Saturday 26.11 *[41 marks]*

1a. Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^4$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. *[3 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

uses the binomial theorem on $(\cos\theta + \mathrm{i}\sin\theta)^4$ **M1**

$= {}_4C_0\cos^4\theta + {}_4C_1\cos^3\theta(\mathrm{i}\sin\theta) + {}_4C_2\cos^2\theta\left(\mathrm{i}^2\sin^2\theta\right) + {}_4C_3\cos\theta\left(\mathrm{i}^3\sin^3\theta\right) + {}_4C$
A1

$= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$ **A1**

[3 marks]

1b. Use de Moivre’s theorem and the result from part (a) to show that $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. *[5 marks]*

Markscheme

(using de Moivre's theorem with $n = 4$ gives) $\cos 4\theta + \mathrm{i} \sin 4\theta$ **(A1)**

equates both the real and imaginary parts of $\cos 4\theta + \mathrm{i} \sin 4\theta$ and $\left(\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta\right) + \mathrm{i}\left(4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta\right)$ **M1**

$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$ and
$\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$

recognizes that $\cot 4\theta = \frac{\cos 4\theta}{\sin 4\theta}$ **(A1)**

substitutes for $\sin 4\theta$ and $\cos 4\theta$ into $\frac{\cos 4\theta}{\sin 4\theta}$ **M1**

$\cot 4\theta = \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$

divides the numerator and denominator by $\sin^4 \theta$ to obtain

$\cot 4\theta = \frac{\frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{\sin^4\theta}}{\frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\sin^4\theta}}$ **A1**

$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$ **AG**

[5 marks]

1c. Use the identity from part (b) to show that the quadratic equation $x^2 - 6x + 1 = 0$ has roots $\cot^2 \frac{\pi}{8}$ and $\cot^2 \frac{3\pi}{8}$. *[5 marks]*

Markscheme

setting $\cot\,4\theta = 0$ and putting $x = \cot^2\,\theta$ in the numerator of $\cot\,4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$ gives $x^2 - 6x + 1 = 0$ **M1**

attempts to solve $\cot\,4\theta = 0$ for θ **M1**

$4\theta = \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots\left(4\theta = \frac{1}{2}(2n+1)\pi,\ \mathrm{n} = 0,\ 1,\ \ldots\right)$ **(A1)**

$\theta = \frac{\pi}{8},\ \frac{3\pi}{8}$ **A1**

Note: Do not award the final **A1** if solutions other than $\theta = \frac{\pi}{8},\ \frac{3\pi}{8}$ are listed.

finding the roots of $\cot\,4\theta = 0\left(\theta = \frac{\pi}{8},\ \frac{3\pi}{8}\right)$ corresponds to finding the roots of $x^2 - 6x + 1 = 0$ where $x = \cot^2\,\theta$ **R1**

so the equation $x^2 - 6x + 1 = 0$ as roots $\cot^2\frac{\pi}{8}$ and $\cot^2\frac{3\pi}{8}$ **AG**

[5 marks]

1d. Hence find the exact value of $\cot^2\frac{3\pi}{8}$. *[4 marks]*

Markscheme

attempts to solve $x^2 - 6x + 1 = 0$ for x **M1**

$x = 3 \pm 2\sqrt{2}$ **A1**

since $\cot^2\frac{\pi}{8} > \cot^2\frac{3\pi}{8}$, $\cot^2\frac{3\pi}{8}$ has the smaller value of the two roots **R1**

Note: Award **R1** for an alternative convincing valid reason.

so $\cot^2\frac{3\pi}{8} = 3 - 2\sqrt{2}$ **A1**

[4 marks]

1e. Deduce a quadratic equation with integer coefficients, having roots $\operatorname{cosec}^2\,\frac{\pi}{8}$ and $\operatorname{cosec}^2\,\frac{3\pi}{8}$. *[3 marks]*

Markscheme

let $y = \operatorname{cosec}^2 \theta$

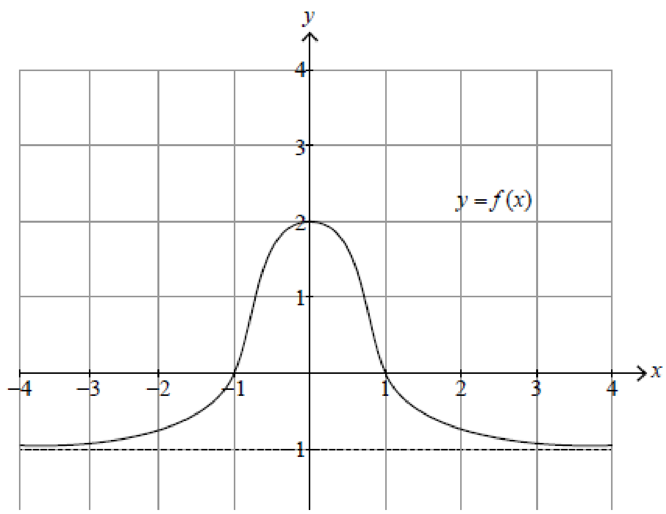
uses $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$ where $x = \cot^2 \theta$ **(M1)**

$x^2 - 6x + 1 = 0 \Rightarrow (y-1)^2 - 6(y-1) + 1 = 0$ **M1**

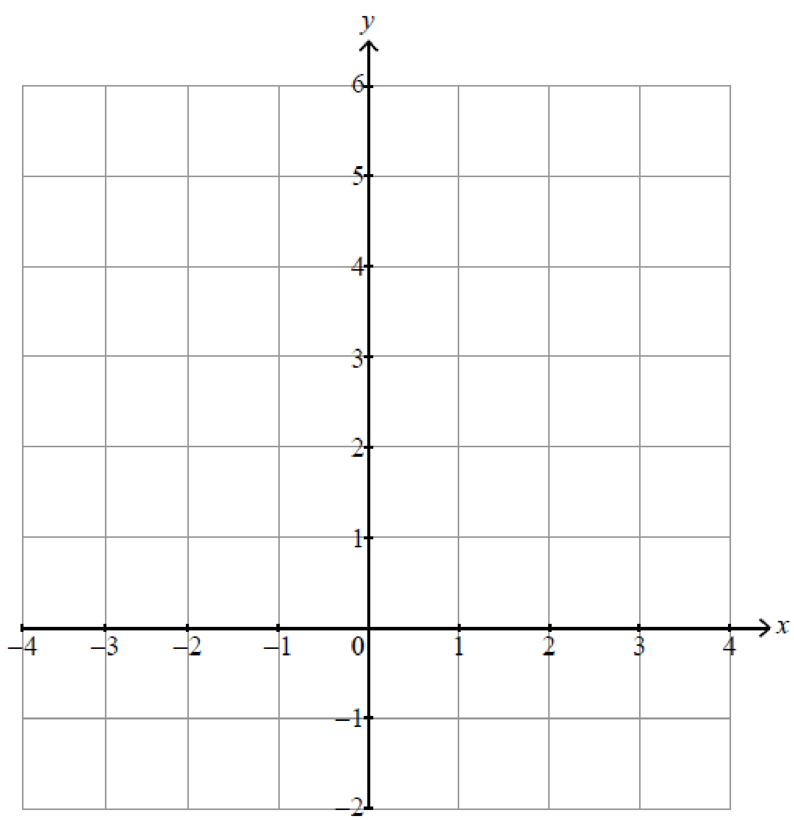
$y^2 - 8y + 8 = 0$ **A1**

[3 marks]

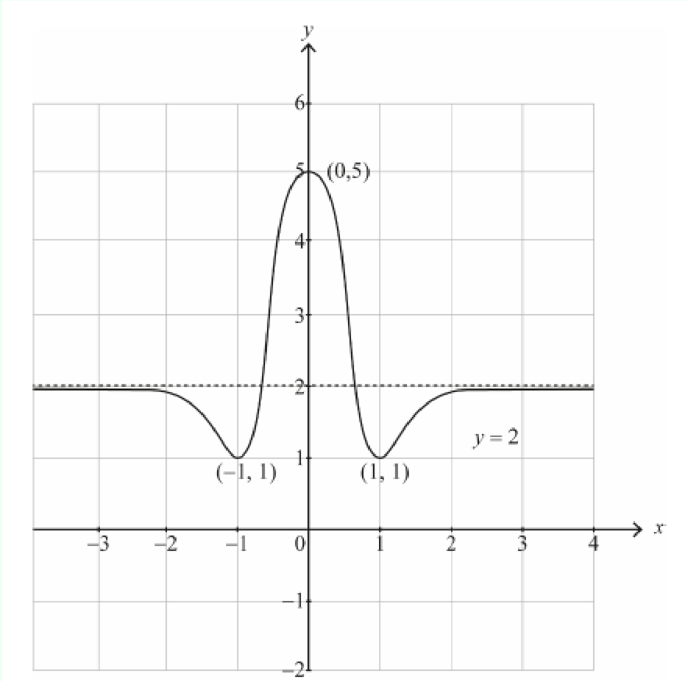
2. The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$. *[5 marks]*



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



Markscheme



no y values below 1 ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ ***A1***

(±1,1) local minima ***A1***

(0,5) local maximum ***A1***

smooth curve and smooth stationary points ***A1***

[5 marks]

3. The plane *Π* has the Cartesian equation $2x + y + 2z = 3$ *[7 marks]*

The line *L* has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line *L* and the plane *Π* is 30°.

Find the possible values of p.

Markscheme

recognition that the angle between the normal and the line is 60° (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9}\times\sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4\,|p|$

attempt to square both sides ***M1***

$9\left(5+p^2\right) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

[7 marks]

4. A discrete random variable X has the probability distribution given by the following table. *[4 marks]*

x	0	1	2	3
$\mathrm{P}(X=x)$	p	$\frac{1}{4}$	$\frac{1}{6}$	q

Given that $\mathrm{E}(X) = \frac{19}{12}$, determine the value of p and the value of q.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\mathrm{E}(X) = (0 \times p) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{6}\right) + 3q\left(= \frac{19}{12}\right)$ ***(M1)***

$\left(\Rightarrow \frac{1}{4} + \frac{1}{3} + 3q = \frac{19}{12}\right)$

$q = \frac{1}{3}$ ***A1***

$p + \frac{1}{4} + \frac{1}{6} + q = 1$ ***(M1)***

$\left(\Rightarrow p + q = \frac{7}{12}\right)$

$p = \frac{1}{4}$ ***A1***

[4 marks]

5. Let $f'(x) = \frac{8x}{\sqrt{2x^2+1}}$. Given that $f(0) = 5$, find $f(x)$. *[5 marks]*

Markscheme

attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}} \mathrm{d}x = \int \frac{2}{\sqrt{u}} \mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

correct substitution into **their** integrated function (must have *C*) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

[5 marks]

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