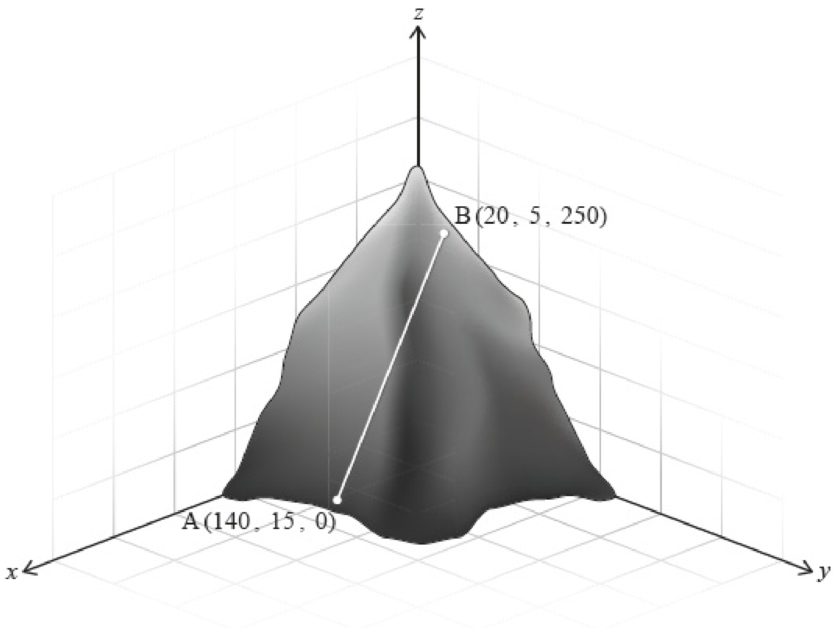
Geometry and trigonometry 29.11 *[47 marks]*

An inclined railway travels along a straight track on a steep hill, as shown in the diagram.

diagram not to scale



The locations of the stations on the railway can be described by coordinates in reference to x, y, and z-axes, where the x and y axes are in the horizontal plane and the z-axis is vertical.

The ground level station A has coordinates $(140, 15, 0)$ and station B, located near the top of the hill, has coordinates $(20, 5, 250)$. All coordinates are given in metres.

1a. Find the distance between stations A and B. *[2 marks]*

Markscheme

attempt at substitution into 3D distance formula ***(M1)***

$\text{AB} = \sqrt{(140-20)^2 + (15-5)^2 + 250^2} \left(= \sqrt{77\,000}\right)$

$= 277 \text{ m} \left(10\sqrt{770}, 277.488\ldots\right)$ ***A1***

[2 marks]

Station M is to be built halfway between stations A and B.

1b. Find the coordinates of station M. *[2 marks]*

Markscheme

attempt at substitution in the midpoint formula ***(M1)***

$\left(\frac{140+20}{2}, \frac{15+5}{2}, \frac{0+250}{2}\right)$

$(80, 10, 125)$ ***A1***

[2 marks]

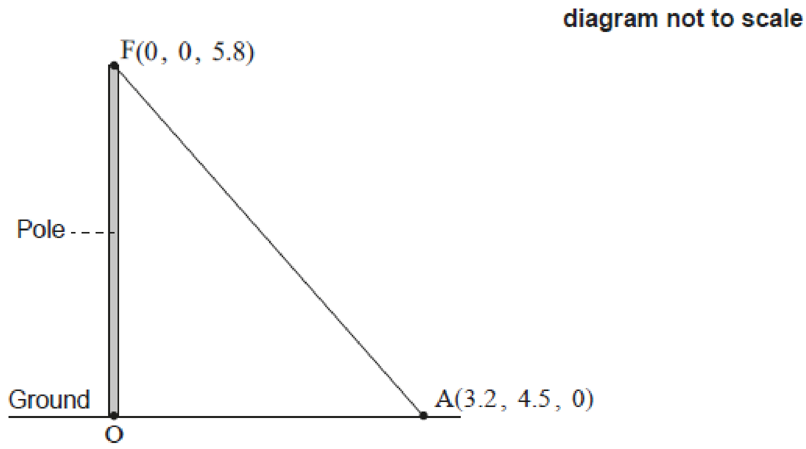
1c. Write down the height of station M, in metres, above the ground. *[1 mark]*

Markscheme

125 m ***A1***

[1 mark]

A vertical pole stands on horizontal ground. The bottom of the pole is taken as the origin, O, of a coordinate system in which the top, F, of the pole has coordinates $(0,\ 0,\ 5.8)$. All units are in metres.



The pole is held in place by ropes attached at F.

One of the ropes is attached to the ground at a point A with coordinates $(3.2,\ 4.5,\ 0)$. The rope forms a straight line from A to F.

2a. Find the length of the rope connecting A to F. *[2 marks]*

Markscheme

$\sqrt{3.2^2+4.5^2+5.8^2}$ ***(M1)***

$= 8.01 \quad (8.00812\ldots)\ \mathrm{m}$ ***A1***

[2 marks]

2b. Find $\mathrm{F\hat{A}O}$, the angle the rope makes with the ground. *[2 marks]*

Markscheme

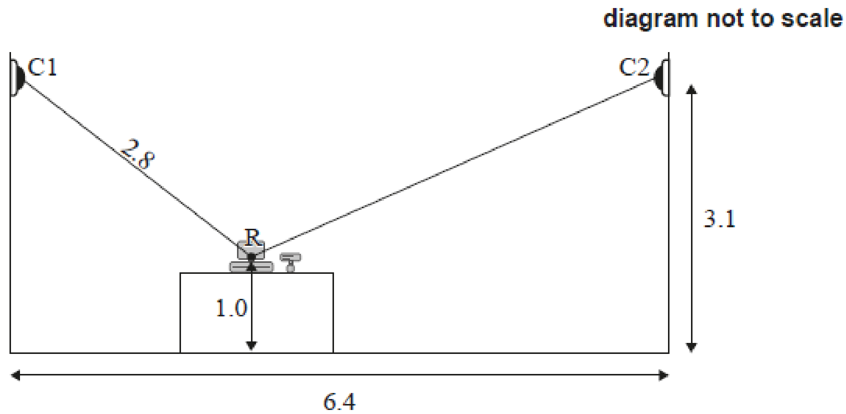
$\mathrm{F\hat{A}O} = \sin^{-1}\left(\frac{5.8}{8.00812\ldots}\right)$ **OR** $\cos^{-1}\left(\frac{5.52177\ldots}{8.00812\ldots}\right)$ **OR** $\tan^{-1}\left(\frac{5.8}{5.52177\ldots}\right)$ ***(M1)***

$46.4^\circ \quad (46.4077\ldots^\circ)$ ***A1***

[2 marks]

The owner of a convenience store installs two security cameras, represented by points $\mathrm{C1}$ and $\mathrm{C2}$. Both cameras point towards the centre of the store's cash register, represented by the point R.

The following diagram shows this information on a cross-section of the store.



The cameras are positioned at a height of $3.1\ \mathrm{m}$, and the horizontal distance between the cameras is $6.4\ \mathrm{m}$. The cash register is sitting on a counter so that its centre, R, is $1.0\ \mathrm{m}$ above the floor.

The distance from Camera 1 to the centre of the cash register is $2.8\ \mathrm{m}$.

3a. Determine the angle of depression from Camera 1 to the centre of the cash register. Give your answer in degrees. *[2 marks]*

Markscheme

$\sin\theta = \frac{2.1}{2.8}$ **OR** $\tan\theta = \frac{2.1}{1.85202\ldots}$ ***(M1)***

$(\theta =)\ 48.6^\circ \quad (48.5903\ldots^\circ)$ ***A1***

[2 marks]

3b. Calculate the distance from Camera 2 to the centre of the cash register. *[4 marks]*

Markscheme

METHOD 1

$\sqrt{2.8^2 - 2.1^2}$ **OR** $2.8\cos(48.5903\ldots)$ **OR** $\frac{2.1}{\tan(48.5903\ldots)}$ ***(M1)***

Note: Award ***M1*** for attempt to use Pythagorean Theorem with 2.1 seen or for attempt to use cosine or tangent ratio.

$1.85\,(\mathrm{m})\quad(1.85202\ldots)$ ***(A1)***

Note: Award the ***M1A1*** if 1.85 is seen in part (a).

$(6.4 - 1.85202\ldots)$

$4.55\ \mathrm{m}\quad(4.54797\ldots)$ ***(A1)***

Note: Award ***A1*** for 4.55 or equivalent seen, either as a separate calculation or in Pythagorean Theorem.

$\sqrt{(4.54797\ldots)^2 + 2.1^2}$

$5.01\ \mathrm{m}\quad(5.00939\ldots\ \mathrm{m})$ ***A1***

METHOD 2

attempt to use cosine rule ***(M1)***

$(c^2 =)\ 2.8^2 + 6.4^2 - 2(2.8)(6.4)\cos(48.5903\ldots)$ ***(A1)(A1)***

Note: Award ***A1*** for $48.5903\ldots{}^\circ$ substituted into cosine rule formula, ***A1*** for correct substitution.

$(c =)\ 5.01\ \mathrm{m}\quad(5.00939\ldots\ \mathrm{m})$ ***A1***

[4 marks]

3c. Without further calculation, determine which camera has the largest angle of depression to the centre of the cash register. Justify your response. *[2 marks]*

Markscheme

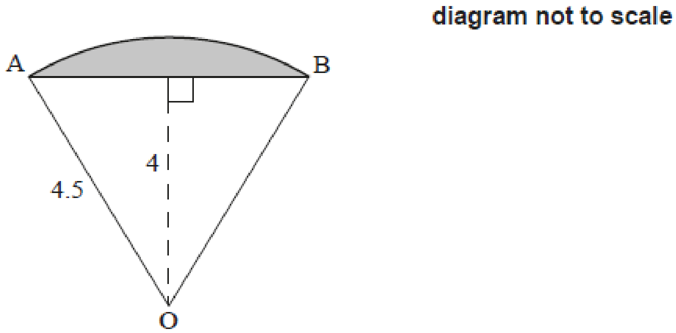
camera 1 is closer to the cash register (than camera 2 and both cameras are at the same height on the wall) ***R1***

the larger angle of depression is from camera 1 ***A1***

Note: Do not award ***R0A1***. Award ***R0A0*** if additional calculations are completed and used in their justification, as per the question. Accept "$1.85 < 4.55$" or "$2.8 < 5.01$" as evidence for the ***R1***.

[2 marks]

A sector of a circle, centre O and radius $4.5\ \mathrm{m}$, is shown in the following diagram.



4a. Find the angle $\mathrm{A\hat{O}B}$. *[3 marks]*

Markscheme

$\left(\frac{1}{2}\mathrm{A\hat{O}B} =\right) \arccos\left(\frac{4}{4.5}\right) = 27.266\ldots$ ***(M1)(A1)***

$\mathrm{A\hat{O}B} = 54.532\ldots \approx 54.5^\circ$ $(0.951764\ldots \approx 0.952$ radians) ***A1***

Note: Other methods may be seen; award ***(M1)(A1)*** for use of a correct trigonometric method to find an appropriate angle and then ***A1*** for the correct answer.

[3 marks]

4b. Find the area of the shaded segment. *[5 marks]*

Markscheme

finding area of triangle

EITHER

area of triangle $= \frac{1}{2} \times 4.5^2 \times \sin(54.532\ldots)$ ***(M1)***

Note: Award ***M1*** for correct substitution into formula.

$= 8.24621\ldots \approx 8.25\ \mathrm{m}^2$ ***(A1)***

OR

$\mathrm{AB} = 2 \times \sqrt{4.5^2 - 4^2} = 4.1231\ldots$

area triangle $= \frac{4.1231\ldots \times 4}{2}$ ***(M1)***

$= 8.24621\ldots \approx 8.25\ \mathrm{m}^2$ ***(A1)***

finding area of sector

EITHER

area of sector $= \frac{54.532\ldots}{360} \times \pi \times 4.5^2$ ***(M1)***

$= 9.63661\ldots \approx 9.64\ \mathrm{m}^2$ ***(A1)***

OR

area of sector $= \frac{1}{2} \times 0.9517641\ldots \times 4.5^2$ ***(M1)***

$= 9.63661\ldots \approx 9.64\ \mathrm{m}^2$ ***(A1)***

THEN

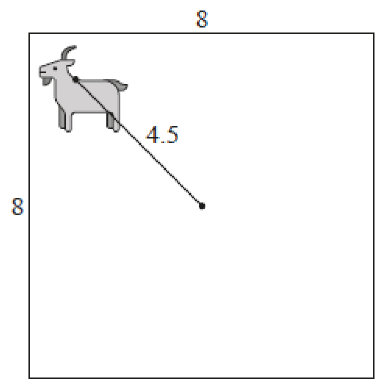
area of segment $= 9.63661\ldots - 8.24621\ldots$

$= 1.39\ \mathrm{m}^2\ \ (1.39040\ldots)$ ***A1***

[5 marks]

A square field with side 8 m has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to 4.5 m from the post.

diagram not to scale



4c. Find the area of a circle with radius 4.5 m. *[2 marks]*

Markscheme

$\pi \times 4.5^2$ ***(M1)***

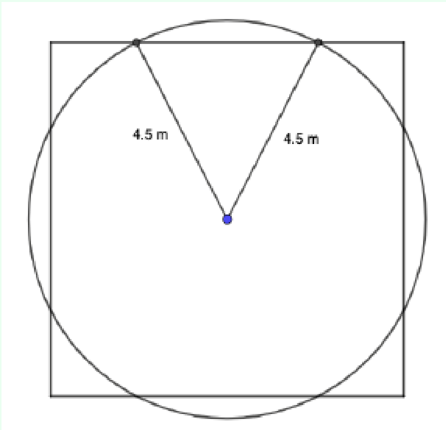
$63.6 \text{ m}^2 \quad (63.6172\ldots \text{ m}^2)$ ***A1***

[2 marks]

4d. Find the area of the field that can be reached by the goat. *[3 marks]*

Markscheme

METHOD 1



$4 \times 1.39040\ldots \quad (5.56160)$ ***(A1)***

subtraction of four segments from area of circle ***(M1)***

$= 58.1 \text{ m}^2 \quad (58.055\ldots)$ ***A1***

METHOD 2

$4\left(0.5 \times 4.5^2 \times \sin 54.532\ldots\right) + 4\left(\frac{35.4679}{360} \times \pi \times 4.5^2\right)$ ***(M1)***

$= \quad 32.9845\ldots + 25.0707$ ***(A1)***

$= 58.1 \text{ m}^2 \quad (58.055\ldots)$ ***A1***

[3 marks]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.3\, t\mathrm{e}^{-t}$.

4e. Find the value of t at which the goat is eating grass at the greatest rate. *[2 marks]*

Markscheme

sketch of $\frac{\mathrm{d}V}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.110363\ldots$ **OR** attempt to find where $\frac{\mathrm{d}^2V}{\mathrm{d}t^2} = 0$ ***(M1)***

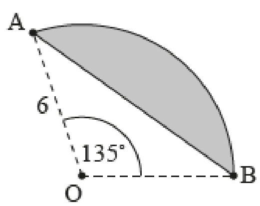
$t = 1$ hour ***A1***

[2 marks]

A garden includes a small lawn. The lawn is enclosed by an arc AB of a circle with centre O and radius $6\ \mathrm{m}$, such that $\mathrm{A\hat{O}B} = 135^\circ$. The straight border of the lawn is defined by chord $[\mathrm{AB}]$.

The lawn is shown as the shaded region in the following diagram.

diagram not to scale



5a. A footpath is to be laid around the curved side of the lawn. Find the length of the footpath. *[3 marks]*

Markscheme

$135^\circ \times \frac{12\pi}{360^\circ}$ ***(M1)(A1)***

$14.1\,(\mathrm{m})(14.1371\ldots)$ ***A1***

[3 marks]

5b. Find the area of the lawn. *[4 marks]*

Markscheme

evidence of splitting region into two areas ***(M1)***

$135^\circ \times \frac{\pi 6^2}{360^\circ} - \frac{6 \times 6 \times \sin 135^\circ}{2}$ ***(M1)(M1)***

Note: Award ***M1*** for correctly substituting into area of sector formula, ***M1*** for evidence of substituting into area of triangle formula.

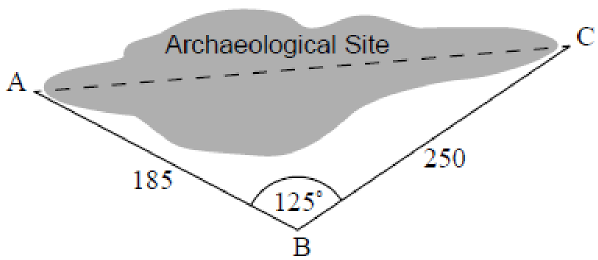
$42.4115\ldots - 12.7279\ldots$

$29.7\ \mathrm{m}^2 (29.6835\ldots)$ ***A1***

[4 marks]

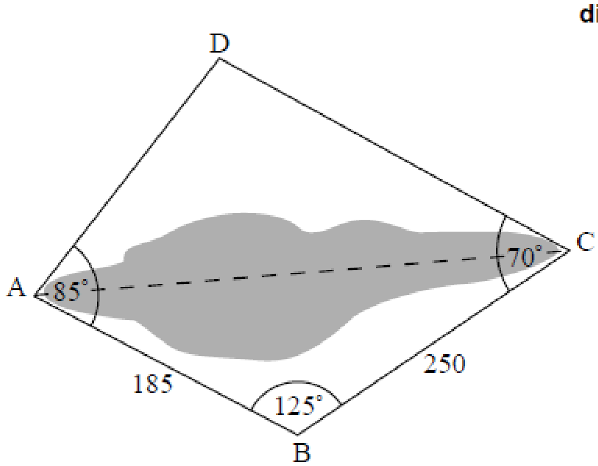
An archaeological site is to be made accessible for viewing by the public. To do this, archaeologists built two straight paths from point A to point B and from point B to point C as shown in the following diagram. The length of path AB is 185 m, the length of path BC is 250 m, and angle $\mathrm{A\hat{B}C}$ is 125°.

diagram not to scale



The archaeologists plan to build two more straight paths, AD and DC. For the paths to go around the site, angle $\mathrm{B\hat{A}D}$ is to be made equal to 85° and angle $\mathrm{B\hat{C}D}$ is to be made equal to 70° as shown in the following diagram.

diagram not to scale



6a. Find the size of angle $\mathrm{C\hat{A}D}$. *[1 mark]*

Markscheme

(CAD =) 53.1° (53.0521...°) ***(A1)*****(ft)**

Note: Follow through from their part (b)(i) only if working seen.

[1 mark]

6b. Find the size of angle $\mathrm{A\hat{C}D}$. *[2 marks]*

Markscheme

(ACD =) 70° − (180° − 125° − 31.9478°...)   ***(M1)***

Note: Award ***(M1)*** for subtracting their angle $\mathrm{A}\hat{\mathrm{C}}\mathrm{B}$ from 70°.

OR

(ADC =) 360 − (85 + 70 + 125) = 80

(ACD =) 180 − 80 − 53.0521...   ***(M1)***

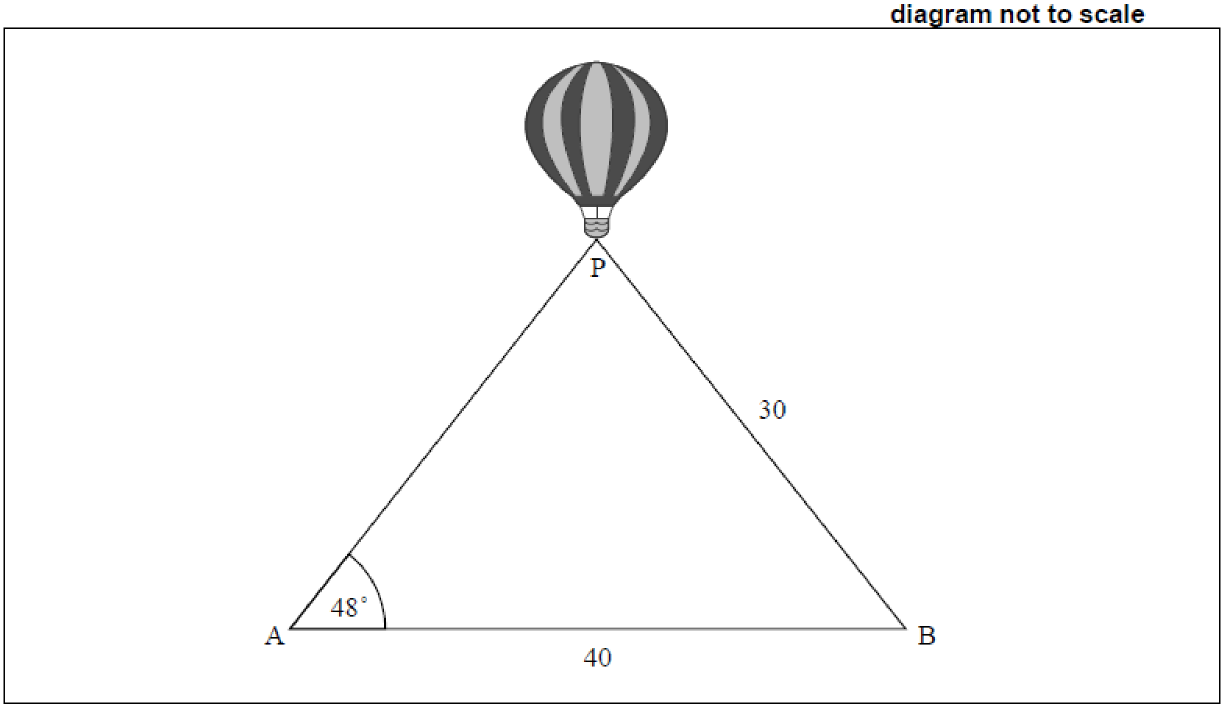
46.9° (46.9478...°)   ***(A1)*****(ft)*****(G2)***

Note: Follow through from part (b)(i).

[2 marks]

Two fixed points, A and B, are 40 m apart on horizontal ground. Two straight ropes, AP and BP, are attached to the same point, P, on the base of a hot air balloon which is vertically above the line AB. The length of BP is 30 m and angle BAP is 48°.

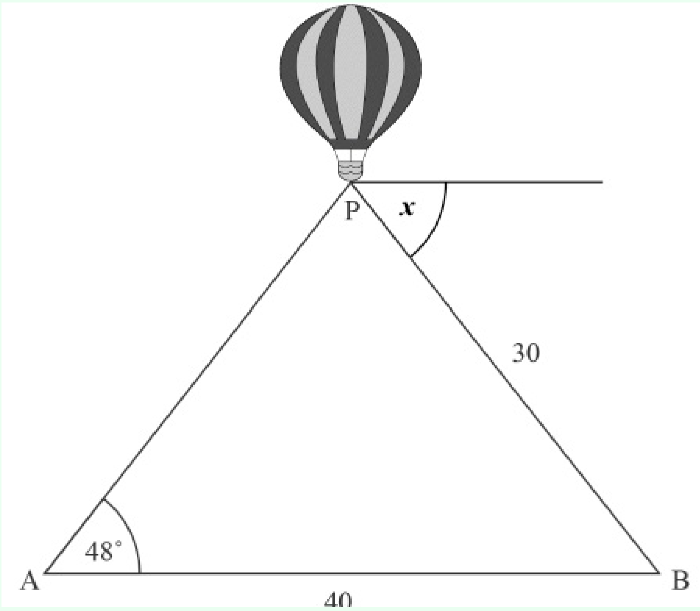
diagram not to scale



7. On the diagram, draw and label with an *x* the angle of depression of B from P. *[1 mark]*

Markscheme

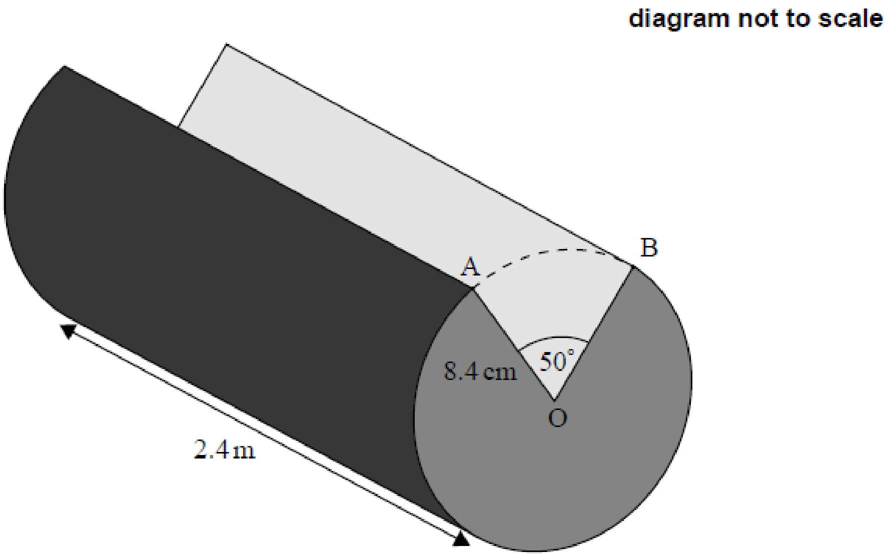
* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.



(A1) (C1)

[1 mark]

8. Helen is building a cabin using cylindrical logs of length 2.4 m and radius 8.4 cm. A wedge is cut from one log and the cross-section of this log is illustrated in the following diagram. *[4 marks]*



Find the volume of this log.

Markscheme

volume $= 240\left(\pi \times 8.4^2 - \frac{1}{2} \times 8.4^2 \times 0.872664\ldots\right)$ ***M1M1M1***

Note: Award ***M1*** 240 × area, award ***M1*** for correctly substituting area sector formula, award ***M1*** for subtraction of their area of the sector from area of circle.

= 45800 (= 45811.96071) ***A1***

[4 marks]