Name:

1. (5 points) Consider the polynomial $P(x) = x^3 - 2x^2 - x + 2$.

(a) Show that $x = 1$ is a root of $P(x)$.

$$P(1) = 1 - 2 - 1 + 2 = 0$$

$P(1) = 0$, so 1 is a root of $P(x)$.

(b) Hence, or otherwise, factorize $P(x)$ into product of linear factors.

We know one root, so we can use synthetic division to get that:

$$P(x) = (x - 1)(x^2 - x - 2)$$

and then factorize to get:

$$P(x) = (x - 1)(x + 1)(x - 2)$$

Alternatively we could've just factorized $P(x)$ straight away:

$P(x) = x^3-2x^2-x+2 = x^2(x-2)-(x-2) = (x^2-1)(x-2) = (x-1)(x+1)(x-2)$

Consider another polynomial $Q(x)$. The remainders when $Q(x)$ is divided by $(x - 1)$, $(x + 1)$ and $(x - 2)$ are 2, -8 and 10 respectively.

(c) Find the remainder when $Q(x)$ is divided by $P(x)$.

$P(x)$ is a degree 3 polynomial, so the remainder will be of degree at most 2, so $R(x) = ax^2 + bx + c$.

We know that $Q(1) = 2$, $Q(-1) = -8$ and $Q(2) = 10$, note that we also have $P(1) = P(-1) = P(2) = 0$.

We write

$$Q(x) = S(x)P(x) + ax^2 + bx + c$$

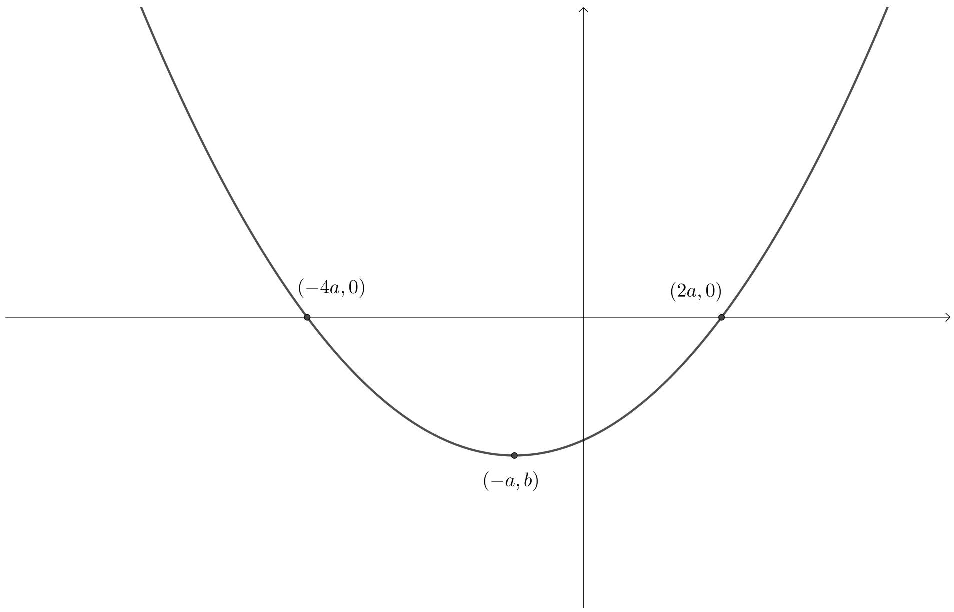
and set $x = 1$, $x = -1$ and $x = 2$, to get three equations with three unknowns.

$$\begin{cases} Q(1) = 2 \\ Q(-1) = -8 \\ Q(2) = 10 \end{cases}$$

$$\begin{cases} a + b + c = 2 \\ a - b + c = -8 \\ 4a + 2b + c = 10 \end{cases}$$

Subtracting the second equation from the first one gives $b = 5$, then first from the third gives $a = 1$ and finally we get $c = -4$. So the remainder is $R(x) = x^2 + 5x - 4$.

2. (5 points) The diagram below shows the graph of a function $f(x)$.

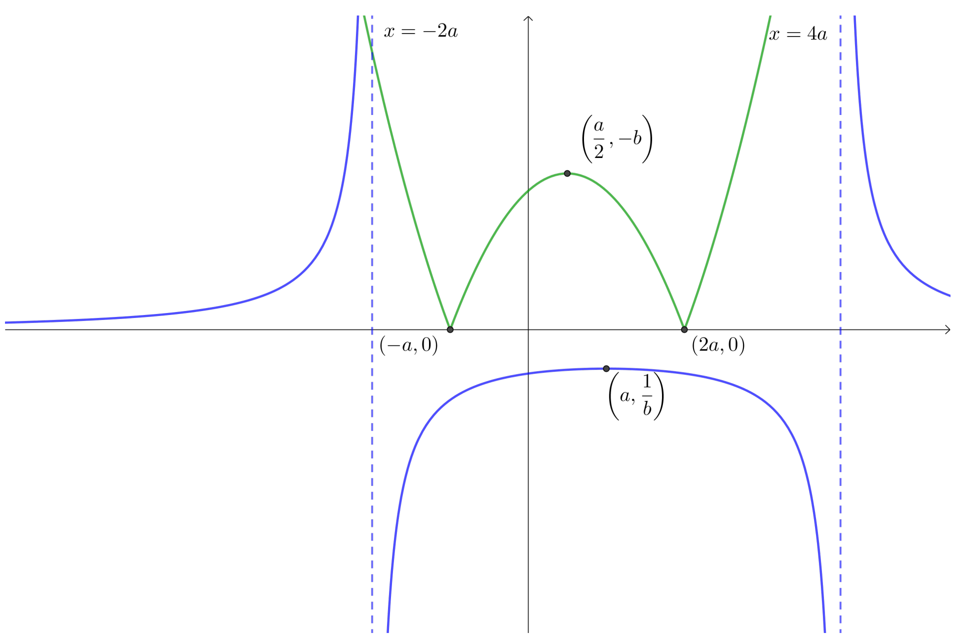


with $a > 0$ and $b < -1$.

Use the diagram below to sketch the graphs of

(i) $g(x) = |f(-2x)|$ (ii) $h(x) = \dfrac{1}{f(x-2a)}$

Clearly indicate all the x-axis intercepts, maxima and minima and asymptotes.



3. (4 points) Consider the function

$$f(x) = \sqrt{\arcsin x + \frac{\pi}{6}}$$

(a) Find the domain and range of $f(x)$.

Because of $\arcsin x$ we have $-1 \leqslant x \leqslant 1$. Now we also need:

$$\arcsin x + \frac{\pi}{6} \geqslant 0$$

which gives

$$\arcsin x \geqslant -\frac{\pi}{6}$$

Apply sin() (restricted to I and IV quadrants, so increasing) and get:

$$x \geqslant -\frac{1}{2}$$

So the domain is $-\frac{1}{2} \leqslant x \leqslant 1$. Of course we could have also drawn graphs of $y = \arcsin x$ and $y = -\frac{\pi}{6}$ and get the same result.

Now we want to range. The minimum of $\arcsin x$ is $-\frac{\pi}{6}$, the maximum is $\frac{\pi}{2}$, this gives the range of $f(x)$ as $0 \leqslant y \leqslant \sqrt{\frac{2\pi}{3}}$

(b) Find the $f^{-1}(x)$, the inverse of $f(x)$.

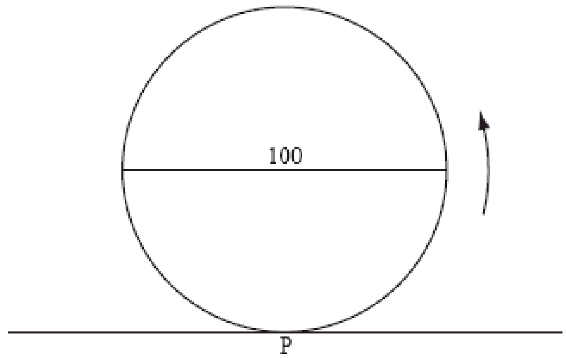
$$y = \sqrt{\arcsin x + \frac{\pi}{6}} \quad \Rightarrow \quad \sin\left(y^2 - \frac{\pi}{6}\right) = x$$

So we have $f^{-1}(x) = \sin\left(x^2 - \frac{\pi}{6}\right)$

(c) Write down the domain and range of $f^{-1}(x)$.

The domain of the inverse function is the range of the original function and *vice versa*, so the domain is $0 \leqslant x \leqslant \sqrt{\frac{2\pi}{3}}$ and the range of the inverse function is $-\frac{1}{2} \leqslant y \leqslant 1$.

4. (7 points) The following diagram represents a large Ferris wheel, with a diameter of 100 metres.



Let P be a point on the wheel. The wheel starts with P at the lowest point, at ground level. The wheel rotates at a constant rate, in an counter-clockwise direction. One revolution takes 20 minutes.

(a) Write down the height of P above ground level after

(i) 10 minutes; $100m$

(ii) 15 minutes; $50m$

Let $h(t)$ metres be the height of P above ground level after t minutes.

(b) Given that h can be expressed in the form $h(t) = a \cos bt + c$, find a, b and c.

The principle axis is $h = 50$, so $c = 50$. The amplitude is 50, but we start at the bottom and go up, so the function is reflected, which means that $a = -50$. Finally the period is 20, which gives $b = \frac{\pi}{10}$.

(c) Sketch the graph of $h(t)$ for $0 \leqslant t \leqslant 40$.