polynomials 3IB HL *[88 marks]*

1. Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants. *[5 marks]*

The remainder when $f(x)$ is divided by $(x + 1)$ is 7, and the remainder when $f(x)$ is divided by $(x - 2)$ is 1. Find the value of p and the value of q.

2. Consider the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where a, b, c, $d \in \mathbb{R}$ *[7 marks]* and $z \in \mathbb{C}$.

Two of the roots of the equation are $\log_2 6$ and $i\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$.

Show that $6a + d + 12 = 0$.

3. The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$. *[5 marks]*

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

4. The quadratic equation $x^2 - 2kx + (k-1) = 0$ has roots α and β such that $\alpha^2 + \beta^2 = 4$. Without solving the equation, find the possible values of the real number k. *[6 marks]*

5. The polynomial $x^4 + px^3 + qx^2 + rx + 6$ is exactly divisible by each of $(x - 1)$, $(x - 2)$ and $(x - 3)$. *[5 marks]*

Find the values of p, q and r.

Consider the polynomial $q(x) = 3x^3 - 11x^2 + kx + 8$.

6a. Given that $q(x)$ has a factor $(x - 4)$, find the value of k. *[3 marks]*

6b. Hence or otherwise, factorize $q(x)$ as a product of linear factors. *[3 marks]*

It is given that $f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$ where a and b are positive integers.

7a. Given that $x^2 - 1$ is a factor of $f(x)$ find the value of a and the value of b. *[4 marks]*

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7b. Factorize $f(x)$ into a product of linear factors. *[3 marks]*

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7c. Using your graph state the range of values of c for which $f(x) = c$ has exactly two distinct real roots. *[3 marks]*

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Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.

The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.

The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

8a. Show that $abc = 8$. *[5 marks]*

The values a, b, and c are consecutive terms in a geometric sequence.

8b. Show that one of the real roots is equal to 1. *[3 marks]*

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8c. Given that $q = 8d^2$, find the other two real roots. *[9 marks]*

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p,\ q,\ r\ \in\ \mathbb{R}$, has roots $\alpha,\ \beta$ and γ.

9a. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that: *[3 marks]*

$p = -(\alpha + \beta + \gamma)$

$q = \alpha\beta + \beta\gamma + \gamma\alpha$

$r = -\alpha\beta\gamma$.

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9b. Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. *[3 marks]*

9c. Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. *[3 marks]*

9d. Given that $p^2 < 3q$, deduce that $\alpha,\ \beta$ and γ cannot all be real. *[2 marks]*

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Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

9e. Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. *[2 marks]*

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Noah believes that if $p^2 \geq 3q$ then $\alpha,\ \beta$ and γ are all real.

9f. By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. *[2 marks]*

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9g. Explain why the equation will have at least one real root for all values of q. *[1 mark]*

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Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p,\ q,\ r,\ s \in \mathbb{R}$, has roots $\alpha,\ \beta,\ \gamma$ and δ.

In a similar way to the cubic equation, it can be shown that:

$p = -(\alpha + \beta + \gamma + \delta)$

$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$

$s = \alpha\beta\gamma\delta$.

9h. Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. *[3 marks]*

9i. Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. *[1 mark]*

9j. Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. *[1 mark]*

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The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

9k. State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. *[1 mark]*

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9l. Write down the integer root of this equation. *[1 mark]*

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9m. By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. *[4 marks]*