{q}=\begin{pmatrix}-1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}-1+2\lambda\\1+2\lambda\\2+\lambda\end{pmatrix}$$

We can express vector $\overrightarrow{BQ}$ in terms of λ and then set its magnitude equal to 15

$$\overrightarrow{BQ}=\underline{q}-\underline{b}$$

$$=\begin{pmatrix}-1+2\lambda\\1+2\lambda\\2+\lambda\end{pmatrix}-\begin{pmatrix}3\\5\\4\end{pmatrix}=\begin{pmatrix}2\lambda-4\\2\lambda-4\\\lambda-2\end{pmatrix}$$

$$\left|\overrightarrow{BQ}\right|=15$$

It is easier to work without the square root, so we square the magnitude equation

$$\therefore (2\lambda-4)^2+(2\lambda-4)^2+(\lambda-2)^2=15^2$$

$$\Leftrightarrow \quad 9\lambda^2-36\lambda-189=0$$

$$\Leftrightarrow \quad \lambda=-3 \text{ or } 7$$

We can now find the position vector of Q by substituting the values of λ in to the line equation

$$\therefore \underline{q}=\begin{pmatrix}-7\\-5\\-1\end{pmatrix} \text{ or } \begin{pmatrix}13\\15\\9\end{pmatrix}$$

Exercise 14A

1. Find the vector equation of each line in the given direction through the given point.

(a) (i) Direction $\begin{pmatrix}1\\4\end{pmatrix}$, point $(4,-1)$

(ii) Direction $\begin{pmatrix}2\\-3\end{pmatrix}$, point $(4,1)$

(b) (i) Point $(1,0,5)$, direction $\begin{pmatrix}1\\3\\-3\end{pmatrix}$

(ii) Point $(-1,1,5)$, direction $\begin{pmatrix}3\\-2\\2\end{pmatrix}$

(c) (i) Point $(4,0)$, direction $2\boldsymbol{i}+3\boldsymbol{j}$

(ii) Point $(0,2)$, direction $\boldsymbol{i}-3\boldsymbol{j}$

(d) (i) Direction $\boldsymbol{i}-3\boldsymbol{k}$, point $(0,2,3)$

(ii) Direction $2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, point $(4,-3,0)$

2. Find a vector equation of the line through each pair of points.

(a) (i) $(4,1)$ and $(1,2)$ (ii) $(2,7)$ and $(4,-2)$

(b) (i) $(-5,-2,3)$ and $(4,-2,3)$ (ii) $(1,1,3)$ and $(10,-5,0)$

3. Decide whether or not the given point lies on the given line.

(a) (i) Line $\boldsymbol{r}=\begin{pmatrix}2\\1\\5\end{pmatrix}+t\begin{pmatrix}-1\\2\\2\end{pmatrix}$, point $(0,\ 5,\ 9)$

(ii) Line $\boldsymbol{r}=\begin{pmatrix}-1\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\1\\5\end{pmatrix}$, point $(-1,\ 0,\ 3)$

(b) (i) Line $\boldsymbol{r}=\begin{pmatrix}4\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\0\\3\end{pmatrix}$, point $(0,\ 0,\ 0)$

(ii) Line $\boldsymbol{r}=\begin{pmatrix}-1\\5\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\7\end{pmatrix}$, point $(-1,\ 3,\ 8)$

4. (a) Show that the points A (4, −1, −8) and B (2, 1, −4) lie on the line l with equation $r = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

(b) Find the coordinates of the point C on the line l such that $AB = BC$. *[6 marks]*

5. (a) Find the vector equation of line l through points P (7, 1, 2) and Q (3, −1, 5).

(b) Point R lies on l and $PR = 3PQ$. Find the possible coordinates of R. *[6 marks]*

6. (a) Write down the vector equation of the line l through the point A (2, 1, 4) parallel to the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

(b) Calculate the magnitude of the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

(c) Find the possible coordinates of point P on l such that $AP = 35$. *[8 marks]*

14B Solving problems with lines

In this section we will use vector equations of lines to solve problems involving angles and intersections.

We will also see how vector equations of lines can be used to describe paths of moving objects in mechanics.

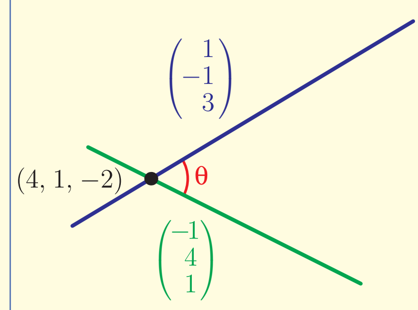
Worked example 14.5

Find the acute angle between lines with equations $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

We know the formula for the angle between two vectors (see Section 13D)

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Draw a diagram to identify which two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ make the required angle



continued . . .

The two vectors are in the directions of the two lines. So we take ***a*** and ***b*** to be the direction vectors of the two lines

$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

We can now use the formula to calculate the angle

$$\therefore \quad \cos\theta = \frac{-1-4+3}{\sqrt{(1+1+9)}\sqrt{1+16+1}}$$

$$= -\frac{2}{\sqrt{11}\sqrt{18}}$$

$$\theta = 98.2°$$

The angle found is obtuse; the question asked for the acute angle

acute angle $= 180° - 98.2° = 81.8°$

The example above illustrates the general method for finding an angle between two lines.

KEY POINT 14.2

The angle between two lines is equal to the angle between their direction vectors.

Now that we know that the angle between two lines is the angle between their direction vectors, it is easy to identify parallel and perpendicular lines.

KEY POINT 14.3

Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are:

- parallel if $\boldsymbol{d}_1 = k\,\boldsymbol{d}_2$
- perpendicular if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.

Parallel and perpendicular vectors were covered in Sections 13B and 13E.

Worked example 14.6

Decide whether the following pairs of lines are parallel, perpendicular, or neither:

(a) $\boldsymbol{r}=\begin{pmatrix}2\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}4\\-1\\2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-2\\0\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\-3\end{pmatrix}$

(b) $\boldsymbol{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+t\begin{pmatrix}1\\0\\-3\end{pmatrix}$

(c) $\boldsymbol{r}=\begin{pmatrix}2\\-1\\5\end{pmatrix}+t\begin{pmatrix}4\\-6\\2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-2\\0\\3\end{pmatrix}+s\begin{pmatrix}-10\\15\\-5\end{pmatrix}$

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(a) If $\begin{pmatrix}4\\-1\\2\end{pmatrix}=k\begin{pmatrix}1\\-2\\-3\end{pmatrix}$ then

$$\begin{cases}4=k\times1\Rightarrow k=4\\-1=k\times(-2)\Rightarrow k=\frac{1}{2}\end{cases}$$

$4\neq\frac{1}{2}$

∴ They are not parallel.

Is $\boldsymbol{d}_1\cdot\boldsymbol{d}_2=0$?

$$\begin{pmatrix}4\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-3\end{pmatrix}=4+2-6=0$$

∴ The lines are perpendicular.

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(b) If $\begin{pmatrix}2\\1\\1\end{pmatrix}=k\begin{pmatrix}1\\0\\-3\end{pmatrix}$ then

$$\begin{cases}2=k\times1\Rightarrow k=2\\1=k\times0 \text{ impossible}\end{cases}$$

∴ They are not parallel.

Is $\boldsymbol{d}_1\cdot\boldsymbol{d}_2=0$?

$$\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\end{pmatrix}=2+0+6=8\neq0$$

∴ The lines are neither parallel nor perpendicular.

continued . . .

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(c) If $\begin{pmatrix}4\\-6\\2\end{pmatrix} = k\begin{pmatrix}-10\\15\\-5\end{pmatrix}$ then

$$\begin{cases} 4 = k\times(-10) \Rightarrow k = -\frac{2}{5} \\ -6 = k\times 15 \Rightarrow k = -\frac{2}{5} \\ 2 = k\times(-5) \Rightarrow k = -\frac{2}{5} \end{cases}$$

∴ The lines have parallel directions.

Check to see if they are the same line

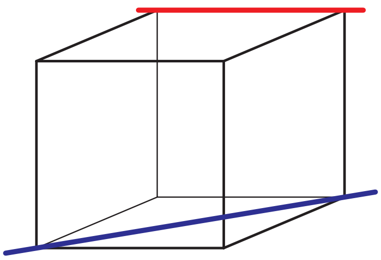
Point $\begin{pmatrix}-2\\0\\3\end{pmatrix}$ does not lie on the first line:

$$\begin{cases} 2+4t = -2 \Rightarrow t = -1 \\ -1-6t = 0 \Rightarrow t = -\frac{1}{6} \end{cases}$$

They are not the same line.
∴ The lines are parallel.

We will now see how to find the point of intersection of two lines. Suppose two lines have vector equations $\boldsymbol{r}_1 = \boldsymbol{a} + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r}_2 = \boldsymbol{b} + \mu\boldsymbol{d}_2$. If they intersect, then there must be a point which lies on both lines. Remembering that the position vector of a point on the line is given by the vector $\boldsymbol{r}$, this means that we need to find the values of λ and μ which make $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

In two dimensions, two straight lines either intersect or are parallel. However, in three dimensions it is possible to have two lines which are not parallel but do not intersect, as illustrated by the red and blue lines in the diagram. Such lines are called **skew lines**.



With skew lines we will see that we cannot find values of λ and μ such that $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

Worked example 14.7

Find the coordinates of the point of intersection of the following pairs of lines.

(a) $\boldsymbol{r}=\begin{pmatrix}0\\-4\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\mu\begin{pmatrix}4\\-2\\-2\end{pmatrix}$ (b) $\boldsymbol{r}=\begin{pmatrix}-4\\1\\3\end{pmatrix}+t\begin{pmatrix}1\\1\\4\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\2\end{pmatrix}$

We need to make $\boldsymbol{r}_1=\boldsymbol{r}_2$

(a) $\begin{pmatrix}0\\-4\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\mu\begin{pmatrix}4\\-2\\-2\end{pmatrix}$

$$\Leftrightarrow\begin{pmatrix}0+\lambda\\-4+2\lambda\\1+\lambda\end{pmatrix}=\begin{pmatrix}1+4\mu\\3-2\mu\\5-2\mu\end{pmatrix}$$

If two vectors are equal, then all their components are equal

$$\Rightarrow\begin{cases}0+\lambda=1+4\mu\\-4+2\lambda=3-2\mu\\1+\lambda=5-2\mu\end{cases}$$

$$\Rightarrow\begin{cases}\lambda-4\mu=1 & (1)\\2\lambda+2\mu=7 & (2)\\\lambda+2\mu=4 & (3)\end{cases}$$

Solve two simultaneous equations in two variables. Use eqn (1) and (3) (as subtracting them eliminates λ)

$(3)-(1)\quad 6\mu=3$

$\therefore \mu=\frac{1}{2}, \lambda=3$

We need to check that the values of λ and μ also satisfy the second equation otherwise the lines do not actually meet

$(2): 2\times3+2\times\frac{1}{2}=7$

∴ the lines intersect

The position of the intersection point is given by the vector $\boldsymbol{r}_1$ (or $\boldsymbol{r}_2$ - they should be the same)

$$\underline{r}_1=\begin{pmatrix}0\\-4\\1\end{pmatrix}+3\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}3\\2\\4\end{pmatrix}$$

The lines intersect at the point $(3, 2, 4)$

Make $\boldsymbol{r}_1=\boldsymbol{r}_2$

(b) $\begin{pmatrix}-4\\3\\3\end{pmatrix}+t\begin{pmatrix}1\\1\\4\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\2\end{pmatrix}$

$$\Rightarrow\begin{cases}t-2\lambda=6 & (1)\\t+3\lambda=-2 & (2)\\4t-2\lambda=-2 & (3)\end{cases}$$

→

continued . . .

We can find t and λ from eqn (1) and (2)

(1) and $(2) \Rightarrow \lambda = -\frac{8}{5}, t = \frac{14}{5}$

We need to check that the values found also satisfy the third equation

$(3) \quad 4 \times \frac{14}{5} - 2 \times \left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$

This tells us that it is impossible to find t and λ to make $\boldsymbol{r}_1 = \boldsymbol{r}_2$

The two lines do not intersect.

EXAM HINT

You can use your calculator to solve simultaneous equations.
See Calculator sheet 6 on the CD-ROM.

Vector questions often ask you to find a point on a given line which satisfies certain conditions. We have already seen how we can use the position vector $\boldsymbol{r}$ for a general point on the line, and then use the condition to write an equation for λ.

See Worked example 14.4.

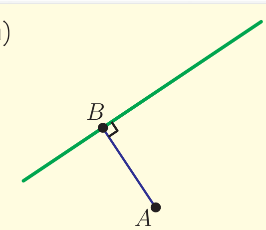
Worked example 14.8

Line l has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and point A has coordinates (3, 9, –2).

(a) Find the coordinates of point B on l so that AB is perpendicular to l.
(b) Hence find the shortest distance from A to l.
(c) Find the coordinates of the reflection of the point A in l.

Draw a diagram. The line AB should be perpendicular to the direction vector of l

(a)



$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \qquad (1)$

continued . . .

We know that B lies on l, so its position vector is given by the equation for $\boldsymbol{r}$

$$\overrightarrow{OB} = \boldsymbol{r} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix}$$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

Now find the value of λ for which the two lines are perpendicular

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow (\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\Rightarrow \lambda = -4$$

Use value of λ in the equation of the line to give the position vector of B

$$r = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

$\therefore$ B has coordinates $(-1, 3, -4)$

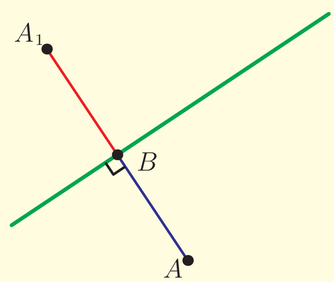
The shortest distance from a point to a line is the perpendicular distance AB. Again, use $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{16+36+4} = 2\sqrt{14}$$

The reflection A_1 lies on the line (AB). Since $BA_1 = AB$ and they are also in the same direction, $\overrightarrow{BA_1} = \overrightarrow{AB}$

(c)



$$\overrightarrow{BA_1} = \overrightarrow{AB}$$

$$\Rightarrow \underline{a}_1 - \underline{b} = \overrightarrow{AB}$$

$$\therefore \underline{a}_1 = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

So A_1 has coordinates $(-5, -3, -6)$

Worked example 14.8(c) illustrates the power of vectors. As vectors contain both distance and direction information, just one equation ($\overrightarrow{BA_1} = \overrightarrow{AB}$) was needed to express both the fact that A_1 lies on the line (AB) and that $BA_1 = AB$.

We have already mentioned that vectors have many applications, particularly in physics. One such application is describing positions, displacements and velocities. These are all vector quantities, since they have both magnitude and direction.

You are probably familiar with the rule that, for an object moving with constant velocity, displacement = velocity × time. If we are working in two or three dimensions, the positions of points also need to be described by vectors. Suppose an object has constant velocity $\boldsymbol{v}$ and in time t moves from the point with position $\boldsymbol{a}$ to the point with position $\boldsymbol{r}$. Then its displacement is $\boldsymbol{r} - \boldsymbol{a}$, so we can write:

$$\boldsymbol{r} - \boldsymbol{a} = \boldsymbol{v}t$$

This equation can be rearranged to $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$, which looks very much like a vector equation of a line with direction vector $\boldsymbol{v}$. This makes sense, as the object will move in the direction given by its velocity vector. As t changes, $\boldsymbol{r}$ gives position vectors of different points along the object's path.

Note that the speed is the magnitude of the velocity, $|\boldsymbol{v}|$, and the distance travelled is the magnitude of the displacement, $|\boldsymbol{r} - \boldsymbol{a}|$.

In 1896 the physicist Lord Kelvin wrote: "'vector' is a useless survival, or offshoot from quaternions, and has never been of the slightest use to any creature".

(Quaternions are a special type of number linked to complex numbers.)

They are now one of the most important tools in physics. Even great mathematicians cannot always predict what will be useful!

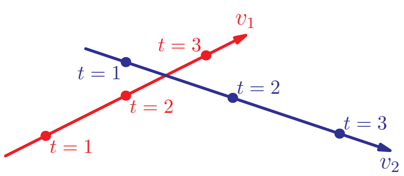
KEY POINT 14.4

For an object moving with constant velocity $\boldsymbol{v}$ from an initial position $\boldsymbol{a}$, the position at time t is given by $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$.

The object moves along the straight line with equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$.

The speed of the object is equal to $|\boldsymbol{v}|$.

When we wanted to find the intersection of two lines, we had to use different parameters (for example, λ and μ) in the two equations. If we have two objects, we can write an equation for $\boldsymbol{r}(t)$ for each of them. In this case, we should use the same t in both equations, as both objects are moving at the same time. For the two objects to meet, they need to be at the same place at the same time. Notice that it is possible for the objects' paths to cross without the objects themselves meeting, if they pass through the intersection point at different times.



Worked example 14.9

Two objects, A and B, have velocities $\boldsymbol{v}_A = 6\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v}_B = -2\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$. Object A starts from the origin and object B from the point with position vector $13\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$. Distance is measured in kilometres and time in hours.

(a) What is the speed of object B?

(b) Find the distance between the two objects after 5 hours.

(c) Show that the two objects do not meet.

Speed is the magnitude of velocity

(a)

$$|v_B| = \sqrt{2^2 + 1^2 + 7^2} = \sqrt{54}$$

So the speed of B is 7.35 km/h.

We need an equation for the position of each object in terms of t

(b)

Using $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$:

$$\boldsymbol{r}_A(t) = t(6\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{r}_B(t) = 13\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} + t(-2\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k})$$

We can then find the position of each object when $t = 5$

When $t = 5$:

$$\boldsymbol{r}_A = 30\boldsymbol{i} + 15\boldsymbol{j} + 5\boldsymbol{k}$$

$$\boldsymbol{r}_B = 3\boldsymbol{i} + 4\boldsymbol{j} + 38\boldsymbol{k}$$

The distance is the magnitude of $\boldsymbol{r}_A - \boldsymbol{r}_B$

$$|\boldsymbol{r}_A - \boldsymbol{r}_B| = \sqrt{27^2 + 11^2 + 33^2}$$

$$= 44.0 \text{ km}$$

If the two objects meet then $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$

(c)

If $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$:

$$\begin{cases} 6t = 13 - 2t \Rightarrow t = \frac{13}{8} \\ 3t = -1 + t \Rightarrow t = -\frac{1}{2} \\ t = 3 + 7t \Rightarrow t = -\frac{1}{2} \end{cases}$$

The three coordinates are not equal at the same time, so the objects do not meet.

Exercise 14B

1. Find the acute angle between the following pairs of lines, giving your answer in degrees.

(a) (i) $\boldsymbol{r} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + s \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 6 \\ 6 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

2. For each pair of lines, state whether they are parallel, perpendicular, the same line, or none of the above.

(a) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \text{»} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

(b) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$

(c) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

(d) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 5 \\ -1 \\ 10 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

3. Determine whether the following pairs of lines intersect and, if they do, find the coordinates of the intersection point.

(a) (i) $\boldsymbol{r} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -4 \\ -4 \\ -11 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

4. Line l has equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates (7, 2, 3).

Point C lies on l and PC is perpendicular to l. Find the coordinates of C. *[6 marks]*

5. Find the shortest distance from the point (−1, 1, 2) to the line with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. *[6 marks]*

6. Two lines are given by $l_1 : \boldsymbol{r} = \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and

$l_2 : \boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

(a) l_1 and l_2 intersect at P, find the coordinates of P.

(b) Show that the point Q (5, 2, 5) lies on l_2.

(c) Find the coordinates of point M on l_1 such that QM is perpendicular to l_1.

(d) Find the area of the triangle PQM. *[10 marks]*

7. In this question, unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ point due East and North, respectively.

A port is located at the origin. One ship starts from the port and moves with velocity $\boldsymbol{v}_1 = (3\boldsymbol{i} + 4\boldsymbol{j})$ kmh^{-1}.

(a) Write down the position vector at time t hours.

At the same time, a second ship starts 18 km north of the port and moves with velocity $\boldsymbol{v}_2 = (3\boldsymbol{i} - 5\boldsymbol{j})$ kmh^{-1}.

(b) Write down the position vector of the second ship at time t hours.

(c) Show that after half an hour, the distance between the two ships is 13.5 km.

(d) Show that the ships meet, and find the time when this happens.

(e) How long after the meeting are the ships 18 km apart?

[12 marks]

8. At time $t = 0$, two aircraft have position vectors $5\boldsymbol{j}$ and $7\boldsymbol{k}$. The first moves with velocity $3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and the second with velocity $5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

(a) Write down the position vector of the first aircraft at time t.

(b) Show that at time t the distance, d, between the two aircraft is given by $d^2 = 44t^2 - 88t + 74$.

(c) Show that the two aircraft will not collide.

(d) Find the minimum distance between the two aircraft.

[12 marks]

9. Find the distance of the line with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ from the origin.

[7 marks]

10. Two lines with equations $l_1 : \boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $l_2 : \boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ intersect at point P.

(a) Find the coordinates of P.

(b) Find, in degrees, the acute angle between the two lines.

Point Q has coordinates (−1, 5, 10).

(c) Show that Q lies on l_2.

(d) Find the distance PQ.

(e) Hence find the shortest distance from Q to the line l_1. *[12 marks]*

11. Given line $l: \boldsymbol{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ and point P (21, 5, 10):

(a) Find the coordinates of point M on l such that PM is perpendicular to l.

(b) Show that the point Q (15, −14, 17) lies on l.

(c) Find the coordinates of point R on l_1 such that $|PR| = |PQ|$. *[10 marks]*

12. Two lines have equations $l_1: \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and

$l_2: \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and intersect at point P.

(a) Show that Q (5, 2, 6) lies on l_2.

(b) R is a point on l_1 such that $|PR| = |PQ|$. Find the possible coordinates of R. *[8 marks]*

14C Other forms of equation of a line

You know that in two dimensions, a straight line has equation of the form $y = mx + c$ or $ax + by = c$. How is this related to the vector equation of the line we introduced in this chapter?

Let us look at an example of a vector equation of a line in two dimensions. A line with direction vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ passing through the point (1, 4) has vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Vector $\boldsymbol{r}$ is the position vector of a point on the line; in other words, it gives coordinates (x, y) of a point on the line. So $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}+\lambda\begin{pmatrix}3\\2\end{pmatrix}$. This vector equation represents two equations:

$$\begin{cases}x=1+3\lambda\\y=4+2\lambda\end{cases}$$

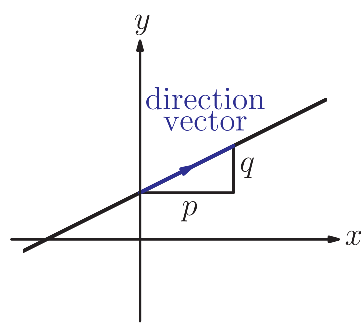
These are called **parametric equations**, because x and y are given in terms of a parameter λ. We can eliminate λ to obtain an equation relating x and y:

$$\lambda=\frac{x-1}{3}$$

$$\therefore\ y=4+\frac{2x-2}{3}$$

$$\Leftrightarrow y=\frac{2}{3}x+\frac{10}{3}$$

This is the more familiar **Cartesian equation** of the line.



It is easy to see how the gradient of line is related to the direction vector: if the direction vector is $\begin{pmatrix}p\\q\end{pmatrix}$ then the gradient is $\frac{q}{p}$, as illustrated in the diagram.

What happens if we try to apply the same method to find a Cartesian equation of a line in three dimensions?

Consider the line with vector equation $\boldsymbol{r}=\begin{pmatrix}1\\4\\-1\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\5\end{pmatrix}$.

The parametric equations of this line are

$$\begin{cases}x=1+3\lambda\\y=4+2\lambda\\z=-1+5\lambda\end{cases}$$

We can substitute λ from the first equation into the other two to express, for example, z in terms of x and y.

$$y=\frac{2}{3}x+\frac{10}{3},\quad z=\frac{5}{3}x-\frac{8}{3}$$

It looks like we cannot obtain a single equation relating x, y and z.

This also means that there is no concept of a gradient in three dimensions, which is why we have introduced the notion of a direction vector.

A better way of writing one Cartesian equation for the line is to write three parametric equations with λ as the subject:

$$\begin{cases} \lambda = \dfrac{x-1}{3} \\ \lambda = \dfrac{y-4}{2} \\ \lambda = \dfrac{z+1}{5} \end{cases}$$

Equating all three expressions for λ gives another form of the Cartesian equation of the line:

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z+1}{5}$$

EXAM HINT

The Formula booklet shows all three forms of equation of a line (vector, parametric and Cartesian), but it does not tell you how to change between them.

KEY POINT 14.5

To find the Cartesian equation of a line from its vector equation:

- Write $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of λ, giving three equations.
- Make λ the subject of each equation.
- Equate the three expressions for λ to get an equation of the form $\dfrac{x-x_0}{l} = \dfrac{y-y_0}{m} = \dfrac{z-z_0}{n}$.

Sometimes a Cartesian equation cannot be written in the above form, as shown in the following example.

Worked example 14.10

Find the Cartesian equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ 5 \\ 0 \end{pmatrix}$.

$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so write an equation involving x, y and z

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ 5 \\ 0 \end{pmatrix}$$

continued . . .

Express λ in terms of x, y and z

$$\begin{cases} x = 1 + \frac{1}{3}\lambda \Rightarrow \lambda = \frac{x-1}{1/3} \\ y = \frac{1}{2} + 5\lambda \Rightarrow \lambda = \frac{y - \frac{1}{2}}{5} \\ z = -3 \end{cases}$$

Equate the expressions for λ from the first two equations. The third equation does not contain λ, so leave it as a separate equation

$$\frac{x-1}{1/3} = \frac{y - \frac{1}{2}}{5}, \; z = -3$$

It will look neater if we rewrite the equation without 'fractions within fractions'

$$\Leftrightarrow \frac{3x-3}{1} = \frac{2y-1}{10}, \; z = -3$$

EXAM HINT

The Cartesian equation can sometimes be 'read off' the vector equation; if the vector equation is $\boldsymbol{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} k \\ m \\ n \end{pmatrix}$ then the Cartesian equation is $\frac{x-a}{k} = \frac{y-b}{m} = \frac{z-c}{n}$.

However, if any of the components of the direction vector is 0, we have to complete the whole procedure described in the Key point 14.5.

We can reverse the above procedure to go from Cartesian to vector equation. Vector equations are convenient if we need to identify the direction vector of the line, or to use methods from Section 14B to solve problems involving lines.

KEY POINT 14.6

To find a vector equation of a line from a Cartesian equation in the form $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$:

- Set each of the three expressions equal to λ.
- Express x, y and z in terms of λ.
- Write $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to obtain $\boldsymbol{r}$ in terms of λ.

We can adapt this procedure for Cartesian equations that are not of the above form, as in the next example.

Worked example 14.11

Find a vector equation of the line with Cartesian equation $x = -2, \frac{3y+1}{4} = \frac{2-z}{5}$. Hence write down the direction vector of the line, making all its components integers.

Introduce a parameter λ. As the two expressions involving y and z are equal, set them both equal to λ

$$\begin{cases} \frac{3y+1}{4} = \lambda \\ \frac{2-z}{5} = \lambda \end{cases}$$

Now express x, y and z in terms of λ

$$\therefore \begin{cases} x = -2 \\ y = \frac{4\lambda - 1}{3} \\ z = 2 - 5\lambda \end{cases}$$

The vector equation is an equation for $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of λ. Separate the expression into a part without λ and a part involving λ

$$\therefore \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -\frac{1}{3} + \frac{4}{3}\lambda \\ 2 \quad -5\lambda \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} -2 \\ -\frac{1}{3} \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \frac{4}{3} \\ -5 \end{pmatrix}$$

Now identify the direction vector

The direction vector is $\begin{pmatrix} 0 \\ \frac{4}{3} \\ 5 \end{pmatrix}$

We can change the magnitude of the direction vector so that it does not contain fractions (multiply by 3 in this case)

or $\begin{pmatrix} 0 \\ 4 \\ -15 \end{pmatrix}$

You can solve problems involving lines given by Cartesian equations by changing into the vector equation and using methods from the previous section. However, there are some problems that can be solved more quickly by using the Cartesian equation directly.

Worked example 14.12

Does the point A (3, −2, 2) lie on the line with equation $\frac{x+1}{2}=\frac{4-y}{3}=\frac{2z}{3}$?

If the point lies on the line, the coordinates should satisfy the Cartesian equation.
This means that all three expressions should be equal

$$\begin{cases} \frac{x+1}{2}=\frac{3+1}{2}=2 \\ \frac{4-y}{3}=\frac{4+2}{3}=2 \\ \frac{2z}{3}=\frac{2\times 2}{3}=\frac{4}{3} \end{cases}$$

The second equality is not satisfied

$$2=2\neq\frac{4}{3}$$

∴ The point does not lie on the line.

Intersections of a line with the coordinate axes are also easy to find using the Cartesian equation.

Worked example 14.13

(a) Find the coordinates of the point where the line with equation $\frac{x-6}{2}=\frac{y+1}{7}=\frac{z+9}{-3}$ intersects the y-axis.

(b) Show that the line does not intersect the z-axis.

A point on the y-axis has $x = z = 0$

(a) A point on the y-axis has coordinates $(0, m, 0)$

Substitute coordinates into the Cartesian equation

$$\frac{0-6}{2}=\frac{m+1}{7}=\frac{0+9}{-3}$$

$$\Leftrightarrow \quad -3=\frac{m+1}{7}=-3$$

Find m

$$m+1=-21$$

$$\therefore m=-22$$

The point of intersection is $(0, -22, 0)$

→

continued . . .

A point on the z-axis has $x = y = 0$

(b) A point on the z-axis has coordinates $(0, 0, m)$

Substitute coordinates into the Cartesian equation

$$\frac{0-6}{2} \neq \frac{0+1}{7}$$

The first equality is not satisfied

The line does not intersect the z-axis.

Exercise 14C

1. (a) Write down the Cartesian equation of the line

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

(b) Write down the Cartesian equation of the line

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}.$$

(c) Write down a vector equation of the line with Cartesian equation $\frac{x-3}{2} = \frac{y+1}{-4} = \frac{z}{5}$.

(d) Write down a vector equation of the line with Cartesian equation $\frac{x+1}{5} = \frac{3-z}{2}, y = 1$.

2. Determine whether the following pairs of lines are parallel, perpendicular, the same line, or none of the above.

(a) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = 4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k} + t(5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

(b) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -6 \\ 9 \\ 2 \end{pmatrix}$ and $\frac{2x-1}{4} = \frac{y-2}{-3} = \frac{6-3z}{2}$

(c) $\frac{x-5}{7} = \frac{y-2}{-1} = 4 - z$ and $x = 2\lambda + 1, y = 4, z = 5 - \lambda$

(d) $x = 2t + 1, y = 1 - 4t, z = 3$ and $\boldsymbol{r} = \begin{pmatrix} 8 \\ -13 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

3. (a) Find the Cartesian equation of the line with parametric equation $x = 3\lambda + 1, \quad y = 4 - 2\lambda, \quad z = 3\lambda - 1$.

(b) Find the unit vector in the direction of the line. *[5 marks]*

4. (a) Find a vector equation of the line with Cartesian equation $\frac{2x-1}{4} = \frac{y+2}{3} = \frac{4-3z}{6}$.

(b) Determine whether the line intersects the x-axis.

(c) Find the angle the line makes with the x-axis. *[8 marks]*

5. (a) Find, in degrees, the angle between the lines $\frac{x-3}{5} = y - 2 = \frac{3-2z}{2}$ and $\frac{x+1}{3} = 3 - z, y = 1$.

(b) Determine whether the lines intersect. *[7 marks]*

6. (a) Find the coordinates of the point of intersection of the lines with Cartesian equations $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z+1}{1}$ and $5 - x = \frac{y+2}{-3} = \frac{z-7}{2}$.

(b) Show that the line with equation $\boldsymbol{r} = \begin{pmatrix} 7 \\ 8 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ passes through the above intersection point. *[6 marks]*

14D Equation of a plane

We next look at different ways of representing all points that lie in a given plane.

We will first try an approach like to the one used to derive a vector equation of a line, where we noted that every point on the line can be reached from the origin by going to one particular point on the line and then moving along the line using the direction vector. This gave us an equation for the position vector of any point on the line in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$.

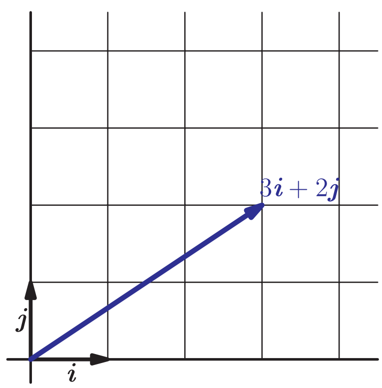
Can we similarly describe a way to get from the origin to any point in a plane?

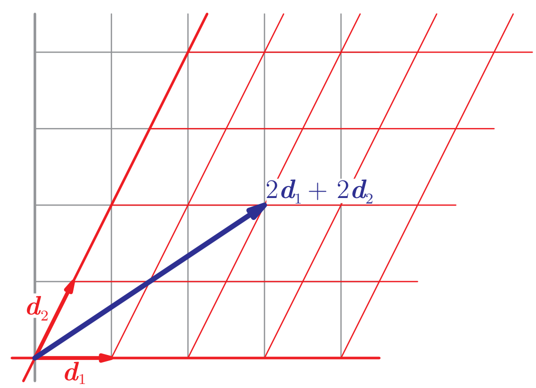
We describe positions of points in the x-y plane using coordinates. To reach the point with coordinates (p, q) from the origin, we move p units in x-direction and q units in y-direction.

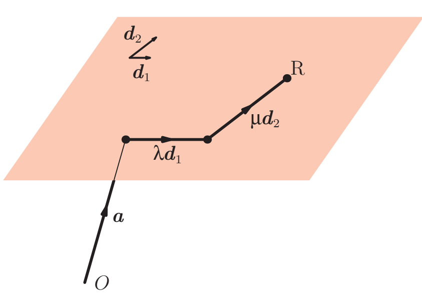
We can also express this using unit vectors parallel to the x-and y-axes; the position vector of the point P (2, 3) is $\boldsymbol{r}_p = 3\boldsymbol{i} + 2\boldsymbol{j}$.

However, it is possible to use directions other than those of $\boldsymbol{i}$ and $\boldsymbol{j}$. In the second diagram below, the same point P is reached from the origin by moving 2 units in the direction of vector $\boldsymbol{d}_1$ and 2 units in the direction of vector $\boldsymbol{d}_2$. Hence its position vector is $\boldsymbol{r}_p = 2\boldsymbol{d}_1 + 2\boldsymbol{d}_2$.

You can see that any point in the plane can be reached by going a certain number of units in the direction of $\boldsymbol{d}_1$ and a certain number of units in the direction of $\boldsymbol{d}_2$; hence every point in the plane has a position vector of the form $\lambda\boldsymbol{d}_1 + \mu\boldsymbol{d}_2$, where λ and μ are scalars.







Now consider a plane that does not pass through the origin. To reach a point in the plane starting from the origin, we go to some other point in the plane first, and then move along two directions which lie in the plane, as illustrated in the diagram alongside.

This means that every point in the plane has a position vector of the form $\boldsymbol{a} + \lambda\boldsymbol{d}_1 + \mu\boldsymbol{d}_2$, where $\boldsymbol{a}$ is the position vector of one point in the plane, and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are two vectors parallel to the plane (but not parallel to each other). $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ do not need to be perpendicular to each other.

KEY POINT 14.7

The plane containing point $\boldsymbol{a}$ and parallel to the directions of vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ has a vector equation:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}_1 + \mu\boldsymbol{d}_2$$

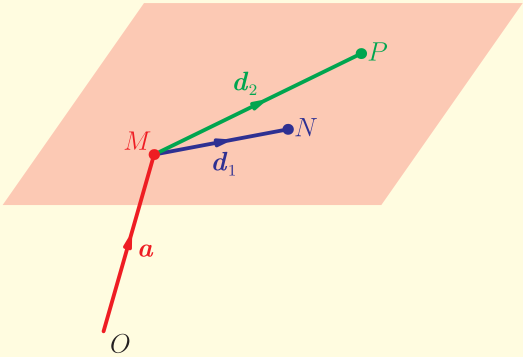
Worked example 14.14

Find a vector equation of the plane containing points M (3, 4, –2), N (1, –1, 3) and P (5, 0, 2).

We need one point and two vectors parallel to the plane

$$\boldsymbol{r} = \underline{a} + \lambda\underline{d}_1 + \mu\underline{d}_2$$

Draw a diagram to see which vectors to use



Choose any of the three given points, as they all lie in the plane

$$\underline{a} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

Vectors $\overrightarrow{MN}$ and $\overrightarrow{MP}$ are parallel to the plane

$$\underline{d}_1 = \overrightarrow{MN} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$$

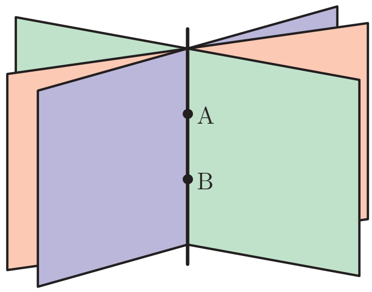
$$\underline{d}_2 = \overrightarrow{MP} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

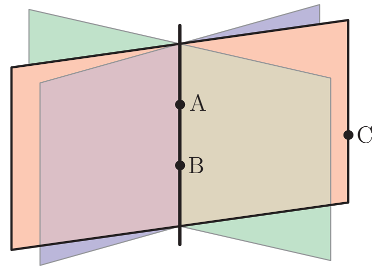
We can now write down the equation

$$\therefore \underline{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

How many points are needed to determine a plane? In the above example, the plane was determined by three points. It should be clear from the diagram below that two points do not determine a plane: there is more than one plane containing the line determined by points A and B.

We can pick out one of these planes by requiring that it also passes through a third point which is not on the line (AB), as illustrated in the second diagram. This suggests that a plane can also be determined by a line and a point outside of that line.





Worked example 14.15

Find a vector equation of the plane containing the line $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ and point $A\,(4, -1, 2)$.

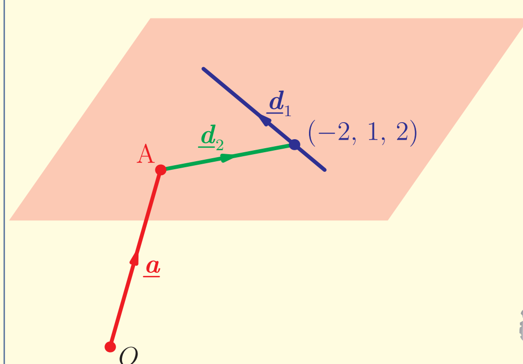
Point A lies in the plane

$$\underline{a} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

The direction vector of the line is parallel to the plane

$$\underline{d}_1 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

We need another vector parallel to the plane. We can use any vector between two points in the plane. One point in the plane is A and for the second point, we can pick any point on the line: for example, (–2, 1, 2)



continued . . .

$$\underline{d}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

We can now write down the equation of the plane

$$\therefore \underline{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

The following example looks at what happens if we have four points.

Worked example 14.16

Determine whether points A (2, −1, 3), B (4, 1, 1), C (3, 3, 2) and D (−3, 1, 5) lie in the same plane.

We know how to find an equation of the plane containing points A, B and C

Plane containing A, B and C

$$\boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

For D to also lie in the plane, we need values of λ and μ which make $\boldsymbol{r}$ equal to the position vector of D

$$\boldsymbol{r} = \overrightarrow{OD}:$$

$$\begin{cases} 2 + 2\lambda + \mu = -3 & (1) \\ -1 + 2\lambda + 4\mu = 1 & (2) \\ 3 - 2\lambda - \mu = 5 & (3) \end{cases}$$

Solve the first two equations

$$(1) \text{ and } (2) \quad \begin{cases} 2\lambda + \mu = -5 \\ 2\lambda + 4\mu = 2 \end{cases}$$

$$\Rightarrow \lambda = -\frac{11}{3}, \quad \mu = \frac{7}{3}$$

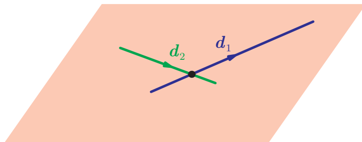
Then check whether the solutions satisfy the third

$$(3) \quad 3 - 2 \times \left(-\frac{11}{3}\right) - \frac{7}{3} = 8 \neq 5$$

There are no values of λ and μ which satisfy all three equations

D does not lie in the same plane as A, B and C

Worked example 14.16 shows that it is not always possible to find a plane containing four given points. However, we can always find a plane containing three given points, as long as they do not lie on the same straight line.

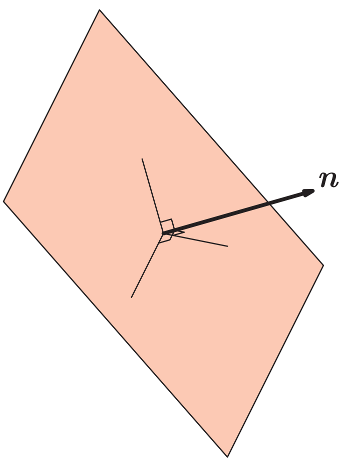


A plane can also be determined by two intersecting lines. In that case, vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ can be taken to be the direction vectors of the two lines. We will see how to find the equation of the plane in Section 14G.

KEY POINT 14.8

To uniquely determine a plane we need:

- three points, not on the same line, OR
- a line and a point outside that line, OR
- two intersecting lines.



A vector equation of the plane can be difficult to work with, as it contains two parameters. It is also difficult to see whether two equations actually describe the same plane, because there are many pairs of vectors parallel to the plane which can be used in the equation. So it is reasonable to ask: Is there a way we can determine the 'direction' of the plane using just one direction vector?

The diagram alongside shows a plane and a vector $\boldsymbol{n}$ which is perpendicular to it. This vector is perpendicular to every line in the plane, and it is called the **normal vector** of the plane.

If we know one point, A, in the plane and the normal vector, what can we say about the position vector of any other point, P, in the plane? The normal vector is perpendicular to the line (AP), so $\overrightarrow{AP}\bullet\boldsymbol{n}=0$. This means that $(\boldsymbol{r}-\boldsymbol{a})\bullet\boldsymbol{n}=0$, which gives us another form of an equation of the plane.

KEY POINT 14.9

A plane with a normal vector $\boldsymbol{n}$ and containing a point with position vector $\boldsymbol{a}$ has a **scalar product equation**:
$\boldsymbol{r}\bullet\boldsymbol{n}=\boldsymbol{a}\bullet\boldsymbol{n}$

Worked example 14.17

Vector $\boldsymbol{n} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ is perpendicular to the plane Π which contains point A (3, −5, 1).

(a) Write an equation of Π in the form $\boldsymbol{r} \cdot \boldsymbol{n} = d$.

(b) Find the Cartesian equation of the plane.

EXAM HINT

The letter Π (capital π) is often used as the name for a plane.

The equation of the plane is $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

(a) $\underline{r} \cdot \underline{n} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

$= 6 - 20 - 1$

$\therefore \underline{r} \cdot \underline{n} = -15$

The Cartesian equation involves x, y and z (the coordinates of P), which are the components of the general position vector $\boldsymbol{r}$

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = -15$

$\Rightarrow 2x + 4y - z = -15$

KEY POINT 14.10

The Cartesian equation of a plane has the form

$$ax + by + cz = d$$

where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector of the plane.

The next example shows how to convert from vector to Cartesian equation of the plane.

Worked example 14.18

Find the Cartesian equation of the plane with vector equation $\boldsymbol{r}=\begin{pmatrix}1\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-3\\5\end{pmatrix}$.

To find the Cartesian equation we need the normal vector and one point

$\underline{r}\cdot\underline{n}=\underline{a}\cdot\underline{n}$

Point (1, –2, 5) lies in the plane

$\underline{a}=\begin{pmatrix}1\\-2\\5\end{pmatrix}$

n is perpendicular to all lines in the plane, so it is perpendicular to the two vectors $\begin{pmatrix}1\\1\\3\end{pmatrix}$ and $\begin{pmatrix}2\\-3\\5\end{pmatrix}$ which are parallel to the plane. The cross product of two vectors is perpendicular to both of them

$\underline{n}=\begin{pmatrix}1\\1\\3\end{pmatrix}\times\begin{pmatrix}2\\-3\\5\end{pmatrix}$

$=\begin{pmatrix}14\\1\\-5\end{pmatrix}$

Cross product was introduced in Section 13G

To get the Cartesian equation, write ***r*** as $\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}14\\1\\-5\end{pmatrix}=\begin{pmatrix}1\\-2\\5\end{pmatrix}\cdot\begin{pmatrix}14\\1\\-5\end{pmatrix}=13$

$\Leftrightarrow \quad 14x+y-5z=13$

The Cartesian equation is very convenient for checking whether a point lies in the plane; we just need to check that the coordinates of the point satisfy the equation. In the next section we will see how to use it to examine the relationship between a line and a plane.

Exercise 14D

1. Write down the vector equation of the plane parallel to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and containing point P.

(a) (i) $\boldsymbol{a} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $P(1, 0, 2)$

(ii) $\boldsymbol{a} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$, $P(0, 2, 0)$

(b) (i) $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{p} = \boldsymbol{j} + \boldsymbol{k}$

(ii) $\boldsymbol{a} = 5\boldsymbol{i} - 6\boldsymbol{j}$, $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{P} = \boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}$

2. Find a vector equation of the plane containing points A, B and C.

(a) (i) $A(3, -1, 3)$, $B(1, 1, 2)$, $C(4, -1, 2)$

(ii) $A(-1, -1, 5)$, $B(4, 1, 2)$, $C(-7, 1, 1)$

(b) (i) $A(9, 0, 0)$, $B(-2, 1, 0)$, $C(1, -1, 2)$

(ii) $A(11, -7, 3)$, $B(1, 14, 2)$, $C(-5, 10, 0)$

3. Find a vector equation of the plane containing line l and point P.

(a) (i) $l: \boldsymbol{r} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, $P(-1, 4, 3)$

(ii) $l: \boldsymbol{r} = \begin{pmatrix} 9 \\ -3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$, $P(11, 12, 13)$

(b) (i) $l: \boldsymbol{r} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $P(-3, 1, 0)$

(ii) $l: \boldsymbol{r} = t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $P(4, 0, 2)$

4. A plane has normal vector $\boldsymbol{n}$ and contains point A. Find the equation of the plane in the form $\boldsymbol{r} \bullet \boldsymbol{n} = d$, and the Cartesian equation of the plane.

(a) (i) $\boldsymbol{n} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, A\ (3, 3, 1)$

(ii) $\boldsymbol{n} = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}, A\ (4, 3, -1)$

(b) (i) $\boldsymbol{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, A\ (-3, 0, 2)$

(ii) $\boldsymbol{n} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}, A\ (0, 0, 2)$

5. Find a normal vector to the plane given by the vector equation:

(a) (i) $\boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$

6. Find the equations of the planes from question 5 in the form $\boldsymbol{r} \bullet \boldsymbol{n} = d$.

7. Find the Cartesian equations of the planes from question 5.

8. Find the Cartesian equation of the plane containing points A, B and C.

(a) (i) $A\,(7, 1, 2)$, $B\,(-1, 4, 7)$, $C\,(5, 2, 3)$

(ii) $A\,(1, 1, 2)$, $B\,(4, -6, 2)$, $C\,(12, 12, 2)$

(b) (i) $A\,(12, 4, 10)$, $B\,(13, 4, 5)$, $C\,(15, -4, 0)$

(ii) $A\,(1, 0, 0)$, $B\,(0, 1, 0)$, $C\,(0, 0, 1)$

9. Show that point P lies in the plane Π.

(a) $P\,(-4, 8, 9)$, $\Pi : \boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$

(b) $P\,(4, 7, 5)$, $\Pi : \boldsymbol{r} \bullet \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 19$

(c) $P\,(1, 1, -2)$, $\Pi : 2x - 3y - 7z = 12$

10. Show that plane Π contains line l.

(a) $\Pi : x + 6y + 2z = 7$, $l : \boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(b) $\Pi : 5x + y - 2z = 15$, $l : \dfrac{x-4}{1} = \dfrac{y+1}{1} = \dfrac{z-2}{3}$

(c) $\Pi : \boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = -5$, $l : \boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}$

(d) $\Pi : \boldsymbol{r} \bullet \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix}$, $l : \dfrac{x-3}{2} = \dfrac{y}{3} = \dfrac{z+1}{2}$

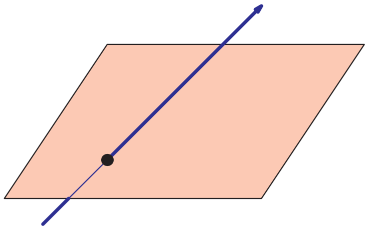
14E Angles and intersections between lines and planes

In this section we will look at angles and intersections between a line and a plane and between two planes. You will be expected to recall and use the four methods shown here. In the examination you will meet much longer and more complicated problems, where you are expected to combine these common techniques.

When finding the intersection between a line and a plane, it is most convenient if the equation of the line is in the vector form and the equation of the plane in the Cartesian form. In all examples in this section the planes will be given by their Cartesian equations, but in the examination you may need to convert them into this form first. The key idea we use is that the coordinates of the intersection point (if there is one) must satisfy both the line equation and the plane equation.

You saw how to find the angle and intersection of two lines in Section 14B.

We will discuss strategies for more complicated problems in the final section of this chapter.



Worked example 14.19

Find the intersection between the given line and plane, or show that they do not intersect.

(a) $\boldsymbol{r} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ and $2x - y + 2z = 7$

(b) $\frac{x-1}{-1} = \frac{y}{-3} = \frac{z+4}{2}$ and $x - 3y - 4z = 12$

(c) $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $x - 3y = 6$

The coordinates of the intersection point must satisfy both equations. Remember that the coordinates of a point on the line are given by vector $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so we can substitute x, y, z into the equation of the plane

(a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4+3\lambda \\ -3 \\ 1-2\lambda \end{pmatrix}$

$\Rightarrow 2(4+3\lambda) - (-3) + 2(1-2\lambda) = 7$

$\Leftrightarrow 2\lambda = -6$

$\Leftrightarrow \lambda = -3$

Now use this value of λ to find the coordinates

$\therefore \underline{r} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix}$

$\therefore$ The intersection point is $(-5, -3, 7)$

We would know how to do this if the equation of the line were in vector form. Set each expression $= \mu$

(b) $\begin{cases} x-1 = -\mu \\ y = -3\mu \\ z+4 = 2\mu \end{cases}$

$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-\mu \\ -3\mu \\ -4+2\mu \end{pmatrix}$

Substitute x, y, z into the equation of the plane

$\therefore (1-\mu) - 3(-3\mu) - 4(-4+2\mu) = 12$

$\Rightarrow \quad 17 = 12$

It is impossible to find a value of μ for a point which satisfies both equations. This means that the line is parallel to the plane

Impossible to find μ.

$\therefore$ The line and plane do not intersect.

Substitute x, y, z into the equation of the plane

(c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+3t \\ -1+t \\ 1 \end{pmatrix}$

$(3+3t) - 3(-1+t) = 6$

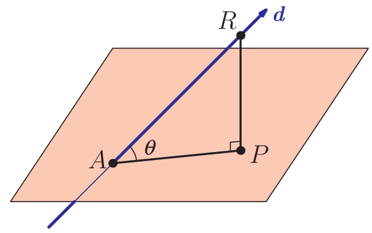
$\Rightarrow \quad 6 = 6$

The equation is satisfied for all values of t. This means that every point on the line also lies in the plane

Every t is a solution.

$\therefore$ The line lies in the plane.

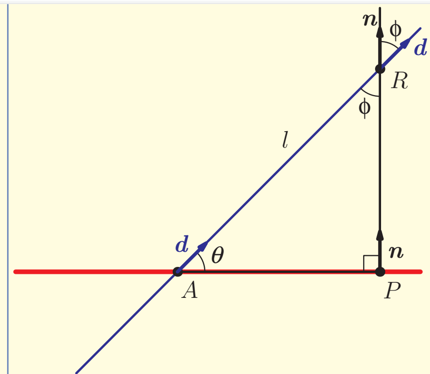
When a line intersects a plane, we can find the angle between them. First we need to decide which angle to find. If we take different lines in the plane, they will make different angles with the given line l. The smallest possible angle θ is with the line $[AP]$ shown in the diagram. Drawing a two-dimensional diagram of triangle APR makes it clearer what the angles are.



Worked example 14.20

Find the angle between line l with equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ and the plane with equation $5x - y + z = 7$.

We want to find the angle marked θ. We don't know the direction of the line (AP), but we do know that $\overrightarrow{AR}$ is in the direction of the line and $\overrightarrow{RP}$ is in the direction of the normal to the plane



Therefore we can find the angle marked ϕ

$$\cos\phi = \frac{\boldsymbol{l} \bullet \boldsymbol{n}}{|\boldsymbol{l}||\boldsymbol{n}|}$$

$$= \frac{\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{9+9+4}\sqrt{25+1+1}}$$

$$= \frac{14}{\sqrt{22}\sqrt{27}}$$

$$\therefore \phi = 54.9°$$

Then we use the fact that $A\hat{P}R$ is a right angle

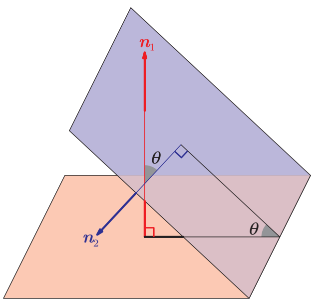
$$\theta = 90° - \phi = 35.1°$$

EXAM HINT
If the angle ϕ between the line and the normal is obtuse, find $\phi_1 = 180° - \phi$, and then $\theta = 90° - \phi_1$

The method shown in Worked example 14.20 can always be used to find the angle between a line and a plane.

KEY POINT 14.11

The angle between line with direction vector $\boldsymbol{d}$ and plane with normal $\boldsymbol{n}$ is $90° - \phi$, where ϕ is the acute angle between $\boldsymbol{d}$ and $\boldsymbol{n}$.



We can also find the angle between two planes. The diagram shows two planes and their normals. Using the fact that the sum of the angles in a quadrilateral is 360°, you can show that the two angles marked θ are equal.

KEY POINT 14.12

The angle between two planes is equal to the angle between their normals.

Worked example 14.21

Find the acute angle between planes with equations $4x - y + 5z = 11$ and $x + y - 3z = 3$.

We need to find the angle between the normals

$$\cos\theta = \frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

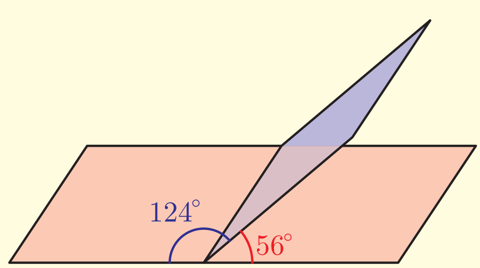
The components of the normal vectors are the coefficients in the Cartesian equations

$$= \frac{\begin{pmatrix}4\\-1\\5\end{pmatrix} \bullet \begin{pmatrix}1\\1\\-3\end{pmatrix}}{\sqrt{16+1+25}\sqrt{1+1+9}}$$

$$= \frac{-12}{\sqrt{42}\sqrt{11}}$$

$$\Rightarrow \theta = 123.9°$$

We need the acute angle



$180° - 123.9° = 56.1°$

The angle between the planes is 56.1°

In our final example we look at the intersection of two planes. We will find the intersection of the planes with equations $4x+5y-z=7$ and $x-4z=-7$.

Two planes intersect along a straight line. Every point on this line must satisfy equations of both planes, so it is a solution of a system of two equations with three unknowns. We saw in Worked example 4.13 that such a system has infinitely many solutions, and we can find the general solution using Gaussian elimination.

We covered Gaussian elimination in Section 4E.

Worked example 14.22

Find the line of intersection of planes with equations $4x+5y-z=7$ and $x-4z=-7$.

The points on the intersection line satisfy both equation, so we need to solve the system using Gaussian elimination. We expect infinitely many solutions

$$\begin{cases} 4x+5y-z=7 & (1) \\ x-4z=-7 & (2) \end{cases}$$

Note that there is no y in equation (2), so we can go straight to back substitution

$$(2) \Rightarrow z=t, x=4t-7$$

$$(1) \Rightarrow 5y = 7+t-4(4t-7)$$

$$= 35-15t$$

$$\therefore y = 7-3t$$

(x, y, z) are coordinates of any point on the line of intersection. To find the vector equation of the line we should write the coordinates as a position vector

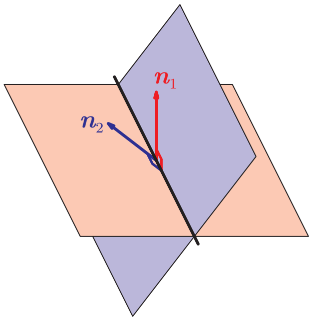
$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4t-7 \\ 7-3t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 4t \\ -3t \\ t \end{pmatrix}$$

This is of the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$, which is a vector equation of a line

The equation of the line is

$$\underline{r} = \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$



It is worth noting that since the line of intersection lies in both planes, it must be perpendicular to the two normals. This can be used to find the direction of the line.

KEY POINT 14.13

The line of intersection of planes with normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ has direction $\boldsymbol{n}_1 \times \boldsymbol{n}_2$.

We can check in the above example that

$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -20 \\ 15 \\ -5 \end{pmatrix}$, which is parallel to the direction $\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ of the intersection line we found.

Worked example 14.23

Find the vector equation of the line of intersection of the planes with equations $3x - y + z = 0$ and $x - 3y - z = 0$.

Both planes contain the origin, so the line of intersection will also pass through the origin. So we only need to find the direction vector

The line passes through $\boldsymbol{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Direction vector:

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix}$$

$$\therefore \boldsymbol{d} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

Vector equation is $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{c}$

Line of intersection:

$$\underline{r} = \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

The only situation when the above methods will fail to find the intersection is if the two planes are parallel. In that case their normal vectors are parallel, and this is easy to see from the Cartesian equations. For example, the planes with equations $2x-4y+2z=5$ and $3x-6y+3z=1$ are parallel, since

$\begin{pmatrix}3\\-6\\3\end{pmatrix}=\frac{3}{2}\begin{pmatrix}2\\-4\\2\end{pmatrix}$. Note that before stating that the two planes:

are parallel you should check that the two equations do not represent the same plane. In this example they are not the same,

as multiplying the first equation by $\frac{3}{2}$ gives $3x-6y+3z=\frac{15}{2}$.

Exercise 14E

1. Find the coordinates of the point of intersection of line l and plane Π.

   (a) (i) $l:\mathbf{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}5\\0\\-1\end{pmatrix}$, $\Pi: 4x+2y-z=29$

   (ii) $l:\mathbf{r}=\begin{pmatrix}-5\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}7\\3\\-3\end{pmatrix}$, $\Pi: x+y+5z=11$

   (b) (i) $l:\frac{x-2}{5}=\frac{y+1}{2}=\frac{z}{6}$, $\Pi:\mathbf{r}\bullet\begin{pmatrix}1\\-4\\1\end{pmatrix}=4$

   (ii) $l:\frac{x-5}{-1}=\frac{y-3}{2}=\frac{z-5}{1}$, $\Pi:\mathbf{r}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}=21$

2. Find the acute angle between line l and plane Π, correct to the nearest $0.1°$.

   (a) (i) $l:\mathbf{r}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\Pi:\mathbf{r}\bullet\begin{pmatrix}4\\-1\\2\end{pmatrix}=7$

   (ii) $l:\mathbf{r}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\1\end{pmatrix}$, $\Pi:\mathbf{r}\bullet\begin{pmatrix}-1\\-2\\2\end{pmatrix}=1$

(b) (i) $l: \frac{x}{2} = \frac{y-1}{5} = \frac{z-2}{5}$, $\Pi: x - y - 3z = 1$

(ii) $l: \frac{x+1}{-1} = \frac{y-3}{3} = \frac{z+2}{-3}$, $\Pi: 2x + y + z = 14$

3. Find the acute angle between the following pairs of planes:

(a) $3x - 7y + z = 4$ and $x + y - 4z = 5$

(b) $x - z = 4$ and $y + z = 1$

4. Find a vector equation of the line of intersection of the following pairs of planes:

(a) (i) $3x + y - z = 3$ and $x - 2y + 4z = -5$

(ii) $x + y = 3$ and $x - y = 5$

(b) (i) $2x - y = 4$ and $2y + z = 5$

(ii) $x + 2y - 5z = 6$ and $z = 0$

5. Plane Π_1 has Cartesian equation $3x - y + z = 7$.

(a) Write down a normal vector of Π_1.

Plane Π_2 has equation $x - 5y + 5z = 11$.

(b) Find, correct to the nearest degree, the acute angle between Π_1 and Π_2. *[6 marks]*

6. Find the coordinates of the point of intersection of line $\frac{x-2}{3} = \frac{y-1}{2} = z$ with the plane $2x - y - 2z = 5$. *[5 marks]*

7. Show that the lines $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ and $\frac{x-1}{4} = \frac{y+2}{3} = \frac{2z-1}{4}$ do not intersect. *[5 marks]*

8. The plane with equation $12x - 3y + 5z = 60$ intersects the x-, y- and z-axes at points P, Q and R respectively. *[7 marks]*

(a) Find the coordinates of P, Q and R.

(b) Find the area of the triangle PQR.

9. Plane Π has equation $5x - 3y - z = 1$.

(a) Show that point P (2, 1, 6) lies in Π.

(b) Point Q has coordinates (7, –1, 2). Find the exact value of the sine of the angle between (PQ) and Π.

(c) Find the exact distance PQ.

(d) Hence find the exact distance of Q from Π. *[10 marks]*

10. Two planes have equations:

$$\Pi_1 : 3x - y + z = 17$$

$$\Pi_2 : x + 2y - z = 4$$

(a) Calculate $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) Show that Π_1 and Π_2 are perpendicular.

(c) Show that the point M (1, 1, 2) does not lie in either of the two planes.

(d) Find a vector equation of the line through M which is parallel to both planes. *[10 marks]*

11. (a) Calculate $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$.

(b) Planes Π_1 and Π_2 have equations:

$$\Pi_1 : x + 3y - 2z = 0$$

$$\Pi_2 : 3x + 5y - z = 0$$

l is the line of intersection of the two planes.

(i) Show that l passes through the origin.

(ii) Write down a vector equation for l.

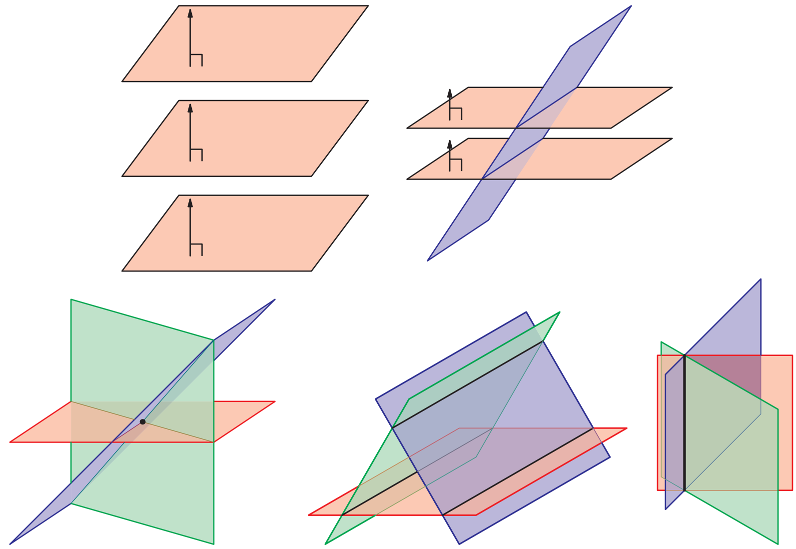
(c) A third plane has equation:

$$\Pi_3 : x - 5z + z = 8$$

Find the coordinates of the intersection of all three planes. *[10 marks]*

14F Intersection of three planes

In the last example of the previous section we saw how to find the line of intersection of two planes. Two different planes can either intersect or be parallel. When we have three planes there are many more possibilities.



Before looking through the examples you may wish to revisit Section 4E on solving systems of equations.

In the first two cases some of the planes are parallel. As we mentioned in the last section, this can be seen from the equations because one normal vector will be a multiple of another. When none of the planes are parallel, there are still three different possibilities for how they can intersect. To find out which of those is the case, we need to try solving the equations.

Any point which lies in the intersection of the three planes must satisfy all three equations. It is therefore a solution of the system of three equations with three unknowns. As we saw in Section 4E, such a system can have a unique solution, no solutions or infinitely many solutions. These three cases correspond to the three possibilities of how three planes can intersect.

EXAM HINT

Remember that you can solve systems of equations either using the simultaneous equation solver, or perform Gaussian elimination using a matrix. See calculator skills sheets 5 and 6 on the CD ROM.

Worked example 14.24

Find the intersection of the planes with equations:

$$3x - y + 4z = 7$$
$$x - 2y + z = 3$$
$$x - y + 4z = -5$$

continued . . .

Points of intersection will be the solutions of the system of equations. We can try solving them on the calculator

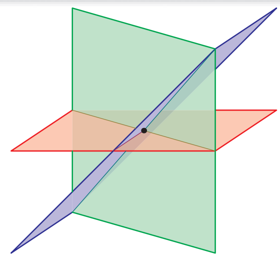
Using GDC:

$x = 6, y = \frac{1}{7}, z = -\frac{19}{7}$

So the three planes intersect at the point $\left(6, \frac{1}{7}, -\frac{19}{7}\right)$.

In this example the system of equations has a unique solution, so the planes intersect at a single point, as in the third diagram.

In the next example the system of equations has no solutions.



Worked example 14.25

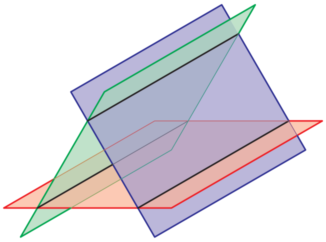
Show that the three planes with equations:

$x + 2y + 3z = 10$

$2x + 3y + 2z = 4$

$4x + 7y + 8z = 7$

do not intersect.



We can attempt to solve the equations and show that it is impossible. Without a calculator, use Gaussian elimination

$$\begin{cases} x + 2y + 3z = 10 & (1) \\ 2x + 3y + 2z = 4 & (2) \\ 4x + 7y + 8z = 7 & (3) \end{cases}$$

$$\begin{matrix} \\ 2\times(1)-(2) \\ 4\times(1)-(3) \end{matrix} \quad \Rightarrow \begin{cases} x + 2y + 3z = 10 & (1) \\ y + 4z = 16 & (4) \\ y + 4z = 23 & (5) \end{cases}$$

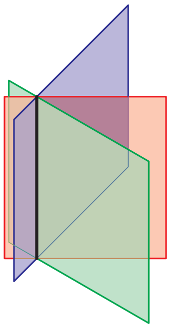
$$\begin{matrix} \\ \\ (5)-(4) \end{matrix} \quad \Rightarrow \begin{cases} x + 2y + 3z = 10 & (1) \\ y + 4z = 16 & (4) \\ 0z = -7 & (6) \end{cases}$$

The last equation is impossible, so we cannot find z

Equation (6) has no solutions, so the three planes do not intersect.

In the above example none of the three planes are parallel, so each two planes intersect along a line. However, the line of intersection of any two planes is parallel to the third plane. For example, the line of intersection of the first two planes has direction $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}-5\\4\\-1\end{pmatrix}$. This is perpendicular to the normal of the third plane: $\begin{pmatrix}-5\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\7\\8\end{pmatrix}=-20+28-8=0$. Hence the line with direction $\begin{pmatrix}-5\\4\\-1\end{pmatrix}$ is parallel to the third plane.

The result for the direction of the line of intersection of two planes was given in Key point 14.13.



The final possibility is for the three planes to intersect along a line, as in the next example. It also shows you a common type of exam question where you have to find an unknown parameter.

Worked example 14.26

Find the value of c for which the planes with equations

$$2x-y-3z=3$$
$$x+y-3z=0$$
$$x+2y-4z=c$$

intersect, and find the equation of the line of intersection.

This is the same as finding the value of the parameter for which the system of equations is consistent, as in Worked example 4.14.

Solve the equations using Gaussian elimination

$$\begin{cases}2x-y-3z=3 & (1)\\ x+y-3z=0 & (2)\\ x+2y-4z=c & (3)\end{cases}$$

$$\begin{matrix}2\times(2)-(1)\\ 2\times(3)-(1)\end{matrix}\Rightarrow\begin{cases}2x-y-3z-3 & (1)\\ 3y-3z=-3 & (4)\\ 5y-5z-2c & (5)\end{cases}$$

continued . . .

Equation (6) only has a solution when RHS=0

Then (6) says $0z = 0$, so z can be any number

(x, y, z) are the coordinates of a point where the three planes intersect. There are infinitely many points, and we can write $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ as a vector equation of a line

$$5\times(4)-3\times(5) \quad \Rightarrow \begin{cases} 2x-y-3z=3 & (1) \\ 3y-3z=-3 & (4) \\ 0z=-6c-6 & (6) \end{cases}$$

This has a solution when

$$-6c-6=0$$
$$\therefore c=-1$$

For any $t \in \mathbb{R}$:

$z = t$

$$(4) \Rightarrow 3y = -3+3t$$
$$\therefore y = -1+t$$
$$(1) \Rightarrow 2x = 3+3t+(-1+t)$$
$$= 2+4t$$
$$\therefore x = 1+2t$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2t \\ t \\ t \end{pmatrix}$$

So the equation of the line of intersection is

$$\underline{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Remember that your calculator can perform Gaussian elimination even when the equations do not have a unique solution. If the last equation is $0=0$ you can use the first two equations to find the general solution.

Exercise 14F

1. Find the coordinates of the point of intersection of the planes:

$$\Pi_1: \quad 3x + \ y + \ z = 8$$
$$\Pi_2: -7x + 3y + \ z = 2$$
$$\Pi_3: \quad x + \ y + 3z = 0$$

[3 marks]

2. Find the coordinates of the point of intersection of the three planes with equations $x = 2,\ x + y - z = 7$ and $2x + y + z = 3$. *[4 marks]*

3. Find the equation of the line of intersection of the planes $2x - z = 1,\ 4x + y - z = 5$ and $y + z = 3$. *[4 marks]*

4. Find the intersection of the planes:

$$x - 2y + z = 5$$
$$2x + y + z = 1$$
$$x + 2y - z = -2$$

[5 marks]

5. Show that the planes with equations $2x - y + z = 6$, $3x + y + 5z = -7$ and $x - 3y - 3z = 8$ do not intersect. *[4 marks]*

6. Three planes have equations:

$$\Pi_1: 2x + y - 2z = 0$$
$$\Pi_2: x - 2y - z = 2$$
$$\Pi_3: 3x + 4y - 3z = d$$

(a) Find the value of d for which the three planes intersect.

(b) For this value of d, find the equation of the line of intersection of the three planes. *[7 marks]*

7. Three planes have equations:

$$\Pi_1: x - y = 4$$
$$\Pi_2: y + z = 1$$
$$\Pi_3: x - z = d$$

Find, in terms of d, the coordinates of the point of intersection of the three planes. *[5 marks]*

8. (a) Explain why the intersection of the planes

$$\Pi_1 : x + y = 0$$
$$\Pi_2 : x - 4y - 2z = 0$$
$$\Pi_3 : \frac{1}{2}x + 3y + z = 0$$

contains the origin.

(b) Show that the intersection of the three planes is a line and find its direction vector in the form $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, where $a, b, c \in \mathbb{Z}$. *[7 marks]*

9. (a) Find the value of a for which the three planes

$$\Pi_1 : x - 2y + z = 7$$
$$\Pi_2 : 2x + y - 3z = 9$$
$$\Pi_3 : x + y - az = 3$$

do not intersect.

(b) Find the Cartesian equation of the line of intersection of Π_1 and Π_2. *[9 marks]*

10. Three planes have equations

$$x - y - z = -2$$
$$2x + 3y - 7z = a + 4$$
$$x + 2y + pz = a^2$$

(a) Find the value of p and the two values of a for which the intersection of the three planes is a line.

(b) For the value of p and the larger value of a found above, find the equation of the line of intersection.

[12 marks]

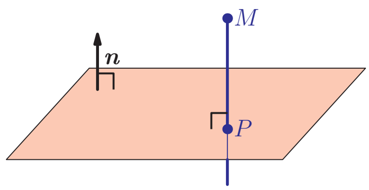
14G Strategies for solving problems with lines and planes

We now have all the tools we need to solve more complex problems involving lines and planes in space. We can find equations of lines and planes determined by points, intersections and angles between two lines, two planes, or a line and a plane. We also know how to calculate the distance between two points and areas of triangles.

Solving a more complex problem requires two things:

- a strategy saying what needs to be calculated
- being able to carry out all the calculations.

The second part is what we have been practising so far. In this section we look at strategies to solve the most common problems. There are no examples – you need to select the most appropriate strategy for each question. For each problem we explain the reasons behind the choice of strategy and then list the required steps.



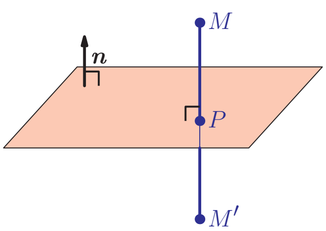
Distance of a point from a plane

Given a plane with equation $\boldsymbol{a} \cdot \boldsymbol{n} = d$ and a point M outside of the plane, the distance from M to the plane is equal to the distance MP, where the line (MP) is perpendicular to the plane. This means that the direction of (MP) is $\boldsymbol{n}$.

To find the distance MP:

- Write down the vector equation of the line with direction $\boldsymbol{n}$ through point M.
- Find the intersection, P, between the line and the plane.
- Calculate the distance MP.

The point P is called the **foot of the perpendicular** from the point to the plane.

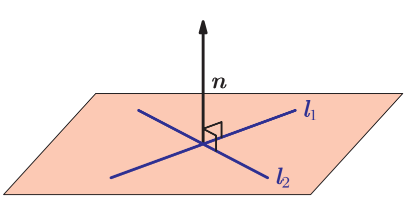


Reflection of a point in a plane

Given a plane Π with equation $\boldsymbol{a} \bullet \boldsymbol{n} = d$ and a point M which is not in the plane, the reflection of M in Π is the point M' such that MM' is perpendicular to the plane and the distance of M' from Π is the same as the distance of M from Π. Calculations with distances can be difficult, so instead we can use the fact that, since MPM' is a straight line, $\overrightarrow{PM'} = \overrightarrow{MP}$.

To find the coordinates of M':

- Write down the vector equation of the line with direction $\boldsymbol{n}$ through point M.
- Find the intersection, P, between the line and the plane.
- Find the point M' such that $\overrightarrow{PM'} = \overrightarrow{MP}$ by using position vectors: $\boldsymbol{m}' - \boldsymbol{p} = \boldsymbol{p} - \boldsymbol{m}$.



Equation of a plane determined by two intersecting lines

We noted in Key point 14.8 that a plane is uniquely determined by two intersecting lines. In other words, if we have equations of two lines that intersect, we should be able to find the equation of the plane containing both of them.

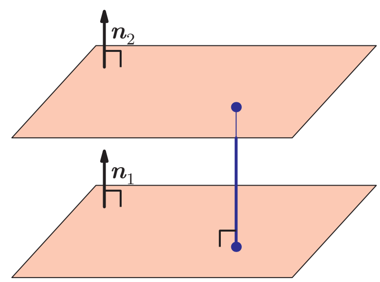
To do this, we note that the normal to the plane must be perpendicular to both lines (this is the definition of the normal;

a vector which is perpendicular to all the lines in the plane). But we know that the vector product of two vectors is perpendicular to *both* of them. So we can take the normal vector to be the vector product of the direction vectors of the two lines.
To complete the scalar product equation of the plane we also need one point. The intersection of the two lines clearly lies in the plane, so we can use this point, or we can use any point on either of the two lines, which may be easier!

So, to find the equation of the plane containing lines $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \lambda \boldsymbol{d}_2$:

- The normal vector is $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2$.
- For a point P in the plane, pick any point on either of the two lines (***a***, ***b*** or the intersection are some possible choices).
- The scalar product equation of the plane is $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{p} \bullet \boldsymbol{n}$.

Distance between parallel planes

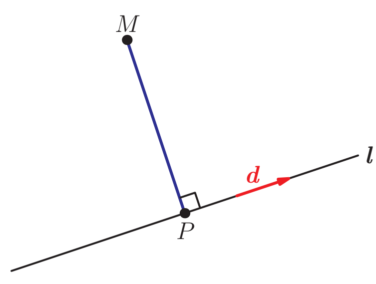


If two planes are parallel, we can find the perpendicular distance between them. To do this, we note that the perpendicular distance is measured in the direction of the normal vector of the two planes. (Since the planes are parallel, their normals are in the same direction!)

One possible strategy is as follows:

- Pick a point in the first plane.
- Write down the equation of the line in the direction of the normal passing through this point.
- Find the intersection point of this line and the second plane.
- Find the distance between the two points.

Distance from a point to a line



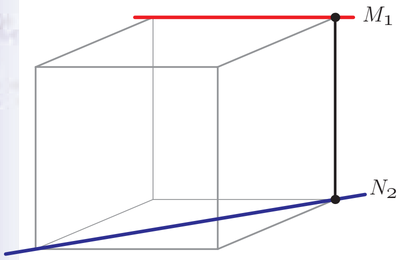
We have already seen an example of this in Worked example 14.8. The strategy is based on the fact that the distance is measured along a direction perpendicular to the line. However, there is more than one direction perpendicular to any given line so we cannot not just write down the required direction. Instead, we use a general point, P, on the line (given by the position vector ***r***) and use scalar product to express the fact that MP is perpendicular to the line.

To find the shortest distance from point M to a line l given by $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$:

- Form the vector $\overrightarrow{MP} = \boldsymbol{r} - \boldsymbol{m}$; it will be in terms of λ.
- $[MP]$ is perpendicular to the line: $\overrightarrow{MP} \bullet \boldsymbol{d} = 0$.
- This is an equation for λ; solve it.

- Use this value of λ to find $\overrightarrow{MP}$.
- The required distance is $\left|\overrightarrow{MP}\right|$.

Distance between two skew lines



Consider points M and N moving along two skew lines, l_1 and l_2 respectively. The distance between them is minimum possible when [MN] is perpendicular to both lines. It may not be immediately obvious that such a position of M and N always exists, but it does. When you sketch a diagram of this, it is useful to imagine a cuboid, where one line runs along an upper edge, and the other runs along the diagonal of the base, as shown. The shortest distance between the two is then the height of the cuboid.

Suppose the two skew lines have equations $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \mu \boldsymbol{d}_2$. The strategy is similar to finding the distance between a line and a plane, except now we have two general points, one on each line.

- Write down position vectors of two general points M and N, one on each line, using the equations for $\boldsymbol{r}$.
- Form the vector $\overrightarrow{MN}$; this will be in terms of both λ and μ.
- Write down two equations: $\overrightarrow{MN} \bullet \boldsymbol{d}_1 = 0$ and $\overrightarrow{MN} \bullet \boldsymbol{d}_2 = 0$.
- These are simultaneous equations for λ and μ; solve them.
- Use the values of λ and μ to find $\overrightarrow{MN}$.
- The required distance is $\left|\overrightarrow{MN}\right|$.

Exam questions usually give you hints to help solve any of the above problems. (Any question that does not would definitely be difficult!) However, questions often ask you to carry out the required calculations, but not tell you how to fit them together to solve the final part. This is why it is extremely useful to draw a diagram and label everything you have found. Remember that the diagrams are just sketches showing relative positions of points, lines and planes – they do not have to be accurate.

The exam-style questions in the following exercise are intended give you an idea how much guidance you can expect to get. Use the strategies described in this section to help you. Some questions will not use any of the above strategies, but you will be given hints.

Exercise 14G

1. Plane Π has equation $2x+2y-z=11$. Line l is perpendicular to Π and passes through the point $P\,(-3, -3, 4)$.

(a) Find the equation of l.

(b) Find the coordinates of the point Q where l intersects Π.

(c) Find the shortest distance from P to Π. *[8 marks]*

2. Two planes have equations:

$$\Pi_1 : x-3y+z=6$$

$$\Pi_2 : 3x-9y+3z=0$$

(a) Show that Π_1 and Π_2 are parallel.

(b) Show that Π_2 passes through the origin.

(c) Write down the equation of the line through the origin which is perpendicular to Π_2.

(d) Hence find the distance between the planes Π_1 and Π_2. *[10 marks]*

3. (a) Calculate $\begin{pmatrix}-1\\0\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix}$.

(b) Two lines have equations:

$$l_1 : \boldsymbol{r}=\begin{pmatrix}7\\-3\\2\end{pmatrix}+t\begin{pmatrix}-1\\0\\2\end{pmatrix} \text{ and } l_2 : \boldsymbol{r}=\begin{pmatrix}1\\1\\26\end{pmatrix}+s\begin{pmatrix}0\\1\\3\end{pmatrix}.$$

(i) Show that l_1 and l_2 intersect.

(ii) Find the coordinates of the point of intersection.

(c) Plane Π contains lines l_1 and l_2 Find the Cartesian equation of Π. *[11 marks]*

4. Four points have coordinates $A\,(7,\ 0,\ 1)$ $B\,(8,-1,\ 4)$, $C\,(9,\ 0,\ 2)$ and $D\,(6,\ 5,\ 3)$

(a) Show that $\overrightarrow{AD}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Write down the equation of the plane Π containing the points A, B and C in the form $\boldsymbol{r}\bullet\boldsymbol{n}=k$.

(c) Find the exact distance of point D from plane Π.

(d) Point D_1 is the reflection of D in Π. Find the coordinates of D_1. *[10 marks]*

5. Two lines are given by Cartesian equations:

$$l_1 : \frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-2}{1}$$

$$l_2 : \frac{x-5}{3} = 1 - y = z + 4$$

(a) Show that l_1 and l_2 are parallel.

(b) Show that the point $A\ (14, -5, 6)$ lies on l_1.

(c) Find the coordinate of point B on l_2 such that (AB) is perpendicular to the two lines.

(d) Hence find the distance between l_1 and l_2, giving your answer to 3 significant figures. *[10 marks]*

6. (a) Find the coordinates of the point of intersection of lines

$$l_1 : \frac{x-1}{3} = \frac{y+1}{4} = \frac{3-z}{3} \text{ and } l_2 : \frac{x+12}{2} = \frac{y}{1} = \frac{z+17}{1}.$$

(b) Find a vector perpendicular to both lines.

(c) Hence find the Cartesian equation of the plane containing l_1 and l_2. *[13 marks]*

7. Points $A\ (8,\ 0,\ 4)$, $B\ (12, -1,\ 5)$ and $C\ (10,\ 0,\ 7)$ lie in the plane Π.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Hence find the area of the triangle ABC, correct to 3 significant figures.

(c) Find the Cartesian equation of Π.

Point D has coordinates $(-7, -28, 11)$.

(d) Find a vector equation of the line through D perpendicular to the plane.

(e) Find the intersection of this line with Π, and hence find the perpendicular distance of D from Π.

(f) Find the volume of the pyramid $ABCD$. *[16 marks]*

8. Line l passes through point $A\ (-1, 1, 4)$ and has direction vector $\boldsymbol{d} = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}$. Point B has coordinates $(3, 3, 1)$. Plane Π has normal vector $\boldsymbol{n}$, and contains the line l and the point B.

(a) Write down a vector equation for l.

(b) Explain why $\overrightarrow{AB}$ and $\boldsymbol{d}$ are both perpendicular to $\boldsymbol{n}$.

(c) Hence find one possible vector $\boldsymbol{n}$.

(d) Find the Cartesian equation of plane Π. *[10 marks]*

9. Plane Π has equation $6x-2y+z=16$. Line l is perpendicular to Π and passes through the origin.

(a) Find the coordinates of the foot of the perpendicular from the origin to Π.

(b) Find the shortest distance of Π from the origin, giving your answer in exact form. *[8 marks]*

10. (a) Show that the planes $\Pi_1 : x-z=4$ and $\Pi_2 : z-x=8$ are parallel.

(b) Write down a vector equation of the line through the origin which is perpendicular to the two planes.

(c) (i) Find the coordinates of the foot of the perpendicular from the origin to Π_1.

(ii) Find the coordinates of the foot of the perpendicular from the origin to Π_2.

(d) Use your answers from part (c) to find the exact distance between the two planes. *[11 marks]*

Summary

- Vector equations give position vectors of points on a line or a plane.
- The **vector equation** of a line has the form $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{d}$, where $\boldsymbol{d}$ is a vector in the direction of the line and $\boldsymbol{a}$ is the position vector of one point on the line. $\boldsymbol{r}$ is the position vector of a general point on the line and the parameter λ gives positions of different points on the line.
- Vector equation of a plane has the form $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{d}_1+\mu\boldsymbol{d}_2$, where $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are two vectors parallel to the plane and $\boldsymbol{a}$ is the position vector of one point in the plane.
- Cartesian equations are equations satisfied by the coordinates of a point on the line or in the plane.
- To uniquely determine a plane we need three points, not on the same line, OR a line and a point outside that line, OR two intersecting lines.
- **Cartesian equation** of a line has the form $\frac{x-a}{k}=\frac{y-b}{m}=\frac{z-c}{n}$, and can be derived from the vector equation by writing three equations for λ in terms of x, y and z. If we express x, y and z in terms of λ instead, we obtain **parametric equations** of the line.
- The Cartesian equation of a plane has the form $n_1x+n_2y+n_3z=k$. This can also be written in the scalar product form $\boldsymbol{r}\bullet\boldsymbol{n}=\boldsymbol{a}\bullet\boldsymbol{n}$, where $\boldsymbol{n}=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ is the **normal vector** of the plane, which is perpendicular to every line in the plane. To derive the Cartesian equation from a vector equation, use $\boldsymbol{n}=\boldsymbol{d}_1\times\boldsymbol{d}_2$.

- The angle between two lines is the angle between their direction vectors.
- Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are parallel if $\boldsymbol{d}_1 = k\boldsymbol{d}_2$, perpendicular if $\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = 0$.
- The angle between two planes is the angle between their normals.
- The angle between a line and a plane is $90° - \theta$, where θ is the angle between the line direction vector and the plane's normal.
- To find the intersection of two lines, set the two position vectors equal to each other and use two of the equations to find λ and μ. If these values do not satisfy the third equation, the lines are **skew lines**.
- To find the intersection between a line and a plane, express x, y and z for the line in terms of λ and substitute into the Cartesian equation of the plane.
- The line of intersection of two planes has direction parallel to $\boldsymbol{n}_1 \times \boldsymbol{n}_2$, and we can use any point which satisfies both plane equations.
- The intersection of two planes or three planes can be found by solving the system of equations given by the Cartesian equations of the planes.
- Three distinct planes may intersect at a single point, along a straight line, or have no intersection at all. These cases correspond to the different possibilities for the solutions of a system of three equations. When the solution is not unique, the straight line corresponds to the general solution of the system.
- The vector equation of a line can be used to describe the path of an object moving with constant velocity. For an object moving with constant velocity $\boldsymbol{v}$ from an initial position $\boldsymbol{a}$, the direction vector of the line can be taken to be the velocity vector, and the position of time $\boldsymbol{t}$ is given by $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$. The object moves along the straight line with equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$. The speed is equal to $|v|$.

In solving problems with lines and planes we often need to set up and solve equations. In doing so we use properties of vectors, in particular the fact that the magnitude of a vector represents distance, and that $\boldsymbol{a} \bullet \boldsymbol{b} = 0$ for perpendicular vectors. We also need to use diagrams, often to identify right angled triangles. In longer questions, we can combine several answers to solve the last part.

Introductory problem revisited

Which is more stable (less wobbly): a three-legged stool or a four-legged stool?

A stool will be stable if the end points of all the legs lie in the same plane. As we have seen, we can always find a plane containing three points, so a three-legged stool is stable, it never wobbles. This is why photographers place their cameras on tripods.

If there are four points, it is possible that the fourth one does not lie in the same plane as the other three. So if the legs are not all of the same length, the four end points could determine four different planes. Equally, if the floor is slightly uneven, only a three-legged stool can be relied upon to be stable. This is why four-legged chairs and tables often wobble. However, we also want furniture that is not easily knocked over – can you see why stool legs which form a square at the base might be better than legs that form a triangle in this respect?

Mixed examination practice 14

Short questions

1. Find a vector equation of the line passing through points $(3, -1, 1)$ and $(6, 0, 1)$. *[4 marks]*

2. The point $(3, -1, 2)$ lies on the line with equation $\frac{x+3}{2} = \frac{y-8}{-3} = \frac{z+13}{p}$.

   Find the value of p. *[4 marks]*

3. The vector $\boldsymbol{n} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is normal to a plane which passes through the point $(3, -1, 2)$.

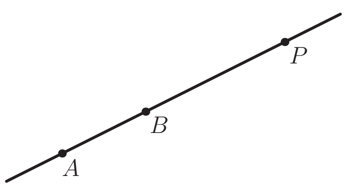
   (a) Find an equation for the plane.

   (b) Find a if the point $(a, 2a, a-1)$ lies on the plane. *[6 marks]*

4. Find the coordinates of the point of intersection of the planes with equations $x - 2y + z = 5$, $2x + y + z = 1$ and $x + 2y - z = -2$. *[6 marks]*

5. Points $A\ (-1, 1, 2)$ and $B\ (3, 5, 4)$ lie on the line with equation $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

   Find the coordinates of point P on the same line such that $AP = 3AB$, as shown in the diagram.



   [5 marks]

6. Point $A\ (-3, 0, 4)$ lies on the line $\boldsymbol{r} = -3\boldsymbol{i} + 4\boldsymbol{k} + \lambda(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$, where λ is a real parameter. Find the coordinates of one point on the line which is 10 units from A. *[6 marks]*

7. Points $A\ (4, 1, 12)$ and $B\ (8, -11, 20)$ lie on the line l.

   (a) Find an equation of line l, giving the answer in parametric form.

   (b) The point P is on l such that $\overrightarrow{OP}$ is perpendicular to l. Find the coordinates of P. *[6 marks]*

8. (a) Given that $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$, show that $\boldsymbol{b} \times \boldsymbol{a} = 3\boldsymbol{i} + 7\boldsymbol{j} + \boldsymbol{k}$.

   Two planes have equations $\boldsymbol{r} \cdot \boldsymbol{a} = 5$ and $\boldsymbol{r} \cdot \boldsymbol{b} = 12$.

   (b) Show that the point $(2, 2, 3)$ lies in both planes.

   (c) Write down the Cartesian equation of the line of intersection of the two planes. *[6 marks]*

9. The plane $3x+2y-z=2$ contains the line $x-3=\frac{2y+2}{5}=\frac{z-5}{k}$.
Find the value of k. *[6 marks]*

10. (a) If $\boldsymbol{u}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{v}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ show that $\boldsymbol{u}\times\boldsymbol{v}=7\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k}$.

(b) Let $\boldsymbol{w}=\lambda\boldsymbol{u}+\mu\boldsymbol{v}$ where λ and μ are scalars. Show that $\boldsymbol{w}$ is perpendicular to the line of intersection of the planes $x+2y+3z=5$ and $2x-y+2z=7$ for all values of λ and μ.

[8 marks]

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11. Find the Cartesian equation of the plane containing the two lines

$$x=\frac{3-y}{2}=z-1 \text{ and } \frac{x-2}{3}=\frac{y+1}{-3}=\frac{z-3}{5}.$$ *[8 marks]*

Long questions

1. Points A and B have coordinates $(4, 1, 2)$ and $(0, 5, 1)$. Line l_1 passes through A and has equation $\boldsymbol{r}_1=\begin{pmatrix}4\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$. Line l_2 passes through B and has equation $\boldsymbol{r}_2=\begin{pmatrix}0\\5\\1\end{pmatrix}+t\begin{pmatrix}4\\-4\\1\end{pmatrix}$.

(a) Show that the line l_2 also passes through A.

(b) Calculate the distance AB.

(c) Find the angle between l_1 and l_2 in degrees.

(d) Hence find the shortest distance from B to l_1. *[10 marks]*

2. (a) Show that the lines $l_1:\boldsymbol{r}=\begin{pmatrix}-3\\3\\18\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\-8\end{pmatrix}$ and $l_2:\boldsymbol{r}=\begin{pmatrix}5\\0\\2\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-1\end{pmatrix}$ do not intersect.

(b) Points P and Q lie on l_1 and l_2 respectively, such that (PQ) is perpendicular to both lines.

(i) Write down $\overrightarrow{PQ}$ in terms of λ and μ.

(ii) Show that $9\mu-69\lambda+147=0$.

(iii) Find a second equation for λ and μ.

(iv) Find the coordinates of P and the coordinates of Q.

(v) Hence find the shortest distance between l_1 and l_2. *[14 marks]*

3. Plane Π has equation $x - 2y + z = 20$ and point A has coordinates $(4, -1, 2)$.

(a) Write down the vector equation of the line l through A which is perpendicular to Π.

(b) Find the coordinates of the point of intersection of line l and plane Π.

(c) Hence find the shortest distance from point A to plane Π. *[10 marks]*

4. In this question, unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ point East and North, and unit vector $\boldsymbol{k}$ is vertically up. The time (t) is measured in minutes and the distance in kilometres.

Two aircraft move with constant velocities $\boldsymbol{v}_1 = (7\boldsymbol{i} + 10\boldsymbol{j} + 3\boldsymbol{k})$ km/min and $\boldsymbol{v}_2 = (3\boldsymbol{i} - 8\boldsymbol{j} - 4\boldsymbol{k})$ km/min. At $t = 0$, the first aircraft is at the point with coordinates $(16, 30, 3)$ and the second aircraft at the point with coordinates $(24, 66, 12)$.

(a) Calculate the speed of the first aircraft.

(b) Write down the position vector of the second aircraft at the time t minutes.

(c) Find the distance between the aircraft after 3 minutes.

(d) Show that there is a time when the first aircraft is vertically above the second one, and find the distance between them at that time.

5. Line L_1 has equation $\boldsymbol{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and line L_2 has equation $\boldsymbol{r} = \begin{pmatrix} 5 \\ 4 \\ 9 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(b) Find the coordinates of the point of intersection of the two lines.

(c) Write down a vector perpendicular to the plane containing the two lines.

(d) Hence find the Cartesian equation of the plane containing the two lines. *[10 marks]*

6. Three planes have equations:

$$\Pi_1 : 3x - y + z = 2$$

$$\Pi_2 : x + 2y - z = -1$$

$$\Pi_3 : 5x - 4y + dz = 3$$

(a) Find the value of d for which the three planes do not intersect.

(b) Find the vector equation of the line l_1 of intersection of Π_1 and Π_2.

(c) For the value of d found in part (a):

(i) Find the value of p so that the point $A\,(0, 1, p)$ lines on l_1.

(ii) Find the vector equation of the line l_2 through A perpendicular to Π_3.

(iii) Hence find the distance between l_1 and Π_3. *[17 marks]*

7. Line l_1 has Cartesian equation $\frac{x-2}{4} = \frac{y+1}{-3} = \frac{z}{3}$. Line l_2 is parallel to l_1 and passes through point $A\ (0,-1,\ 2)$.

(a) Write down a vector equation of l_2.

(b) Find the coordinates of the point B on l_1 such that (AB) is perpendicular to l_1.

(c) Hence find, to three significant figures, the shortest distance between the two lines. *[9 marks]*

8. Line L has equation $\frac{x+5}{3} = \frac{y-1}{3} = \frac{z-2}{-1}$.

(a) Show that the point A with coordinates $(4,10,-1)$ lies on L.

(b) Given that point B has coordinates $(2,1,2)$, calculate the distance AB.

(c) Find the acute angle between L and (AB) in radians.

(d) Find the shortest distance of B from L. *[12 marks]*

9. (a) The plane Π_1 has equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane Π_2 has the equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(i) For points which lie on Π_1 and Π_2, show that $\lambda = \mu$.

(ii) Hence, or otherwise, find a vector equation of the line of intersection of Π_1 and Π_2.

(b) The plane Π_3 contains the line $\frac{2-x}{3} = \frac{y}{-4} = z+1$ and is perpendicular to $3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$. Find the cartesian equation of Π_3.

(c) Find the intersection of Π_1, Π_2 and Π_3. *[12 marks]*

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10. (a) Find the vector equation of the line L through point $A\left(-2, 4, 2\right)$ parallel to the vector $\boldsymbol{l} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(b) Point B has coordinates $(2,\ 3,\ 3)$. Find the cosine of the angle between (AB) and the line L.

(c) Calculate the distance AB.

(d) Point C lies on L and BC is perpendicular to L. Find the exact distance AC. *[10 marks]*

11. Plane Π has equation $x - 4y + 2z = 7$ and point P has coordinates $(9, -7, 6)$.

(a) Show that point R $(5, 1, 3)$ lies in the plane Π.

(b) Find the vector equation of the line (PR).

(c) Write down the vector equation of the line through P perpendicular to Π.

(d) N is the foot of the perpendicular from P to Π. Find the coordinates of N.

(e) Find the exact distance of point P from the plane Π. *[14 marks]*

12. Point A $(3,\ 1, -4)$ lies on line L which is perpendicular to plane $\Pi : 3x - y - z = 1$.

(a) Find the Cartesian equation of L.

(b) Find the intersection of the line L and plane Π.

(c) Point A is reflected in Π. Find the coordinates of the image of A.

(d) Point B has coordinates $(1, 1, 1)$. Show that B lies in Π.

(e) Find the distance between B and L. *[14 marks]*

13. (a) Calculate $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

(b) Plane Π_1 has normal vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and contains point A $(3, 4, -2)$.

Find the Cartesian equation of the plane.

(c) Plane Π_2 has equation $3x + y - z = 15$. Show that Π_2 contains point A.

(d) Write down the vector equation of the line of intersection of the two planes.

(e) A third plane, Π_3, has equation $\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 12$. Find the coordinates of the point of intersection of all three planes.

(f) Find the angle between Π_1 and Π_3 in degrees. *[17 marks]*

(e) 1.06

(f) 0.058, 0.557

4. (a) $r = 2, \alpha = \frac{\pi}{6}$

(b) $[-2, 2]$

(c) $\frac{\pi}{2}, \frac{7\pi}{6}$

5. (a) $(t+1)(t^2 - 4t + 1)$

(c) 1

(d) $\tan 15° = 2 - \sqrt{3}$,

$\tan 75° = 2 + \sqrt{3}$

Chapter 13

Exercise 13A

1. (a) (i) $\boldsymbol{b}$ (ii) $\boldsymbol{a} + \boldsymbol{b}$

(b) (i) $-\boldsymbol{a}$ (ii) $-\frac{1}{2}\boldsymbol{a}$

(c) (i) $\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(ii) $\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

2. (a) (i) $\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}$ (ii) $\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(b) (i) $-\frac{3}{2}\boldsymbol{a} + \boldsymbol{b}$

(ii) $-\frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

(c) (i) $\frac{3}{2}\boldsymbol{a} - \boldsymbol{b}$

(ii) $-\frac{4}{3}\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

3. (a) (i) $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$

(b) (i) $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

4. (a) $\boldsymbol{b} - \boldsymbol{a}$

(b) $\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(c) $4\boldsymbol{a} - 3\boldsymbol{b}$

5. (a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$

(b) $(10, -2)$

6. (a) $\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 3.5 \\ -0.5 \\ 1.5 \end{pmatrix}$

7. $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

8. $\begin{pmatrix} 1.6 \\ 0.8 \\ 1.8 \end{pmatrix}$

9. (a) $\frac{3}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - 2\boldsymbol{k}$

(b) $\left(\frac{1}{2}, \frac{13}{2}, 0\right)$

10. $\begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$

Exercise 13B

1. (a) (i) $\begin{pmatrix} 21 \\ 3 \\ 36 \end{pmatrix}$ (ii) $\begin{pmatrix} 20 \\ -8 \\ 12 \end{pmatrix}$

(b) (i) $\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix}$

(c) (i) $\begin{pmatrix} 11 \\ -3 \\ 8 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 \\ 5 \\ 6 \end{pmatrix}$

(d) (i) $\begin{pmatrix} 10 \\ -3 \\ 11 \end{pmatrix}$ (ii) $\begin{pmatrix} 17 \\ 6 \\ 35 \end{pmatrix}$

2. (a) (i) $-5\boldsymbol{i} + 5\boldsymbol{k}$ (ii) $4\boldsymbol{i} + 8\boldsymbol{j}$

(b) (i) $\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ (ii) $2\boldsymbol{j} + \boldsymbol{k}$

(c) (i) $4\boldsymbol{i} + 7\boldsymbol{k}$

(ii) $5\boldsymbol{i} - 4\boldsymbol{j} + 15\boldsymbol{k}$

3. (a) $-4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

(b) $-\frac{8}{3}\boldsymbol{i} + \frac{4}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k}$

(c) $4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

(d) $-\frac{1}{2}\boldsymbol{i} + \boldsymbol{j} - \frac{1}{2}\boldsymbol{k}$

4. $\begin{pmatrix} 2 \\ 0 \\ \frac{-3}{4} \end{pmatrix}$

5. -2

6. $-\frac{4}{3}$

7. -2

8. $p=\frac{3}{8}, q=\frac{1}{8}$

Exercise 13C

1. $|\boldsymbol{a}|=2\sqrt{5}$ $|\boldsymbol{b}|=\sqrt{26}$ $|\boldsymbol{c}|=2\sqrt{5}$ $|\boldsymbol{d}|=\sqrt{2}$

2. $|\boldsymbol{a}|=\sqrt{21}$ $|\boldsymbol{b}|=\sqrt{2}$ $|\boldsymbol{c}|=\sqrt{21}$ $|\boldsymbol{d}|=\sqrt{2}$

3. (a) (i) $\sqrt{29}$ (ii) $\sqrt{2}$
   (b) (i) $\sqrt{58}$ (ii) $\sqrt{5}$

4. (a) (i) $\sqrt{19}$ (ii) $\sqrt{38}$
   (b) (i) $\sqrt{74}$ (ii) $\sqrt{13}$

5. (a) $\sqrt{53}$ (b) $\sqrt{94}$
   (c) $\sqrt{53}$ (d) $\sqrt{2}$

6. (a) (i) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ (ii) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$
   (b) (i) $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
   (ii) $\frac{1}{5}\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$

7. $\pm 2\sqrt{6}$

8. $\frac{3}{2}$

9. $3, -\frac{5}{3}$

10. (a) $\begin{pmatrix} 4\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$ (b) $\frac{\sqrt{6}}{2}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

11. $-2, -\frac{23}{15}$

12. $t=\frac{1}{3}$, $d=\sqrt{\frac{14}{3}}$

Exercise 13D

1. (a) (i) 1.12 (ii) 1.17
   (b) (i) 1.88 (ii) 1.13
   (c) (i) 1.23 (ii) 1.77

2. (a) (i) $-\frac{5}{2\sqrt{21}}$
   (ii) $-\frac{20}{\sqrt{570}}$
   (b) (i) $-\frac{2}{\sqrt{102}}$
   (ii) $\frac{1}{\sqrt{35}}$
   (c) (i) $\frac{1}{\sqrt{50}}$ (ii) 0

3. (a) $61.0°, 74.5°, 44.5°$
   (b) $94.3°, 54.2°, 31.5°$

4. (a) (i) No (ii) Yes
   (b) (i) Yes (ii) No

5. $92.3°$

6. $40.0°$

7. (b) $107°, 73.2°$
   (c) $\frac{5}{4}$

8. (b) $41.8°, 48.2°$
   (c) $6\sqrt{5}$

Exercise 13E

1. (a) (i) 16 (ii) -56
   (b) (i) 16 (ii) -16
   (c) (i) 9 (ii) 9
   (d) (i) -4 (ii) 0

2. (a) (i) $\frac{7}{3\sqrt{6}}$ (ii) $\frac{5}{\sqrt{39}}$
   (b) (i) $\frac{2}{3}$ (ii) $\frac{1}{\sqrt{10}}$

3. (a) (i) $48.2°$ (ii) $98.0°$

4. (a) 19.2 (b) 3

6. (a) (i) $-\frac{1}{2}$ (ii) $\frac{2}{7}$
   (b) (i) $\frac{4}{5}$ (ii) $0, \frac{3}{2}$

7. (a) $x=-\frac{18}{11}$ $y=\frac{10}{11}$
   (b) (i) $1+11x=-17$
   (ii) $14=4+11y$
   (iii) They are perpendicular to $\begin{pmatrix}3y\\y\end{pmatrix}$ and $\begin{pmatrix}2x\\-3x\end{pmatrix}$, respectively.
   (c) (i) $\begin{pmatrix}1\\2\end{pmatrix}$ or $\begin{pmatrix}-1\\-2\end{pmatrix}$, and $\begin{pmatrix}3\\-5\end{pmatrix}$ or $\begin{pmatrix}-3\\5\end{pmatrix}$
   (ii) $x=-\frac{13}{11}$ $y=-\frac{3}{11}$
8. (a) 19 (b) 7
   (c) 32
9. (a) 2 (b) 6
10. (a) $\frac{52}{9}$
11. (a) 1.6
   (b) 68.7°, 21.3°, 90°
   (c) 88.7
12. **(a)** $\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{b}-\boldsymbol{a}$
   (b) $|\boldsymbol{b}|^2-|\boldsymbol{a}|^2$
13. (b) 2
   (c) $4\sqrt{5}$

Exercise 13F

1. (a) (i) $\begin{pmatrix}-2\\6\\-1\end{pmatrix}$ (ii) $\begin{pmatrix}0\\0\\0\end{pmatrix}$
   (b) (i) $\begin{pmatrix}-5\\-11\\-2\end{pmatrix}$ (ii) $\begin{pmatrix}12\\2\\6\end{pmatrix}$
2. (a) (i) $\frac{1}{2}\sqrt{153}$ (ii) $\sqrt{117}$
   (b) (i) $\frac{15\sqrt{3}}{2}$ (ii) $\frac{9}{2}$
3. (a) $\begin{pmatrix}18\\-12\\72\end{pmatrix}$, $\begin{pmatrix}-18\\12\\-72\end{pmatrix}$
   (b) $\boldsymbol{p}=-\boldsymbol{q}$
4. (a) $(11,2,0)$ (b) 16.8
5. (a) $C(5,4,0), F(5,0,2), G(5,4,2), H(0,4,2)$
   (b) 11.9

Exercise 13G

1. (a) 0.775 (b) 0.128
   (c) 0.630 (d) 0
2. (a) (i) $\begin{pmatrix}-1\\-10\\7\end{pmatrix}$ (ii) $\begin{pmatrix}-9\\-19\\2\end{pmatrix}$
   (b) (i) $\begin{pmatrix}-23\\1\\8\end{pmatrix}$ (ii) $\begin{pmatrix}0\\4\\0\end{pmatrix}$
3. (a) $\frac{1}{\sqrt{14}}\begin{pmatrix}3\\1\\2\end{pmatrix}$
   (b) $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$
4. 17.5
5. (b) $13\boldsymbol{a}\times\boldsymbol{b}$
6. (b) 0

Mixed examination practice 13

Short questions

1. −1
2. (a) $\frac{1}{2}\overrightarrow{AD}-\overrightarrow{AB}$
3. (a) $9\boldsymbol{i}+5\boldsymbol{j}+7\boldsymbol{k}$
   (b) $\sqrt{155}$
4. (a) $\begin{pmatrix}-5\\-3p-1\\15\end{pmatrix}$
   (b) $\frac{19}{3}$
5. 74.4°
6. $\frac{\pi}{2}-2\theta$
7. 0

Long questions

1. (a) $\begin{pmatrix}2\\0\\k-7\end{pmatrix}$ (c) $(3,6,1)$
   (d) $-\frac{1}{\sqrt{10}}$

2. (a) $\begin{pmatrix} -1+\frac{3}{k+1} \\ \frac{3}{4+1} \\ -4 \end{pmatrix}$ (b) 5

(c) $\left(\frac{3}{2},\frac{3}{2},2\right)$ (d) $\sqrt{\frac{33}{2}}$

3. (a) (a,a^2)

(b) $\begin{pmatrix} -a \\ -a^2 \end{pmatrix},\begin{pmatrix} -a \\ 4-a^2 \end{pmatrix}$

(c) $\sqrt{3}$

(d) $2\sqrt{3}$

4. (a) $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$

(b) $|\boldsymbol{c}|\cos\theta$

(d) $\frac{1}{3}$

Chapter 14

Exercise 14A

1. (a) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ (ii) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

(c) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(d) (i) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

2. (a) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (ii) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 7 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -9 \end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

3. (a) (i) Yes (ii) Yes
(b) (i) Yes (ii) No

4. (b) $(0,3,0)$

5. (a) $\boldsymbol{r}=\begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

(b) $(-5,-5,-11)$ or $(19,7,-7)$

6. (a) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

(b) 7

(c) $(-8,16,-26),(12,-14,34)$

Exercise 14B

1. (a) (i) 44.5° (ii) 56.5°
(b) (i) 26.6° (ii) 82.1°

2. (a) Perpendicular (b) Parallel
(c) Parallel (d) Same line

3. (a) (i) $(10,-7,-2)$
(ii) $(4.5,0,0)$
(b) (i) No intersection
(ii) No intersection

4. $\left(\frac{64}{9},\frac{4}{9},\frac{19}{9}\right)$

5. $\sqrt{\frac{6}{11}}$

6. (a) $(4,1,-2)$

(c) $(1,1,2)$ (d) $\frac{5\sqrt{26}}{2}$

7. (a) $\begin{pmatrix} 3t \\ 4t \end{pmatrix}$

(b) $\begin{pmatrix} 3t \\ 18-5t \end{pmatrix}$

(d) $t=2$

(e) 2 hours

8. (a) $\begin{pmatrix} 3t \\ 5-4t \\ t \end{pmatrix}$

(d) 30 km

9. 3

10. (a) $\left(\frac{5}{6},\frac{19}{6},\frac{9}{2}\right)$ (b) 48.5°

(d) $\frac{11\sqrt{11}}{6}\ (=6.08)$

(e) 4.55

11. (a) $(9,-5,8)$

(c) $(3,4,-1)$

12. (b) $\left(2+\sqrt{6},-1-2\sqrt{6},2\sqrt{6}\right)$, or $\left(2-\sqrt{6},2\sqrt{6}-1,-2\sqrt{6}\right)$

Exercise 14C

1. (a) $\frac{x-1}{-1}=\frac{y-7}{1}=\frac{z-2}{2}$ (b) $x=-1, \frac{y-5}{-2}=\frac{z}{2}$

(c) $\mathbf{r}=\begin{pmatrix}3\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\5\end{pmatrix}$ (d) $\mathbf{r}=\begin{pmatrix}-1\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\0\\-2\end{pmatrix}$

2. (a) Perpendicular (b) None
(c) None (d) Parallel

3. (a) $\frac{x-1}{3}=\frac{4-y}{2}=\frac{z+1}{3}$

(b) $\frac{1}{\sqrt{22}}\begin{pmatrix}3\\-2\\3\end{pmatrix}$

4. (a) $\mathbf{r}=\begin{pmatrix}1/2\\-2\\4/3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-2\end{pmatrix}$

(b) Yes at $\left(\frac{11}{6},0,0\right)$

(c) 61.0°

5. (a) 13.2° (b) No

6. (a) $(8,7,1)$

Exercise 14D

1. (a) (i) $\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\5\\2\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\3\end{pmatrix}$

(ii) $\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\4\\-1\end{pmatrix}+\mu\begin{pmatrix}5\\3\\0\end{pmatrix}$

(b) (i) $\mathbf{r}=(\mathbf{j}+\mathbf{k})+\lambda(3\mathbf{i}+\mathbf{j}-3\mathbf{k})+\mu(\mathbf{i}-3\mathbf{j})$

(ii) $\mathbf{r}=(\mathbf{i}-6\mathbf{j}+2\mathbf{k})+\lambda(5\mathbf{i}-6\mathbf{j})+\mu(-\mathbf{i}+3\mathbf{j}-\mathbf{k})$

2. (a) (i) $\mathbf{r}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$

(ii) $\mathbf{r}=\begin{pmatrix}-1\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}-6\\2\\-4\end{pmatrix}$

(b) (i) $\mathbf{r}=\begin{pmatrix}9\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-11\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-8\\-1\\2\end{pmatrix}$

(ii) $\mathbf{r}=\begin{pmatrix}11\\-7\\3\end{pmatrix}+\lambda\begin{pmatrix}-10\\21\\-1\end{pmatrix}+\mu\begin{pmatrix}-16\\17\\-3\end{pmatrix}$

3. (a) (i) $\mathbf{r}=\begin{pmatrix}-1\\4\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\2\end{pmatrix}$

(ii) $\mathbf{r}=\begin{pmatrix}11\\12\\13\end{pmatrix}+\lambda\begin{pmatrix}6\\-3\\1\end{pmatrix}+\mu\begin{pmatrix}2\\15\\6\end{pmatrix}$

(b) (i) $\mathbf{r}=\begin{pmatrix}-3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}+\mu\begin{pmatrix}-7\\-3\\-1\end{pmatrix}$

(ii) $\mathbf{r}=\begin{pmatrix}4\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}4\\0\\2\end{pmatrix}$

4. (a) (i) $3x-5y+2z=-4$
(ii) $6x-y+2z=19$
(b) (i) $3x-y=-9$
(ii) $4x-5z=-10$

5. (a) (i) $\begin{pmatrix}10\\13\\-12\end{pmatrix}$ (ii) $\begin{pmatrix}3\\1\\1\end{pmatrix}$

(b) (i) $\begin{pmatrix}1\\5\\0\end{pmatrix}$ (ii) $\begin{pmatrix}1\\20\\7\end{pmatrix}$

6. (a) (i) $\mathbf{r}\cdot\begin{pmatrix}10\\13\\-12\end{pmatrix}=38$

(ii) $\mathbf{r}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}=3$

(b) (i) $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = 22$

(ii) $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 20 \\ 7 \end{pmatrix} = 152$

7. (a) (i) $10x + 13y - 12z = 38$

(i) $3x + y + z = 1$

(b) (i) $x + 5y = 22$

(ii) $x + 20y + 7z = 152$

8. (a) (i) $x + y + z = 10$

(ii) $z = 2$

(b) (i) $40x + 5y + 8z = 580$

(ii) $x + y + z = 1$

Exercise 14E

1. (a) (i) $(7,1,1)$

(ii) $(-19,-5,7)$

(b) (i) $\left(-\frac{4}{3}, -\frac{7}{3}, -4\right)$

(ii) $(8,-3,2)$

2. (a) (i) 46.4° (ii) 17.5°

(b) (i) 47.6° (ii) 10.8°

3. (a) 75.8° (b) 60°

4. (a) (i) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -13 \\ -7 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(b) (i) $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

5. (a) $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ (b) 57°

6. $(5,3,1)$

8. (a) $(5,0,0), (0,-20,0),$
$(0,0,12)$

(b) 133

9. (b) $\frac{\sqrt{7}}{3}$

(c) $3\sqrt{5}$ (d) $\sqrt{35}$

10. (a) $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$

(d) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$

11. (a) $\begin{pmatrix} 7 \\ -5 \\ -4 \end{pmatrix}$

(b) (ii) $\mathbf{r} = \lambda \begin{pmatrix} 7 \\ -5 \\ -4 \end{pmatrix}$

(c) $\left(2, \frac{-10}{7}, \frac{-8}{7}\right)$

Exercise 14F

1. $\left(\frac{5}{3}, \frac{16}{3}, -\frac{7}{3}\right)$

2. $(2,2,-3)$

3. $\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

4. $\left(\frac{3}{2}, -\frac{11}{6}, -\frac{1}{6}\right)$

6. (a) $d = -2$

(b) $\mathbf{r} = \begin{pmatrix} \frac{-1}{2} \\ \frac{2}{5} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

7. $\left(\frac{5+d}{2}, \frac{d-3}{2}, \frac{5-d}{2}\right)$

8. (b) $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

9. (a) 2

(b) $x - 5 = y + 1 = z$

10. (a) $p = -4, a = 2$ or $-\frac{7}{5}$

(b) $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Exercise 14G

1. (a) $\boldsymbol{r}=\begin{pmatrix}-3\\-3\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$

   (b) $(3,3,1)$ (c) 9

2. (c) $\boldsymbol{r}=\lambda\begin{pmatrix}1\\-3\\1\end{pmatrix}$

   (d) $\frac{6\sqrt{11}}{11}$

3. (a) $\begin{pmatrix}-2\\3\\-1\end{pmatrix}$

   (b) (ii) $(1,-3,14)$

   (c) $2x-3y+z=25$

4. (b) $\boldsymbol{r}\bullet\begin{pmatrix}-1\\5\\2\end{pmatrix}=-5$

   (c) $\sqrt{30}$

   (d) $(8,-5,-1)$

5. (c) $\left(\frac{184}{11},-\frac{-32}{11},-\frac{-1}{11}\right)$

   (d) 6.99

6. (a) $(10,11,-6)$

   (b) $\begin{pmatrix}7\\-9\\-5\end{pmatrix}$

   (c) $7x-9y-5z=1$

7. (a) $\begin{pmatrix}-3\\-10\\2\end{pmatrix}$ (b) 5.32

   (c) $3x+10y-2z=16$

   (d) $\boldsymbol{r}=\begin{pmatrix}-7\\-28\\11\end{pmatrix}+\lambda\begin{pmatrix}3\\10\\-2\end{pmatrix}$

   (e) $(2,2,5)$; 31.9 (3*SF*)

   (f) 56.5

8. (a) $\boldsymbol{r}=\begin{pmatrix}-1\\1\\4\end{pmatrix}+\lambda\begin{pmatrix}6\\1\\5\end{pmatrix}$

   (c) $\begin{pmatrix}-13\\38\\8\end{pmatrix}$

   (d) $-13x+38y+8z=83$

9. (a) $\left(\frac{96}{41},-\frac{32}{41},\frac{16}{41}\right)$

   (b) $\frac{16\sqrt{41}}{41}$

10. (b) $\boldsymbol{r}=\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$

    (c) (i) $(2,0,-2)$ (ii) $(-4,0,4)$

    (d) $6\sqrt{2}$

Mixed examination practice 14

Short questions

1. $\boldsymbol{r}=\begin{pmatrix}3\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\0\end{pmatrix}$

2. 5

3. (a) $3x+y-z=6$

   (b) $\frac{5}{4}$

4. $\left(\frac{3}{2},-\frac{11}{6},-\frac{1}{6}\right)$

5. $(11,13,8)$

6. $\left(\frac{11}{3},\frac{20}{3},\frac{2}{3}\right)$ or $\left(\frac{-29}{3},\frac{-20}{3},\frac{22}{3}\right)$

7. (a) $x=4+\lambda, y=1-3\lambda,$ $z=12+2\lambda$

   (b) $\left(\frac{31}{14},\frac{89}{14},\frac{59}{7}\right)$

8. (c) $\frac{x-2}{3}=\frac{(y-2)}{7}=z-3$

9. $k=8$

11. $7x+2y-3z=3$

Long questions

1. (b) $\sqrt{33}$

   (c) 45.7°

   (d) 4.11

2. (b) (i) $\begin{pmatrix} \mu-2\lambda+8 \\ \mu+\lambda-3 \\ -\mu+8\lambda-16 \end{pmatrix}$

(iii) $3\mu-9\lambda+21=0$

(iv) $(1,1,2)$, $(4,-1,3)$

(v) $\sqrt{14}$

3. (a) $\mathbf{r}=\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(b) $(6,-5,4)$

(c) $2\sqrt{6}$

4. (a) 12.6 km/min

(b) $(24+3t)\mathbf{i}+(66-8t)\mathbf{j}+(12-4t)\mathbf{k}$

(c) 22 km

(d) 5 km (when $t=2$)

5. (a) $\begin{pmatrix} -2 \\ 7 \\ -3 \end{pmatrix}$

(b) $(3,3,8)$

(c) $\begin{pmatrix} -2 \\ 7 \\ -3 \end{pmatrix}$

(d) $2x-7y+3z=9$

6. (a) $d=3$

(b) $\mathbf{r}=\begin{pmatrix} \frac{3}{7} \\ \frac{5}{7} \\ 0 \end{pmatrix}+t\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$

(c) (i) $p=3$

(ii) $\mathbf{r}=\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$

(iii) $\frac{\sqrt{34}}{15}$ (≈ 0.389)

7. (a) $\mathbf{r}=\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

(b) $\left(\frac{30}{17},-\frac{14}{17},-\frac{3}{17}\right)$

(c) 2.81

8. (b) $\sqrt{94}$

(c) 0.551

(d) 5.08

9. (a) (ii) $\mathbf{r}=\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

(b) $3x-2y+z=5$

(c) $(2,1,1)$

10. (a) $\mathbf{r}=\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $\frac{1}{2}$

(c) $3\sqrt{2}$ (d) $\frac{3\sqrt{2}}{2}$

11. (b) $\mathbf{r}=\begin{pmatrix} 9 \\ -7 \\ 6 \end{pmatrix}+\lambda\begin{pmatrix} -4 \\ 8 \\ 3 \end{pmatrix}$

(c) $\mathbf{r}=\begin{pmatrix} 9 \\ -7 \\ 6 \end{pmatrix}+\mu\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$

(d) $(7,1,2)$

(e) $\sqrt{84}$

12. (a) $\frac{x-3}{3}=\frac{y-1}{-1}=\frac{z+4}{-1}$

(b) $(0,2,-3)$

(c) $(-3,3,-2)$

(e) $3\sqrt{2}$

13. (a) $\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$

(b) $2x-y+z=0$

(d) $\mathbf{r}=\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}+\lambda\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$

(e) $(3,4,0)$ (f) 47.1°