# Transformations of trigonometric functions

In this presentation we discuss transformations of trigonometric functions. In particular we will look at the most general forms of four trigonometric functions:

$$f(x) = a\sin(b(x-c)) + d$$

$$f(x) = a\cos(b(x-c)) + d$$

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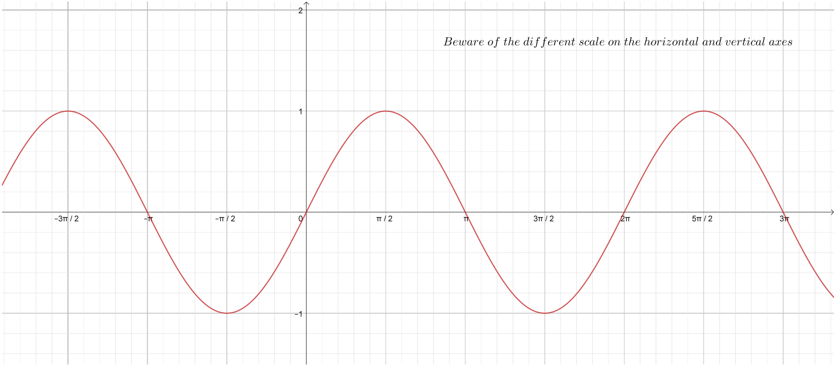
Before you start you need to be familiar with graphs of $\sin x$, $\cos x$, $\tan x$ and transformations of function - in particular: translations, dilations and reflections.

We will start with a brief review of the graphs of the three trig functions.

# Sine function

Graph:



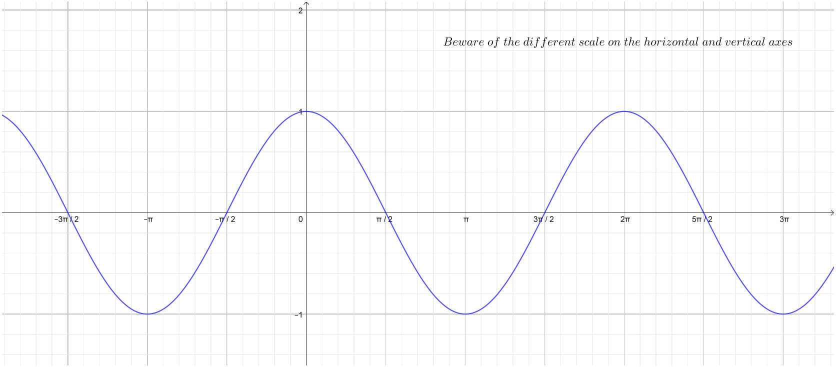

Properties:

Domain: $x \in \mathbb{R}$. Range $y \in [-1, 1]$. Period: $2\pi$.

Zeroes: $x = k\pi$, where $k \in \mathbb{Z}$.

# Cosine function

Graph:



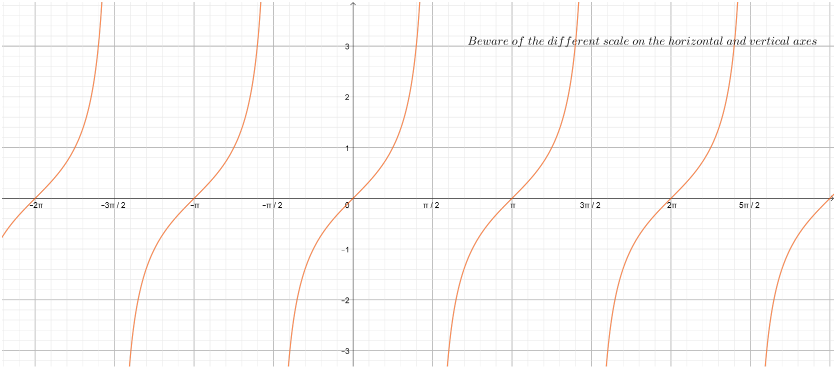

Properties:

Domain: $x \in \mathbb{R}$. Range $y \in [-1, 1]$. Period: $2\pi$.

Zeroes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

# Tangent function

Graph:



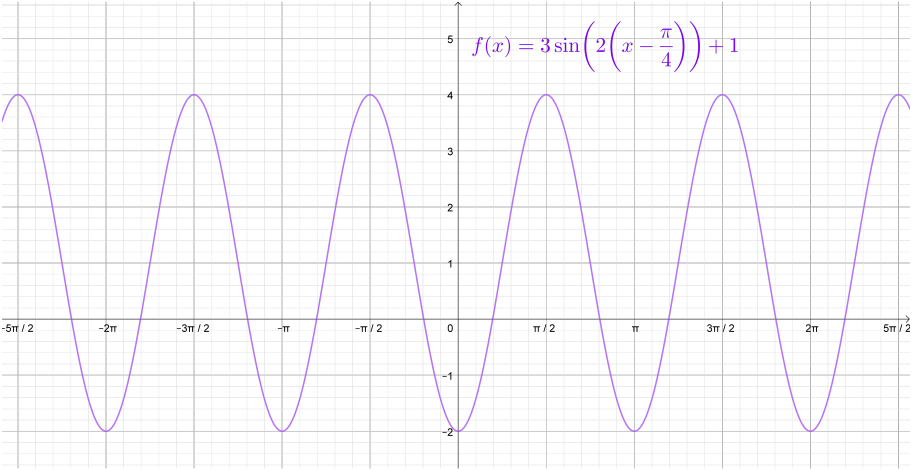

Properties:

Domain: $x \in \mathbb{R} - \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. Range $y \in \mathbb{R}$. Period: $\pi$.

Zeroes: $x = k\pi$, where $k \in \mathbb{Z}$. Asymptotes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

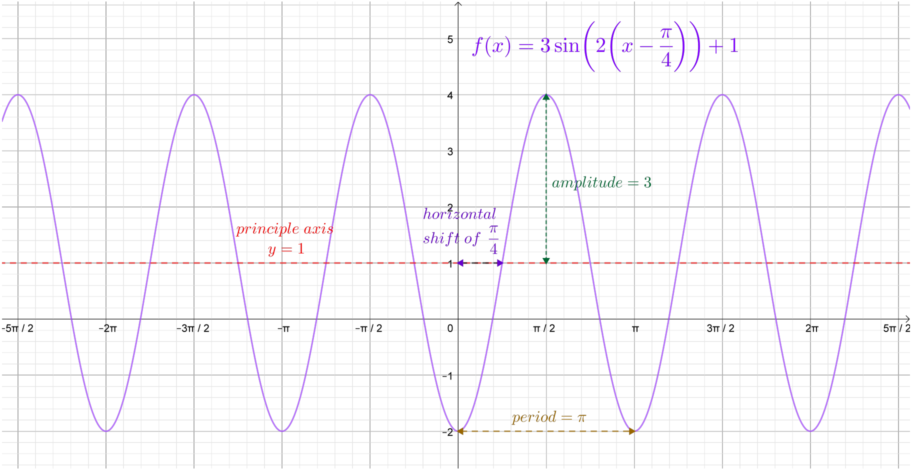
# Intro to transformations of sine and cosine

Consider the following function and its graph:



# Intro to transformations of sine and cosine

Now I will add some useful features of the graph on the diagram:



# Intro to transformations of sine and cosine

The above feature help us deduce the equation from the graph.
If we have a function of the form $f(x) = a\sin(b(x-c)) + d$

- a corresponds to the amplitude, we need to be careful however. We have $|a| = amplitude$ i.e. we need to check if the graph has been reflected.

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- b corresponds to the period. Note that if $b = 2$, then the graph has been stretched (squeezed) horizontally by a factor of $\frac{1}{2}$. It is useful to use the following formula $\frac{old\ period}{b} = new\ period$, where the old period is the period of the original function, so in case of sine and cosine we have $\frac{2\pi}{b} = new\ period$.

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- d corresponds to the principle axis (or "middle line").

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- a corresponds to the amplitude, we need to be careful however. We have $|a| = \textit{amplitude}$ i.e. we need to check if the graph has been reflected.
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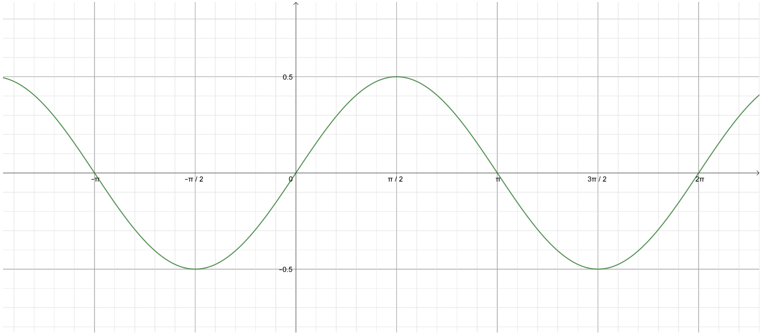
- c corresponds to the horizontal shift.
- d corresponds to the principle axis (or "middle line"). An easy way to find the middle line is to find the average of *max* and *min*.

We will now practice deducing the equation from the graph.

**Example 1**

The graph of the function $f(x) = a \sin x$ is shown below. Find the value of $a$.



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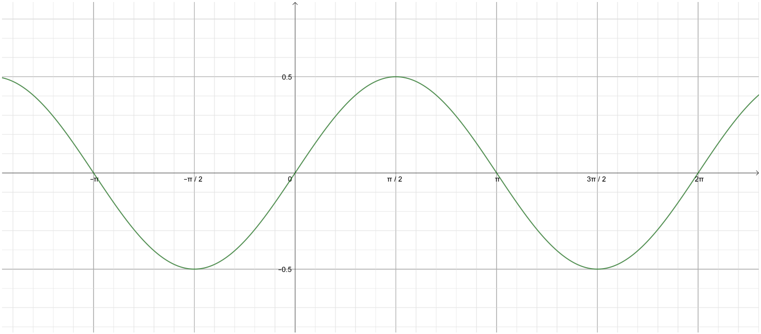
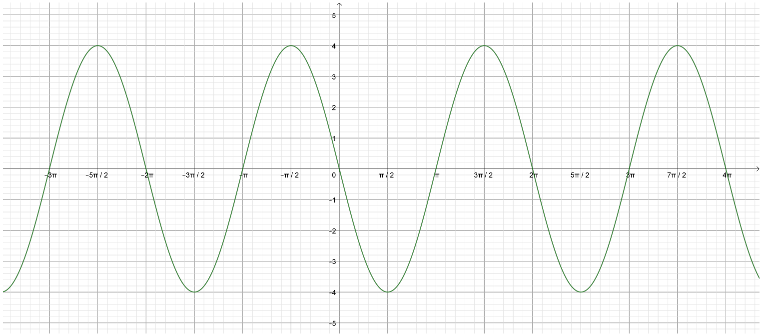



The amplitude of the graph is $\frac{1}{2}$, the graph has not been reflected in the $x$-axis, so $a = \frac{1}{2}$.

## Example 2

The graph of the function $f(x) = a\sin x$ is shown below. Find the value of $a$.



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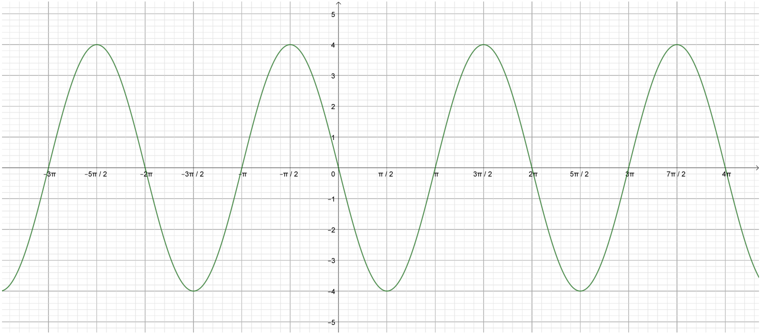
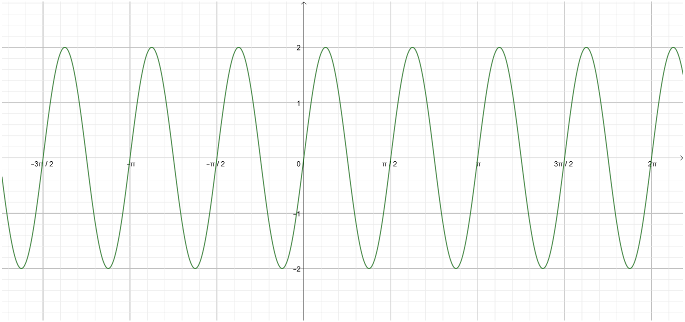



The amplitude of the graph is 4, this time the graph has been reflected in the $x$-axis, so $a = -4$. (To see this note that if we start at the origin and move right, the graph of sine goes up, here the graph goes down.)

## Example 3

The graph of the function $f(x) = a \sin bx$ is shown below. Find the values of $a$ and $b$.



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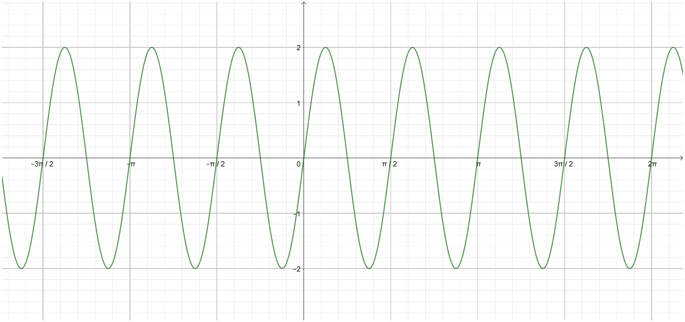


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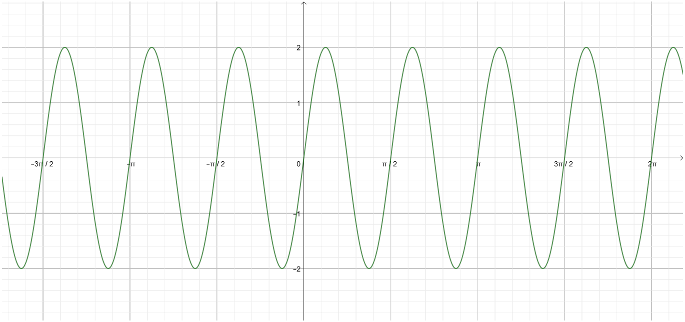


The amplitude of the graph is 2, the graph has not been reflected in the $x$-axis, so $a = 2$. The period of the new function is $\frac{\pi}{2}$. You can see this by looking at the zeroes of the function. This means that the graph of sine has been stretched by a factor of $\frac{1}{4}$, so $b = 4$.

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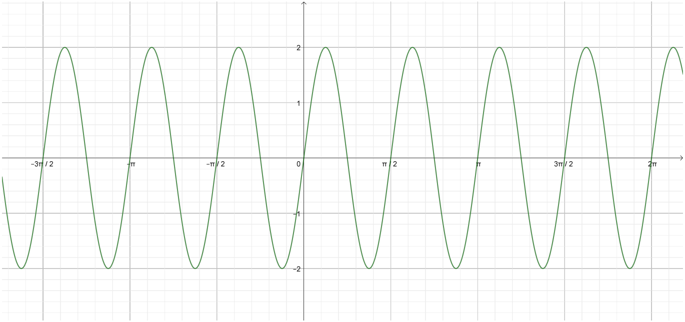


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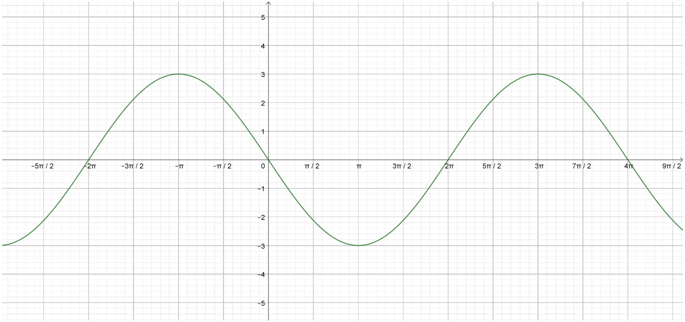

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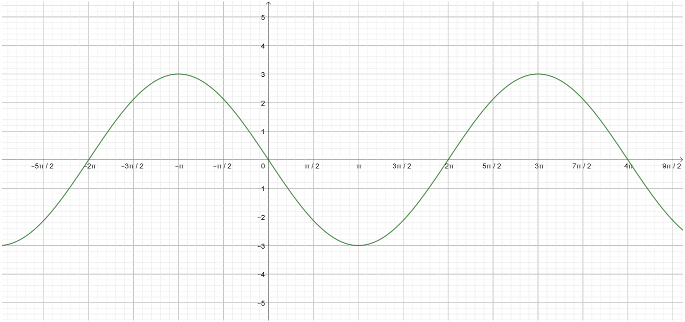


The amplitude of the graph is 3, the graph is reflected in the $x$-axis, so $a = -3$.

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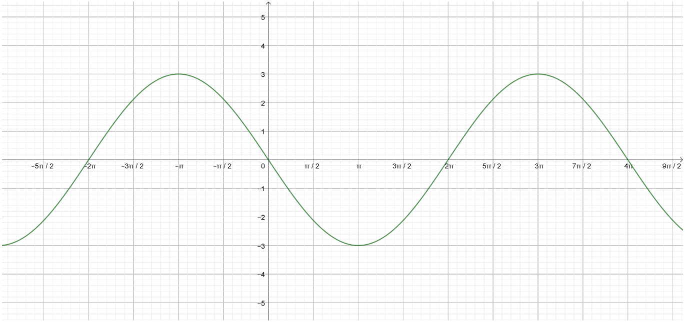


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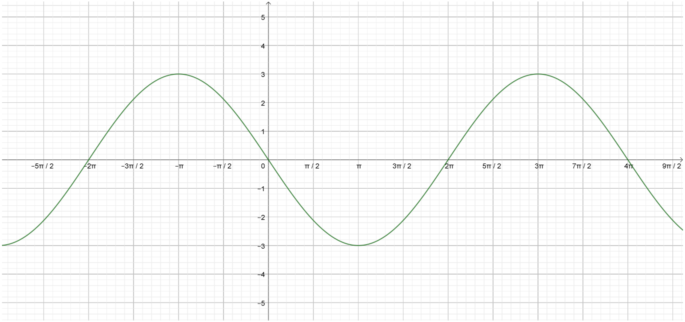


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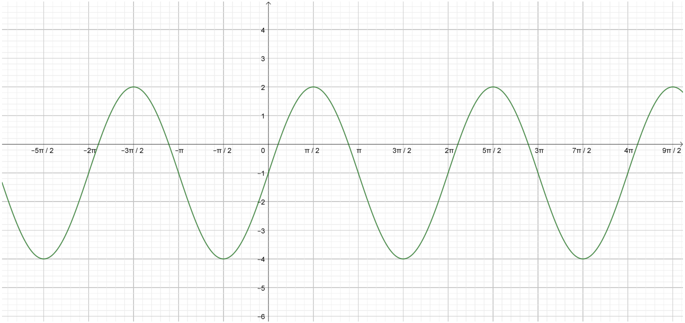
## Example 5

The graph of the function $f(x) = a \sin x + d$ is shown below. Find the values of $a$ and $d$.



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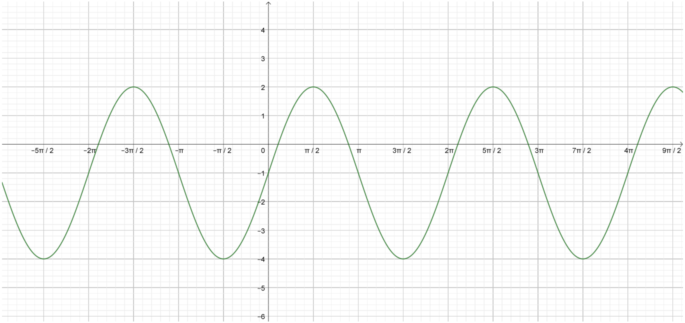
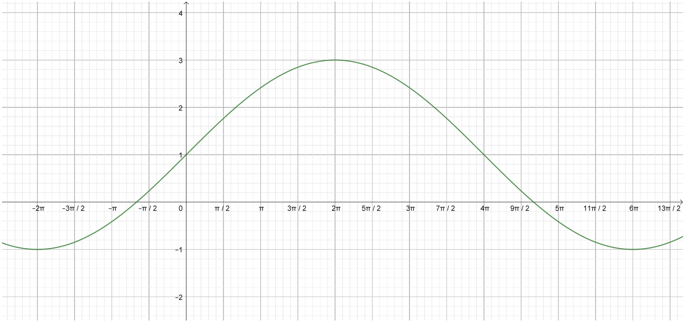



The principle axis is $y = -1$, so $d = -1$. The amplitude is 3 and the graph has not been reflected so $a = 3$.

## Example 6

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of $a$, $b$ and $d$.



## Example 6

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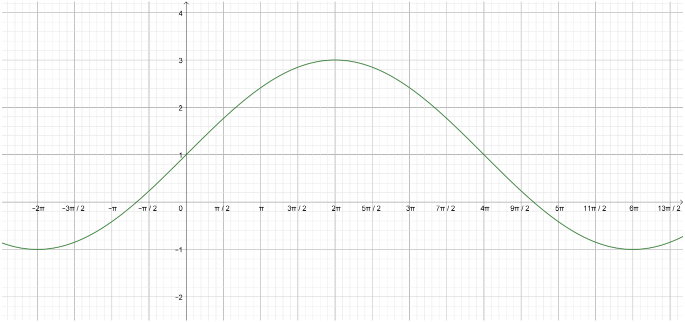
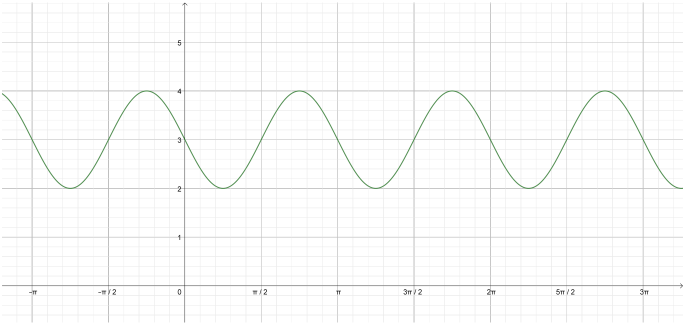



The principle axis is $y = 1$, so $d = 1$. The amplitude is 2 and the graph has not been reflected so $a = 2$. The period is $8\pi$ (we can see clearly that half of the period is $4\pi$). This means that the graph has been stretched by a factor of 4, so $b = \frac{1}{4}$.

## Example 7

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of $a$, $b$ and $d$.



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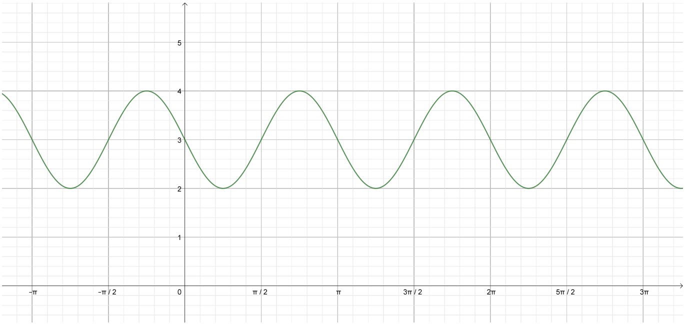
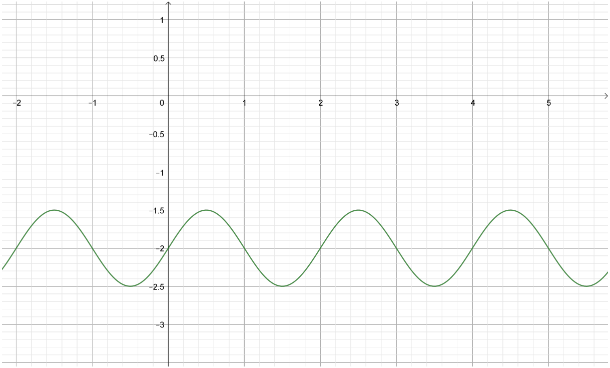



The principle axis is $y = 3$, so $d = 3$. The amplitude is 1 but the graph has been reflected so $a = -1$. The period is $\pi$ so the graph has been stretched by a factor of $\frac{1}{2}$, so $b = 2$.

## Example 8

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of $a$, $b$ and $d$.



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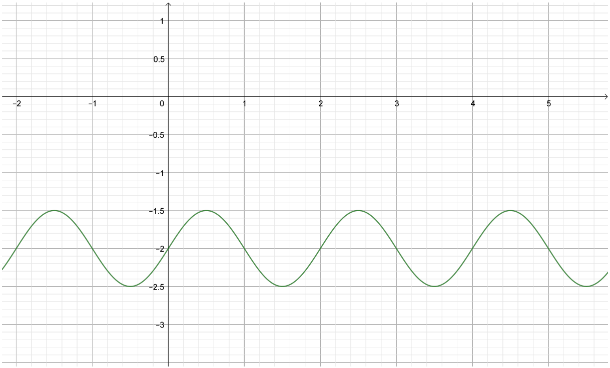
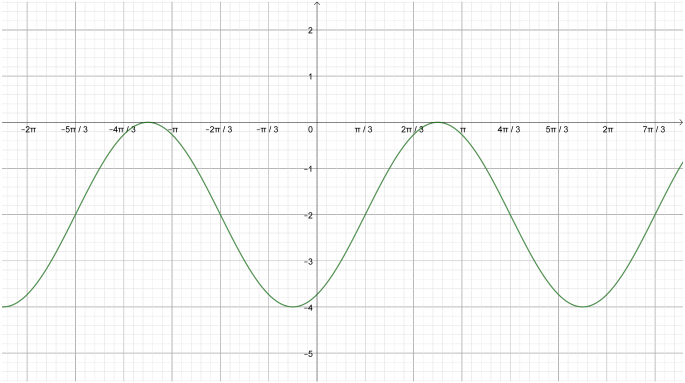



The principle axis is $y = -2$, so $d = -2$. The amplitude is $\frac{1}{2}$, the graph is not reflected so $a = \frac{1}{2}$. The period is 2, it maybe less obvious how the graph was stretched, so let's go straight to the formula $\frac{2\pi}{b} = 2$ and we get that $b = \pi$.

## Example 9

The graph of the function $f(x) = a\sin(x - c) + d$ is shown below. Find the values of $a$, $c$ and $d$.



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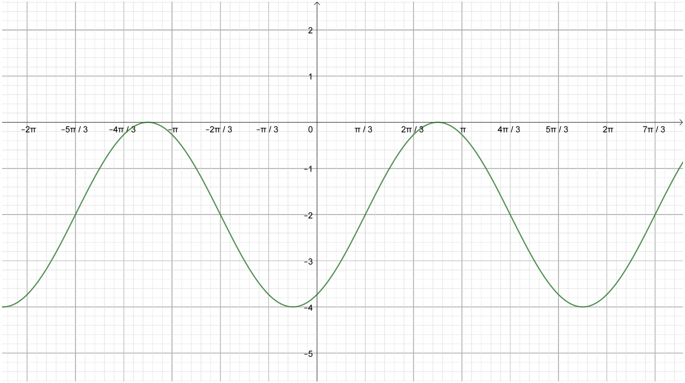
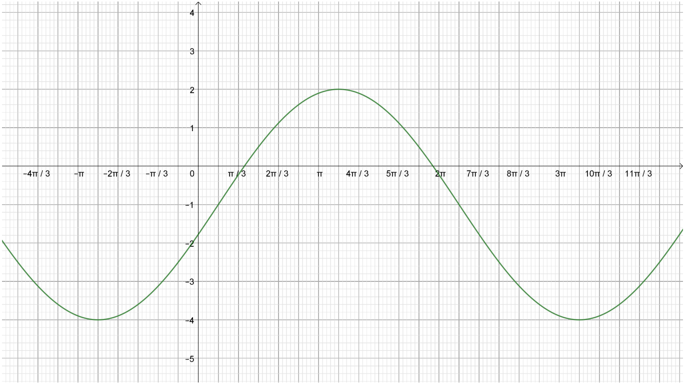



The principle axis is $y = -2$, so $d = -2$. The amplitude is 2, the graph is not reflected so $a = 2$. The graph has been shifted to the right by $\frac{\pi}{3}$, so $c = \frac{\pi}{3}$ (note the negative sign in front of $c$ in the equation).

## Example 10

The graph of the function $f(x) = a\sin(b(x-c)) + d$ is shown below. Find the values of $a$, $b$, $c$ and $d$.



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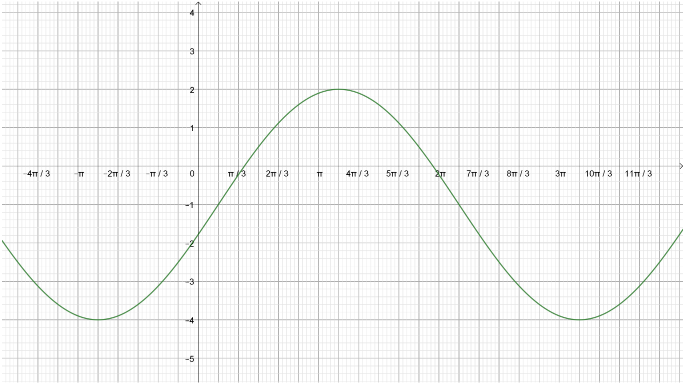


The principle axis $y = -1$, so $d = -1$. Amplitude is 3, not reflected so $a = 3$. Period is $4\pi$ so $b = \frac{1}{2}$. Horizontal shift is $\frac{\pi}{6}$ units to the right so $c = \frac{\pi}{6}$ (again because we're subtracting $c$ from $x$).

You may have noticed that the previous example had more possible answers.

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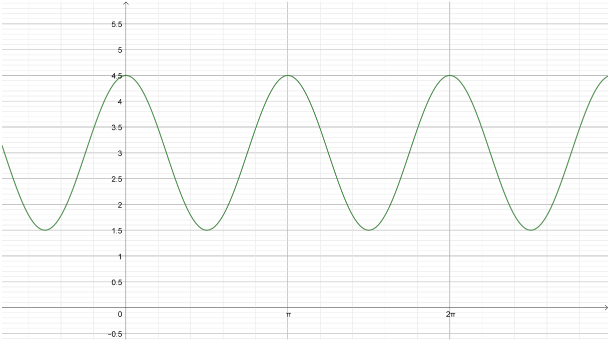
Now we move on to a few examples on cosine functions, but these are very similar in nature.

## Example 11

The graph of the function $f(x) = a\cos bx + d$ is shown below. Find the values of $a$, $b$ and $d$.



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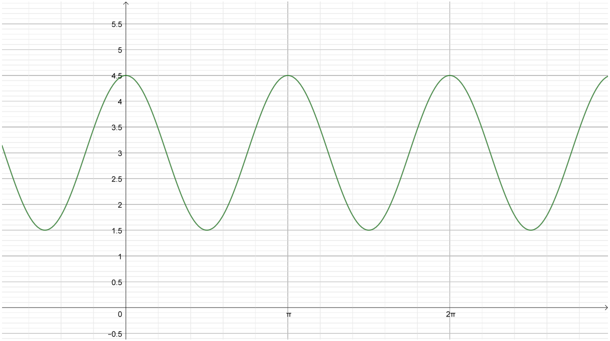
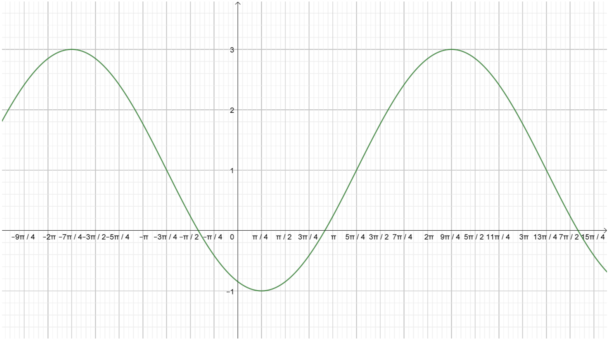



The principle axis is $y = 3$, so $d = 3$. The amplitude is $\frac{3}{2}$, the graph has not been reflected (if we go right from the $y$-axis, the cosine function starts at 1 and goes down, our function also goes down) so $a = \frac{3}{2}$. The period is $\pi$, so $b = 2$.

## Example 12

The graph of the function $f(x) = a\cos(b(x - c)) + d$ is shown below. Find the values of $a$, $b$, $c$ and $d$.



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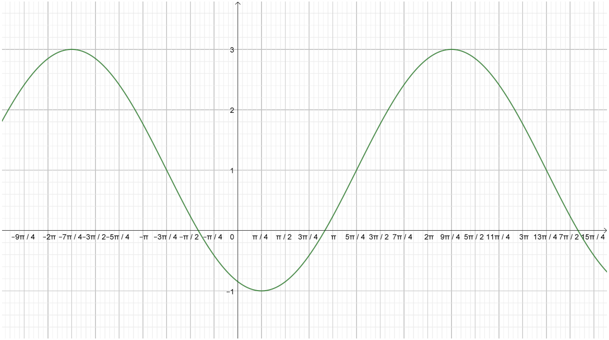


The principle axis is $y = 1$, so $d = 1$. The amplitude is 2, but the graph has been reflected so $a = -2$. The period is $4\pi$ (we can see that half the period is $2\pi$), so $b = \frac{1}{2}$. The graph has been shifted $\frac{\pi}{4}$ units to the right, so $c = \frac{\pi}{4}$. You can look at the bottom peak to see this.

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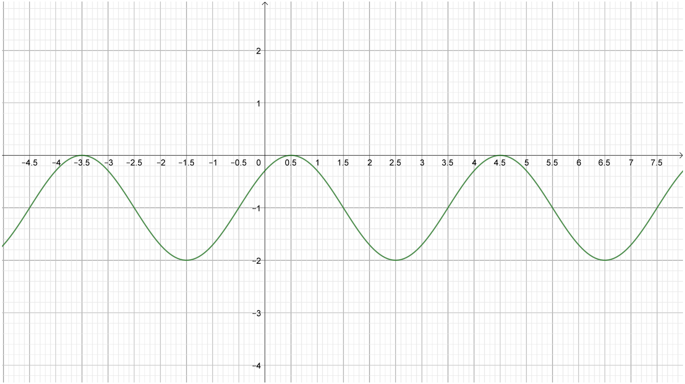
The graph of the function $f(x) = a\cos(b(x - c)) + d$ is shown below. Find the values of $a$, $b$, $c$ and $d$.



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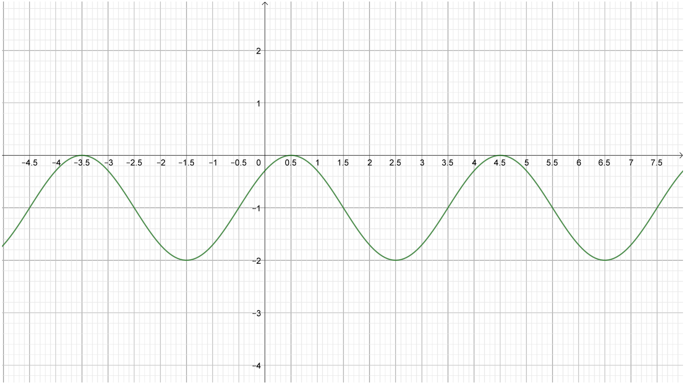


The principle axis is $y = -1$, so $d = -1$. The amplitude is 1, the graph has not been reflected so $a = 1$. The period is 4, we solve $\frac{2\pi}{b} = 4$ to get $b = \frac{\pi}{2}$. Finally the graph has been shifted 0.5 units to the right, so $c = \frac{1}{2}$.

Again notice that the last two examples had multiple solutions. It's a good practice to come up with other solutions, but make sure that you check your answers (by drawing appropriate graph on for instance desmos.com)

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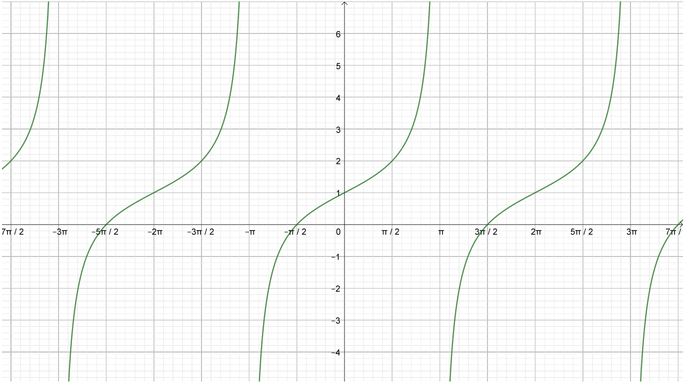
We now turn to tangent function.

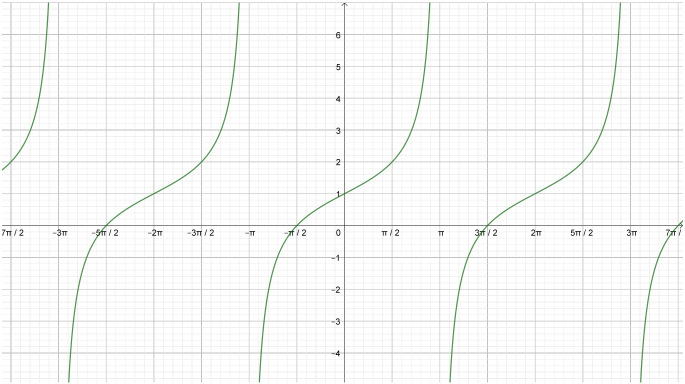
## Example 14

The graph of the function $f(x) = \tan(bx) + d$ is shown below. Find the values of $b$ and $d$.



**Example 14**

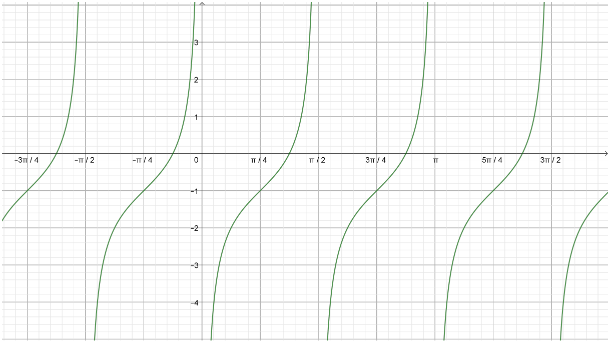
The graph of the function $f(x) = \tan(bx) + d$ is shown below. Find the values of $b$ and $d$.




This may be less obvious, but still we can identify the vertical translation one unit upwards, so $d = 1$. The period is $2\pi$, the period of $\tan x$ is $\pi$, so the it has been stretched by a factor of 2, which means that $b = \frac{1}{2}$.

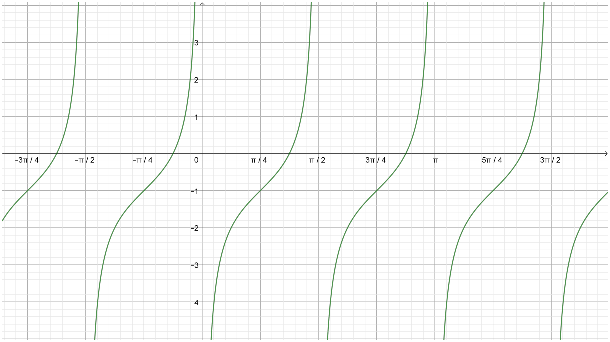
## Example 15

The graph of the function $f(x) = \tan(b(x - c)) + d$ is shown below. Find the values of $b$, $c$ and $d$.



## Example 15

The graph of the function $f(x) = \tan(b(x - c)) + d$ is shown below. Find the values of $b$, $c$ and $d$.




The period is $\frac{\pi}{2}$, so the graph of $\tan x$ has been stretched by a factor of $\frac{1}{2}$, which means that $b = 2$. Now we can see the translation by a vector $\begin{pmatrix} \frac{\pi}{4} \\ -1 \end{pmatrix}$, so $c = \frac{\pi}{4}$ and $d = -1$.

In case of any questions you can email me at T.J.Lechowski@gmail.com.