polynomials 3IB HL *[88 marks]*

1. Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants. *[5 marks]*

   The remainder when $f(x)$ is divided by $(x + 1)$ is 7, and the remainder when $f(x)$ is divided by $(x - 2)$ is 1. Find the value of p and the value of q.

   ## Markscheme

   * This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

   attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

   $1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

   attempt to solve their two equations ***M1***

   $p = -3$, $q = 2$ ***A1***

   [5 marks]

2. Consider the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$. *[7 marks]*

   Two of the roots of the equation are $\log_2 6$ and $i\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$.

   Show that $6a + d + 12 = 0$.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$-i\sqrt{3}$ is a root ***(A1)***

$3+\log_2 3-\log_2 6\left(=3+\log_2 \frac{1}{2}=3-1=2\right)$ is a root ***(A1)***

sum of roots: $-a=3+\log_2 3 \Rightarrow a=-3-\log_2 3$ ***M1***

Note: Award M1 for use of $-a$ is equal to the sum of the roots, do not award if minus is missing.

Note: If expanding the factored form of the equation, award ***M1*** for equating a to the coefficient of z^3.

product of roots: $(-1)^4 d \quad = 2\left(\log_2 6\right)\left(i\sqrt{3}\right)\left(-i\sqrt{3}\right)$ ***M1***

$= 6\log_2 6$ ***A1***

Note: Award ***M1A0*** for $d=-6\log_2 6$

$6a+d+12=-18-6\log_2 3+6\log_2 6+12$

EITHER

$=-6+6\log_2 2=0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

OR

$=-6-6\log_2 3+6\log_2 3+6\log_2 2=0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

[7 marks]

3. The cubic equation $x^3-kx^2+3k=0$ where $k>0$ has roots α, β and $\alpha+\beta$. *[5 marks]*

Given that $\alpha\beta=-\frac{k^2}{4}$, find the value of k.

Markscheme

$\alpha+\beta+\alpha+\beta=k$ ***(A1)***

$\alpha+\beta=\frac{k}{2}$

$\alpha\beta(\alpha+\beta)=-3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right)=-3k\left(-\frac{k^3}{8}=-3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8}+3k=0$ (or equivalent) for k ***(M1)***

$k=2\sqrt{6}\left(=\sqrt{24}\right)(k>0)$ ***A1***

Note: Award ***A0*** for $k=\pm2\sqrt{6}\left(\pm\sqrt{24}\right)$.

[5 marks]

4. The quadratic equation $x^2-2kx+(k-1)=0$ has roots α and β such that $\alpha^2+\beta^2=4$. Without solving the equation, find the possible values of the real number k. *[6 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\alpha+\beta=2k$ ***A1***

$\alpha\beta=k-1$ ***A1***

$(\alpha+\beta)^2=4k^2\Rightarrow\alpha^2+\beta^2+2\overbrace{k-1}^{\alpha\beta}=4k^2$ ***(M1)***

$\alpha^2+\beta^2=4k^2-2k+2$

$\alpha^2+\beta^2=4\Rightarrow4k^2-2k-2=0$ ***A1***

attempt to solve quadratic ***(M1)***

$k=1,\ -\frac{1}{2}$ ***A1***

[6 marks]

5. The polynomial $x^4 + px^3 + qx^2 + rx + 6$ is exactly divisible by each of $(x-1)$, $(x-2)$ and $(x-3)$. *[5 marks]*

Find the values of p, q and r.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

substitute each of x = 1,2 and 3 into the quartic and equate to zero ***(M1)***

$p + q + r = -7$

$4p + 2q + r = -11$ or equivalent ***(A2)***

$9p + 3q + r = -29$

Note: Award ***A2*** for all three equations correct, ***A1*** for two correct.

attempting to solve the system of equations ***(M1)***

$p = -7$, $q = 17$, $r = -17$ ***A1***

Note: Only award ***M1*** when some numerical values are found when solving algebraically or using GDC.

METHOD 2

attempt to find fourth factor ***(M1)***

$(x-1)$ ***A1***

attempt to expand $(x-1)^2(x-2)(x-3)$ ***M1***

$x^4 - 7x^3 + 17x^2 - 17x + 6$ ($p = -7$, $q = 17$, $r = -17$) ***A2***

Note: Award ***A2*** for all three values correct, ***A1*** for two correct.

Note: Accept long / synthetic division.

[5 marks]

Consider the polynomial $q(x) = 3x^3 - 11x^2 + kx + 8$.

6a. Given that $q(x)$ has a factor $(x-4)$, find the value of k. *[3 marks]*

Markscheme

$q(4) = 0$ ***(M1)***

$192 - 176 + 4k + 8 = 0(24 + 4k = 0)$ ***A1***

$k = -6$ ***A1***

[3 marks]

6b. Hence or otherwise, factorize $q(x)$ as a product of linear factors. *[3 marks]*

Markscheme

$3x^3 - 11x^2 - 6x + 8 = (x - 4)(3x^2 + px - 2)$

equate coefficients of x^2: ***(M1)***

$-12 + p = -11$

$p = 1$

$(x - 4)(3x^2 + x - 2)$ ***(A1)***

$(x - 4)(3x - 2)(x + 1)$ ***A1***

Note: Allow part (b) marks if any of this work is seen in part (a).

Note: Allow equivalent methods (*eg*, synthetic division) for the ***M*** marks in each part.

[3 marks]

It is given that $f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$ where a and b are positive integers.

7a. Given that $x^2 - 1$ is a factor of $f(x)$ find the value of a and the value of b. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$g(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$

$g(1) = 0 \Rightarrow a + b = 8$ ***M1A1***

$g(-1) = 0 \Rightarrow -a + b = -6$ ***A1***

$\Rightarrow a = 7,\ b = 1$ ***A1***

[4 marks]

7b. Factorize $f(x)$ into a product of linear factors. *[3 marks]*

Markscheme

$3x^4 + 7x^3 + x^2 - 7x - 4 = (x^2 - 1)(px^2 + qx + r)$

attempt to equate coefficients ***(M1)***

$p = 3,\ q = 7,\ r = 4$ ***(A1)***

$3x^4 + 7x^3 + x^2 - 7x - 4 = (x^2 - 1)(3x^2 + 7x + 4)$

$= (x - 1)(x + 1)^2(3x + 4)$ ***A1***

Note: Accept any equivalent valid method.

[3 marks]

7c. Using your graph state the range of values of c for which $f(x) = c$ has exactly two distinct real roots. *[3 marks]*

Markscheme

$c > 0$ ***A1***

$-6.20 < c < -0.0366$ ***A1A1***

Note: Award ***A1*** for correct end points and ***A1*** for correct inequalities.

Note: If the candidate has misdrawn the graph and omitted the first minimum point, the maximum mark that may be awarded is ***A1FTA0A0*** for $c > -6.20$ seen.

[3 marks]

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where m, n, p, $q \in \mathbb{R}$.

The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.

The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

8a. Show that $abc = 8$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognition of the other root $= -d\mathrm{i}$ ***(A1)***

$\log_2 a + \log_2 b + \log_2 c + d\mathrm{i} - d\mathrm{i} = 3$ ***M1A1***

Note: Award ***M1*** for sum of the roots, ***A1*** for 3. Award ***A0M1A0*** for just $\log_2 a + \log_2 b + \log_2 c = 3$.

$\log_2 abc = 3$ ***(M1)***

$\Rightarrow abc = 2^3$ ***A1***

$abc = 8$ ***AG***

[5 marks]

The values a, b, and c are consecutive terms in a geometric sequence.

8b. Show that one of the real roots is equal to 1. *[3 marks]*

Markscheme

METHOD 1

let the geometric series be u_1, u_1r, u_1r^2

$(u_1r)^3 = 8$ ***M1***

$u_1r = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$

$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***

$b = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

[3 marks]

8c. Given that $q = 8d^2$, find the other two real roots. *[9 marks]*

Markscheme

METHOD 1

product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***

$r_1 \times r_2 = -8$ ***A1***

sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***

$r_1 + r_2 = 2$ ***A1***

solving simultaneously ***(M1)***

$r_1 = -2$, $r_2 = 4$ ***A1A1***

METHOD 2

product of the roots $\log_2 a \times \log_2 b \times \log_2 c \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***M1A1***

$\log_2 a \times \log_2 b \times \log_2 c = -8$ ***A1***

EITHER

a, b, c can be written as $\frac{2}{r}$, 2, $2r$ ***M1***

$\left(\log_2 \frac{2}{r}\right)\left(\log_2 2\right)\left(\log_2 2r\right) = -8$

attempt to solve ***M1***

$\left(1 - \log_2 r\right)\left(1 + \log_2 r\right) = -8$

$\log_2 r = \pm 3$

$r = \frac{1}{8}$, 8 ***A1A1***

OR

a, b, c can be written as a, 2, $\frac{4}{a}$ ***M1***

$\left(\log_2 a\right)\left(\log_2 2\right)\left(\log_2 \frac{4}{a}\right) = -8$

attempt to solve ***M1***

$a = \frac{1}{4}$, 16 ***A1A1***

THEN

a and c are $\frac{1}{4}$, 16 ***(A1)***

roots are −2, 4 ***A1***

[9 marks]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p,\ q,\ r\ \in\ \mathbb{R}$, has roots $\alpha,\ \beta$ and γ.

9a. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that: *[3 marks]*

$p = -(\alpha+\beta+\gamma)$

$q = \alpha\beta+\beta\gamma+\gamma\alpha$

$r = -\alpha\beta\gamma$.

Markscheme

attempt to expand $(x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$= \left(x^2-(\alpha+\beta)x+\alpha\beta\right)(x-\gamma)$ **OR** $=(x-\alpha)\left(x^2-(\beta+\gamma)x+\beta\gamma\right)$ ***A1***

$\left(x^3+px^2+qx+r\right) = x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha+\beta+\gamma)$ ***AG***

$q = (\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$r = -\alpha\beta\gamma$ ***AG***

Note: For candidates who do not include the ***AG*** lines award full marks.

[3 marks]

9b. Show that $p^2 - 2q = \alpha^2+\beta^2+\gamma^2$. *[3 marks]*

Markscheme

$p^2 - 2q = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ ***(A1)***

attempt to expand $(\alpha + \beta + \gamma)^2$ ***(M1)***

$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ or equivalent ***A1***

$= \alpha^2 + \beta^2 + \gamma^2$ ***AG***

Note: Accept equivalent working from RHS to LHS.

[3 marks]

9c. Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. *[3 marks]*

Markscheme

EITHER

attempt to expand $(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***(M1)***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=2\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=2\left(p^2-2q\right)-2q$ or equivalent ***A1***

$=2p^2-6q$ ***AG***

OR

attempt to write $2p^2-6q$ in terms of $\alpha,\ \beta,\ \gamma$ ***(M1)***

$=2\left(p^2-2q\right)-2q$

$=2\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***A1***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***AG***

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

9d. Given that $p^2<3q$, deduce that $\alpha,\ \beta$ and γ cannot all be real. *[2 marks]*

Markscheme

$p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$\Rightarrow (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 < 0$ ***A1***

if all roots were real $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$ ***R1***

Note: Condone strict inequality in the ***R1*** line.
Note: Do not award ***A0R1***.

$\Rightarrow$roots cannot all be real ***AG***

[2 marks]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

9e. Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. *[2 marks]*

Markscheme

$p^2 = (-7)^2 = 49$ and $3q = 51$ ***A1***

so $p^2 < 3q \Rightarrow$ the equation has at least one complex root ***R1***

Note: Allow equivalent comparisons; e.g. checking $p^2 < 6q$

[2 marks]

Noah believes that if $p^2 \geq 3q$ then $\alpha,\ \beta$ and γ are all real.

9f. By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. *[2 marks]*

Markscheme

use of GDC (*eg* graphs or tables) ***(M1)***

$q = 12$ ***A1***

[2 marks]

9g. Explain why the equation will have at least one real root for all values of q. *[1 mark]*

Markscheme

complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).

OR

a cubic curve always crosses the x-axis at at least one point. ***R1***

[1 mark]

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p,\ q,\ r,\ s \in \mathbb{R}$, has roots $\alpha,\ \beta,\ \gamma$ and δ.

In a similar way to the cubic equation, it can be shown that:

$p = -(\alpha + \beta + \gamma + \delta)$

$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$

$s = \alpha\beta\gamma\delta$.

9h. Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. *[3 marks]*

Markscheme

attempt to expand $(\alpha+\beta+\gamma+\delta)^2$ ***(M1)***

$(\alpha+\beta+\gamma+\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$ ***(A1)***

$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$\left(\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2=\right)p^2-2q$ ***A1***

[3 marks]

9i. Hence state a condition in terms of p and q that would imply $x^4+px^3+qx^2+rx+s=0$ has at least one complex root. *[1 mark]*

Markscheme

$p^2<2q$ **OR** $p^2-2q<0$ ***A1***

Note: Allow ***FT*** on their result from part (f)(i).

[1 mark]

9j. Use your result from part (f)(ii) to show that the equation $x^4-2x^3+3x^2-4x+5=0$ has at least one complex root. *[1 mark]*

Markscheme

$4<6$ OR $2^2-2\times 3<0$ ***R1***

hence there is at least one complex root. ***AG***

Note: Allow ***FT*** from part (f)(ii) for the ***R*** mark provided numerical reasoning is seen.

[1 mark]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

9k. State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. *[1 mark]*

Markscheme

$(p^2 > 2q)$ $(81 > 2 \times 24)$ (so) nothing can be deduced ***R1***

Note: Do not allow ***FT*** for the ***R*** mark.

[1 mark]

9l. Write down the integer root of this equation. *[1 mark]*

Markscheme

-1 ***A1***

[1 mark]

9m. By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. *[4 marks]*

Markscheme

attempt to express as a product of a linear and cubic factor ***M1***

$(x+1)\left(x^3-10x^2+34x-12\right)$ ***A1A1***

Note: Award ***A1*** for each factor. Award at most ***A1A0*** if not written as a product.

since for the cubic, $p^2 < 3q\ (100 < 102)$ ***R1***

there is at least one complex root ***AG***

[4 marks]