Tuesday 29.11 *[58 marks]*

The first three terms of an arithmetic sequence are $u_1,\ 5u_1 - 8$ and $3u_1 + 8$.

1a. Show that $u_1 = 4$. *[2 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

EITHER

uses $u_2 - u_1 = u_3 - u_2$ **(M1)**

$(5u_1 - 8) - u_1 = (3u_1 + 8) - (5u_1 - 8)$

$6u_1 = 24$ **A1**

OR

uses $u_2 = \frac{u_1 + u_3}{2}$ **(M1)**

$5u_1 - 8 = \frac{u_1 + (3u_1 + 8)}{2}$

$3u_1 = 12$ **A1**

THEN

so $u_1 = 4$ **AG**

[2 marks]

1b. Prove that the sum of the first n terms of this arithmetic sequence is a square number. *[4 marks]*

Markscheme

$d = 8$ **(A1)**

uses $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ **M1**

$S_n = \frac{n}{2}(8 + 8(n-1))$ **A1**

$= 4n^2$

$= (2n)^2$ **A1**

Note: The final **A1** can be awarded for clearly explaining that $4n^2$ is a square number.

so sum of the first n terms is a square number **AG**

[4 marks]

Consider the function defined by $f(x) = \frac{kx-5}{x-k}$, where $x \in \mathbb{R} \setminus \{k\}$ and $k^2 \neq 5$.

2a. State the equation of the vertical asymptote on the graph of $y = f(x)$. *[1 mark]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$x = k$ ***A1***

[1 mark]

2b. State the equation of the horizontal asymptote on the graph of $y = f(x)$. *[1 mark]*

Markscheme

$y = k$ ***A1***

[1 mark]

2c. Use an algebraic method to determine whether f is a self-inverse function. *[4 marks]*

Markscheme

METHOD 1

$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$ ***M1***

$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$ ***A1***

$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$

$= \frac{k^2x-5x}{k^2-5}$ ***A1***

$= \frac{x(k^2-5)}{k^2-5}$

$= x$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

METHOD 2

$f(x) = \frac{kx-5}{x-k}$

$x = \frac{ky-5}{y-k}$ ***M1***

Note: Interchanging x and y can be done at any stage.

$x(y-k) = ky-5$ ***A1***

$xy-xk = ky-5$

$xy-ky = xk-5$

$y(x-k) = kx-5$ ***A1***

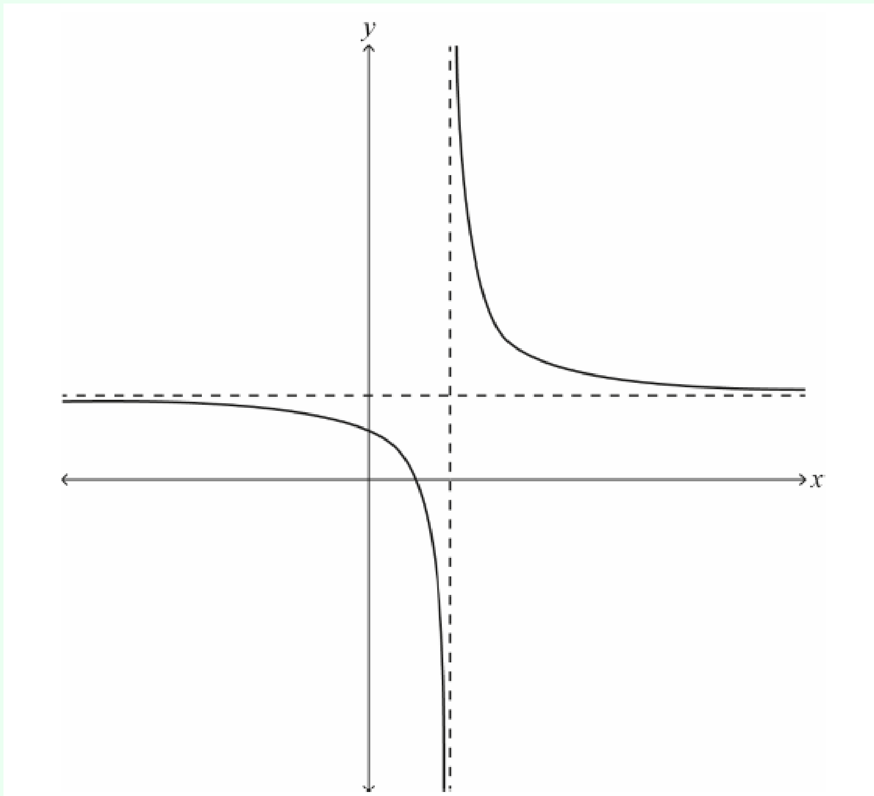
$y = f^{-1}(x) = \frac{kx-5}{x-k}$ (hence f is self-inverse) ***R1***

[4 marks]

Consider the case where $k = 3$.

2d. Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. *[3 marks]*

Markscheme



attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0,\ \frac{5}{3}\right)$ and $\left(\frac{5}{3},\ 0\right)$ ***A1***

[3 marks]

2e. The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b\ \ln 2)$, where $a,\ b \in \mathbb{Z}$. *[6 marks]*

Markscheme

METHOD 1

$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p + \frac{q}{x-3}$ ***M1***

$\frac{3x-5}{x-3} = 3 + \frac{4}{x-3}$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$\left(\frac{3x-5}{x-3}\right)^2 = 9 + \frac{24x-56}{(x-3)^2}$ ***A1***

THEN

$\left(\frac{3x-5}{x-3}\right)^2 = 9 + \frac{24}{x-3} + \frac{16}{(x-3)^2}$ ***A1***

$\text{volume} = \pi\int_5^7 \left(9 + \frac{24}{x-3} + \frac{16}{(x-3)^2}\right) \mathrm{d}x$

$= \pi\left[9x + 24\ln(x-3) - \frac{16}{x-3}\right]_5^7$ ***A1***

$= \pi\lfloor(63 + 24\ln 4 - 4) - (45 + 24\ln 2 - 8)\rfloor$

$= \pi(22 + 24\ln 2)$ ***A1***

METHOD 2

$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

substituting $u = x - 3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$3x - 5 = 3(u+3) - 5 = 3u + 4$

$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$ ***M1***

$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\,\mathrm{d}u$ ***A1***

$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$ ***A1***

Note: Ignore absence of or incorrect limits seen up to this point.

$= \pi(22 + 24\ln 2)$ ***A1***

[6 marks]

The vectors ***a*** and ***b*** are defined by ***a*** = $\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}$, ***b*** = $\begin{pmatrix} 0 \\ -t \\ 4t \end{pmatrix}$, where $t \in \mathbb{R}$.

3a. Find and simplify an expression for ***a* • *b*** in terms of t. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

a* • *b = $(1 \times 0) + (1 \times -t) + (t \times 4t)$ ***(M1)***

= $-t + 4t^2$ ***A1***

[2 marks]

3b. Hence or otherwise, find the values of t for which the angle between ***a*** and ***b*** is obtuse . *[4 marks]*

Markscheme

recognition that ***a* • *b*** = |***a***||***b***|cos θ ***(M1)***

a* • *b < 0 or $-t + 4t^2$ < 0 or cos θ < 0 ***R1***

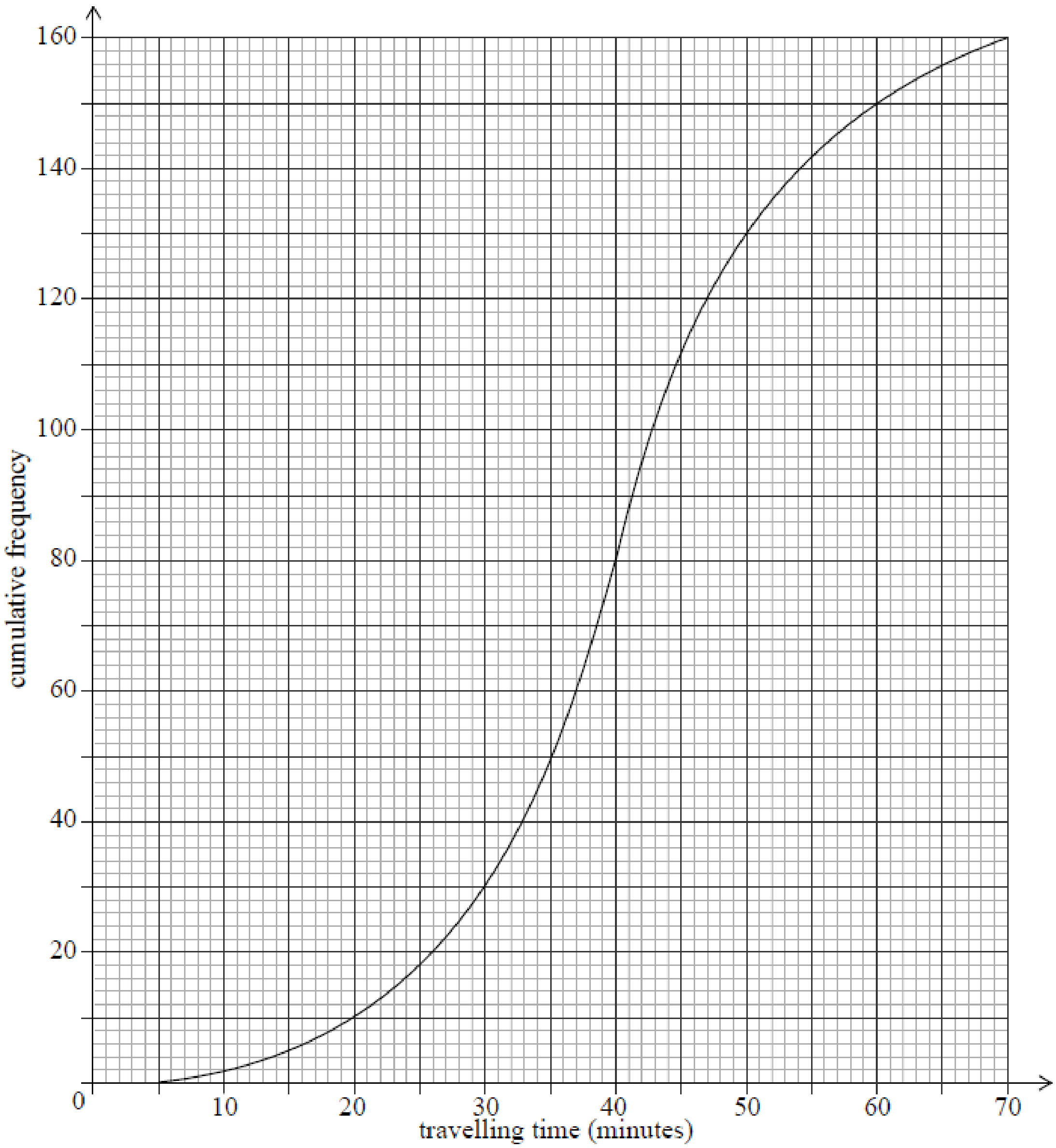
Note: Allow ≤ for ***R1***.

attempt to solve using sketch or sign diagram ***(M1)***

$0 < t < \frac{1}{4}$ ***A1***

[4 marks]

A large company surveyed 160 of its employees to find out how much time they spend traveling to work on a given day. The results of the survey are shown in the following cumulative frequency diagram.



4a. Find the median number of minutes spent traveling to work. *[2 marks]*

Markscheme

evidence of median position ***(M1)***

80th employee

40 minutes ***A1***

[2 marks]

4b. Find the number of employees whose travelling time is within 15 minutes of the median. *[3 marks]*

Markscheme

valid attempt to find interval (25–55) ***(M1)***

18 (employees), 142 (employees) ***A1***

124 ***A1***

[3 marks]

Only 10% of the employees spent more than k minutes traveling to work.

4c. Find the value of k. *[3 marks]*

Markscheme

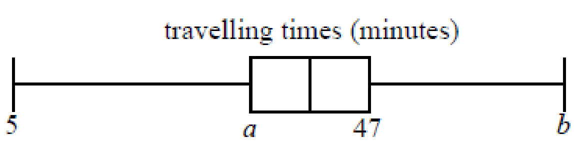
recognising that there are 16 employees in the top 10% ***(M1)***

144 employees travelled more than k minutes ***(A1)***

k = 56 ***A1***

[3 marks]

The results of the survey can also be displayed on the following box-and-whisker diagram.



4d. Write down the value of b. *[1 mark]*

Markscheme

b = 70 ***A1***

[1 mark]

4e. Find the value of a. *[2 marks]*

Markscheme

recognizing a is first quartile value ***(M1)***

40 employees

a = 33 ***A1***

[2 marks]

4f. Hence, find the interquartile range. *[2 marks]*

Markscheme

47 − 33 ***(M1)***

IQR = 14 ***A1***

[2 marks]

4g. Travelling times of less than p minutes are considered outliers. *[2 marks]*

Find the value of p.

Markscheme

attempt to find 1.5 × **their** IQR ***(M1)***

33 − 21

12 ***(A1)***

[2 marks]

Let $f\left(x\right) = \frac{1}{3}x^3 + x^2 - 15x + 17$.

5a. Find $f'\left(x\right)$. *[2 marks]*

Markscheme

$f'(x) = x^2 + 2x - 15$ ***(M1)A1***

[2 marks]

The graph of f has horizontal tangents at the points where $x = a$ and $x = b$, $a < b$.

5b. Find the value of a and the value of b. *[3 marks]*

Markscheme

correct reasoning that $f'(x) = 0$ (seen anywhere) ***(M1)***

$x^2 + 2x - 15 = 0$

valid approach to solve quadratic ***M1***

$(x - 3)(x + 5)$, quadratic formula

correct values for x

3, −5

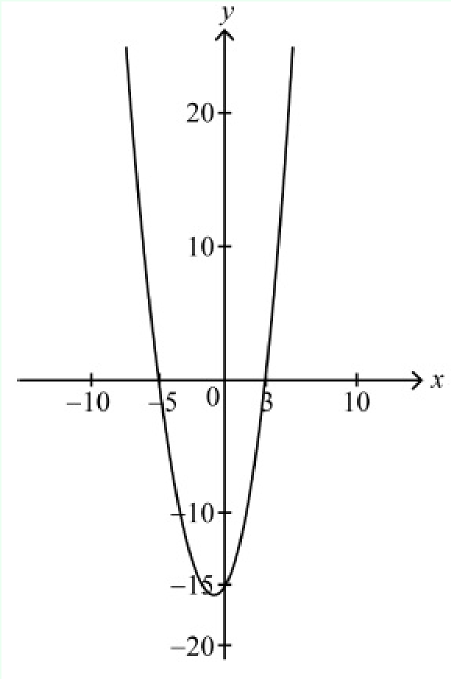
correct values for a and b

$a = -5$ and $b = 3$ ***A1***

[3 marks]

5c. Sketch the graph of $y = f'(x)$. *[1 mark]*

Markscheme



A1

[1 mark]

5d. Hence explain why the graph of f has a local maximum point at $x = a$. *[1 mark]*

Markscheme

first derivative changes from positive to negative at $x = a$ ***A1***

so local maximum at $x = a$ ***AG***

[1 mark]

5e. Find $f''(b)$. *[3 marks]*

Markscheme

$f''(x) = 2x + 2$ ***A1***

substituting **their** b into **their** second derivative ***(M1)***

$f''(3) = 2 \times 3 + 2$

$f''(b) = 8$ ***(A1)***

[3 marks]

5f. Hence, use your answer to part (d)(i) to show that the graph of f has a local minimum point at $x = b$. *[1 mark]*

Markscheme

$f''(b)$ is positive so graph is concave up ***R1***

so local minimum at $x = b$ ***AG***

[1 mark]

5g. The normal to the graph of f at $x = a$ and the tangent to the graph of f at $x = b$ intersect at the point (p, q) . *[5 marks]*

Find the value of p and the value of q.

Markscheme

normal to f at $x = a$ is $x = -5$ (seen anywhere) ***(A1)***

attempt to find y-coordinate at their value of b ***(M1)***

$f(3) = -10$ ***(A1)***

tangent at $x = b$ has equation $y = -10$ (seen anywhere) ***A1***

intersection at $(-5, -10)$

$p = -5$ and $q = -10$ ***A1***

[5 marks]