# vectors and planes [96 marks]

1. The plane  $\Pi$  has the Cartesian equation 2x + y + 2z = 3 [7 marks]

The line *L* has the vector equation r =

$$=egin{pmatrix} 3\-5\1\end{pmatrix}+\muegin{pmatrix}1\-2\p\end{pmatrix},\ \mu,p\in\mathbb{R}.$$
 The acute

angle between the line L and the plane  $\Pi$  is 30°. Find the possible values of p.

# Markscheme

recognition that the angle between the normal and the line is 60° (seen anywhere) **R1** 

attempt to use the formula for the scalar product **M1** 

$$\cos 60^{\circ} = \frac{\left| \begin{pmatrix} 2\\1\\2 \end{pmatrix} \bullet \begin{pmatrix} 1\\-2\\p \end{pmatrix} \right|}{\sqrt{9} \times \sqrt{1+4+p^2}} \quad \textbf{A1}$$

$$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}} \quad \textbf{A1}$$

$$3\sqrt{5+p^2} = 4 |p|$$
attempt to square both sides  $\quad \textbf{M1}$ 

$$9 (5+p^2) = 16p^2 \Rightarrow 7p^2 = 45$$

$$p = \pm 3\sqrt{\frac{5}{7}} \text{ (or equivalent)} \quad \textbf{A1A1}$$

$$[7 \text{ marks]}$$

The lines  $l_1$  and  $l_2$  have the following vector equations where  $\lambda, \ \mu \in \mathbb{R}$  and  $m \in \mathbb{R}$ .

$$l_1: r_1 = \begin{pmatrix} 3\\-2\\0 \end{pmatrix} + \lambda \begin{pmatrix} 2\\1\\m \end{pmatrix} l_2: r_2 = \begin{pmatrix} -1\\-4\\-2m \end{pmatrix} + \mu \begin{pmatrix} 2\\-5\\-m \end{pmatrix}$$

2a. Show that  $l_1$  and  $l_2$  are never perpendicular to each other.

#### [3 marks]

### Markscheme

\* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to calculate 
$$\begin{pmatrix} 2\\1\\m \end{pmatrix} \cdot \begin{pmatrix} 2\\-5\\-m \end{pmatrix}$$
 (M1)

 $=-1-m^2$  Al since  $m^2\geq 0,\ -1-m^2<0$  for  $m\in \mathbb{R}$  Rl so  $l_1$  and  $l_2$  are never perpendicular to each other AG [3 marks]

The plane  $\varPi$  has Cartesian equation x+4y-z=p where  $p\in\mathbb{R}.$ 

Given that  $l_1$  and  $\varPi$  have no points in common, find

2b. the value of m.

[2 marks]

(since  $l_1$  is parallel to  $\varPi$ ,  $l_1$  is perpendicular to the normal of  $\varPi$  and so)

$$\begin{pmatrix} 2\\1\\m \end{pmatrix} \cdot \begin{pmatrix} 1\\4\\-1 \end{pmatrix} = 0 \text{ R1}$$
$$2+4-m=0$$
$$m=6 \text{ A1}$$
[2 marks]

2c. the condition on the value of p.

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[2 marks]
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### **Markscheme** since there are no points in common, (3, -2, 0) does not lie in $\Pi$ **EITHER** substitutes (3, -2, 0) into $x + 4y - z (\neq p)$ (M1) **OR** $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} (\neq p)$ (M1) **THEN** $p \neq -5$ A1 [2 marks]

The points A(5, -2, 5), B(5, 4, -1), C(-1, -2, -1) and D(7, -4, -3) are the vertices of a right-pyramid.

<sup>3a.</sup> Find the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ .

[2 marks]

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$$\overrightarrow{AB} = \begin{pmatrix} 0\\6\\-6 \end{pmatrix} \begin{pmatrix} = 6 \begin{pmatrix} 0\\1\\-1 \end{pmatrix} \end{pmatrix} \textbf{A1}$$
$$\overrightarrow{AC} = \begin{pmatrix} -6\\0\\-6 \end{pmatrix} \begin{pmatrix} = 6 \begin{pmatrix} -1\\0\\-1 \end{pmatrix} \end{pmatrix} \textbf{A1}$$
[2 marks]

<sup>3b.</sup> Use a vector method to show that  ${
m B\widehat{A}C}=60\degree$ .

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[3 marks]
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# Markscheme attempts to use $\cos B\widehat{A}C = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right| \left|\overrightarrow{AC}\right|}$ (M1) $= \frac{\begin{pmatrix} 0\\6\\-6 \end{pmatrix} \cdot \begin{pmatrix} -6\\0\\-6 \end{pmatrix}}{\sqrt{72} \times \sqrt{72}} A1$

$$= \frac{1}{2}$$
 A1  
so  $B\widehat{A}C = 60^{\circ}$  AG  
[3 marks]

3c. Show that the Cartesian equation of the plane  $\Pi$  that contains the [3 marks] triangle ABC is -x + y + z = -2.

attempts to find a vector normal to  $\varPi$   $\mathbf{M1}$ 

for example, 
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -36\\ 36\\ 36 \end{pmatrix} \begin{pmatrix} -1\\ 1\\ 1 \end{pmatrix} \end{pmatrix}$$
 leading to **A1** a vector normal to  $\Pi$  is  $n = \begin{pmatrix} -1\\ 1\\ 1 \end{pmatrix}$ 

#### EITHER

substitutes  $(5,\ -2,\ -5)$  (or  $(5,\ 4,\ -1)$  or  $(-1,\ -2,\ -1)$ ) into -x+y+z=d and attempts to find the value of d

for example, d=-5-2+5(=-2) M1

#### OR

attempts to use  $r\cdot n = a\cdot n$  **M1** 

for example, 
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

#### THEN

leading to the Cartesian equation of  $\varPi$  as -x+y+z=-2 AG

[3 marks]

The line L passes through the point  ${\rm D}$  and is perpendicular to  $\varPi.$ 

3d. Find a vector equation of the line L.

[1 mark]

Markscheme  

$$r = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} (\lambda \in \mathbb{R}) \text{ As}$$
[1 mark]

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Markscheme

substitutes x = 7 - \lambda, y = -4 + \lambda, z = -3 + \lambda into -x + y + z = -2 (M1)

-(7 - \lambda) + (-4 + \lambda) + (-3 + \lambda) = -2(3\lambda = 12)

\lambda = 4 A1

shows a correct calculation for finding d_{\min}, for example, attempts to find

\left| 4 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right| M1

d_{\min} = 4\sqrt{3} (= 6.93) A1

[4 marks]
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3f. Find the volume of right-pyramid  $\operatorname{ABCD}$ .

[4 marks]

let the area of triangle  $\operatorname{ABC}$  be A

#### EITHER

attempts to find  $A=rac{1}{2}\left|\overrightarrow{\mathrm{AB}} imes\overrightarrow{\mathrm{AC}}
ight|$ , for example <code>M1</code>

$$A = \frac{1}{2} \begin{vmatrix} -36\\ 36\\ 36 \end{vmatrix}$$

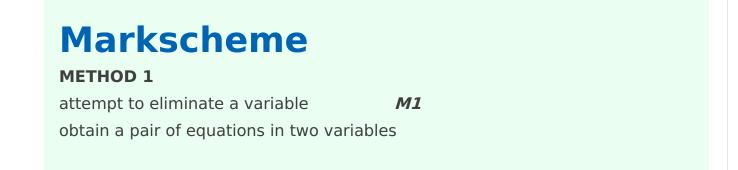
#### OR

attempts to find  $\frac{1}{2} \left| \overrightarrow{AB} \right| \left| \overrightarrow{AC} \right| \sin \theta$ , for example **M1**   $A = \frac{1}{2} \times 6\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$  (where  $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ ) **THEN**   $A = 18\sqrt{3} (= 31.2)$  **A1** uses  $V = \frac{1}{3}Ah$  where A is the area of triangle ABC and  $h = d_{\min}$  **M1**   $V = \frac{1}{3} \times 18\sqrt{3} \times 4\sqrt{3}$  = 72 **A1** [4 marks]

Consider the three planes

4a. Show that the three planes do not intersect.

[4 marks]



#### EITHER

-3x+z=-3 and A1 -3x+z=44 A1

#### OR

-5x+y=-7 and A1 -5x+y=40 A1

#### OR

3x-z=3 and A1  $3x-z=-rac{79}{5}$  A1

#### THEN

the two lines are parallel (
$$-3 
eq 44$$
 or  $-7 
eq 40$  or  $3 
eq -rac{79}{5}$ ) **R1**

**Note:** There are other possible pairs of equations in two variables. To obtain the final *R1*, at least the initial *M1* must have been awarded.

hence the three planes do not intersect **AG** 

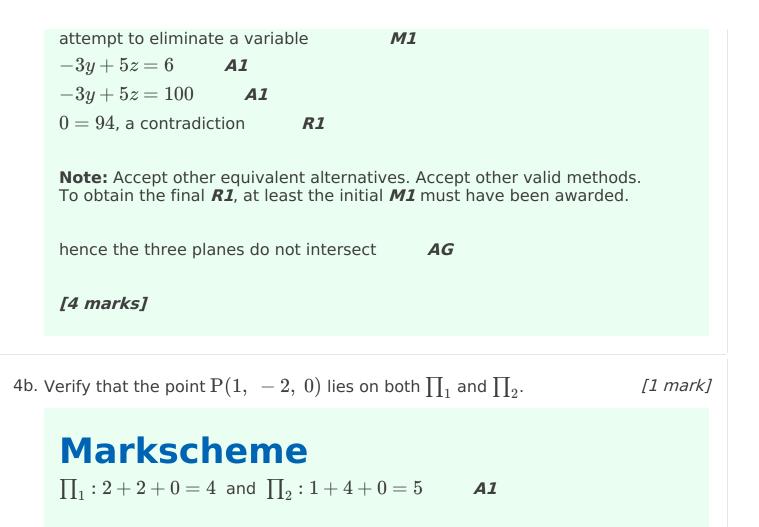
#### METHOD 2

vector product of the two normals 
$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$
 (or equivalent) **A1**  
 $r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$  (or equivalent) **A1**

**Note:** Award **AO** if "r =" is missing. Subsequent marks may still be awarded.

Attempt to substitute  $(1 + \lambda, -2 + 5\lambda, 3\lambda)$  in  $\prod_3$  **M1**  $-9(1 + \lambda) + 3(-2 + 5\lambda) - 2(3\lambda) = 32$ -15 = 32, a contradiction **R1** hence the three planes do not intersect **AG** 

#### METHOD 3



[1 mark]

4c. Find a vector equation of L, the line of intersection of  $\prod_1$  and  $\prod_2$ . [4 marks]

#### METHOD 1

attempt to find the vector product of the two normals **M1** 

$$\begin{pmatrix} 2\\-1\\1 \end{pmatrix} \times \begin{pmatrix} 1\\-2\\3 \end{pmatrix}$$
$$= \begin{pmatrix} -1\\-5\\-3 \end{pmatrix} \quad AI$$
$$r = \begin{pmatrix} 1\\-2\\0 \end{pmatrix} + \lambda \begin{pmatrix} 1\\5\\3 \end{pmatrix} \quad AIAI$$

**Note:** Award **A1A0** if "r =" is missing. Accept any multiple of the direction vector. Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "r =" only once.

#### **METHOD 2**

attempt to eliminate a variable from  $\prod_1$  and  $\prod_2$  **M1** 3x-z=3 OR 3y-5z=-6 OR 5x-y=7Let x=t

substituting x = t in 3x - z = 3 to obtain

 $z=-3+3t\,$  and  $\,y=5t-7$  (for all three variables in parametric form) A1

$$r = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$
 Alal

**Note:** Award **A1A0** if "r =" is missing. Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes  $\prod_1$  and  $\prod_2$ .

[4 marks]

#### METHOD 1

the line connecting L and  $\prod_3$  is given by  $L_1$ 

attempt to substitute position and direction vector to form  $L_1$  (M1)

$$s = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$$
 **A1**  
substitute  $(1 - 9t, -2 + 3t, -2t)$  in  $\prod_3$  **M1**  
 $-9(1 - 9t) + 3(-2 + 3t) - 2(-2t) = 32$   
 $94t = 47 \Rightarrow t = \frac{1}{2}$  **A1**

attempt to find distance between (1, -2, 0) and their point  $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$  *(M1)* 

$$= \begin{vmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \end{vmatrix} = \frac{1}{2} \sqrt{(-9)^2 + 3^2 + (-2)^2}$$
$$= \frac{\sqrt{94}}{2} \qquad \textbf{A1}$$

#### **METHOD 2**

unit normal vector equation of  $\prod_3$  is given by  $\frac{\begin{pmatrix} -9\\3\\2 \end{pmatrix} \cdot \begin{pmatrix} x\\y\\z \end{pmatrix}}{\sqrt{81+9+4}}$ 

(M1)

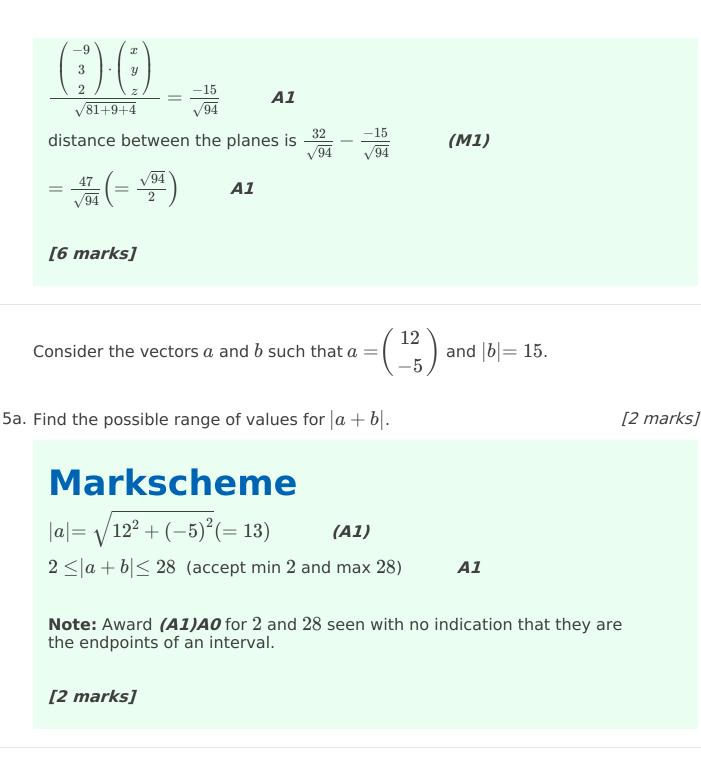
$$=rac{32}{\sqrt{94}}$$
 A1

let  $\prod_4$  be the plane parallel to  $\prod_3$  and passing through P, then the normal vector equation of  $\prod_4$  is given by

$$\begin{pmatrix} -9\\3\\2 \end{pmatrix} \cdot \begin{pmatrix} x\\y\\z \end{pmatrix} = \begin{pmatrix} -9\\3\\2 \end{pmatrix} \cdot \begin{pmatrix} 1\\-2\\0 \end{pmatrix} = -15$$
 M1

unit normal vector equation of  $\prod_4$  is given by

[6 marks]



Consider the vector p such that p = a + b.

5b. Given that |a + b| is a minimum, find p.

[2 marks]

Markschemerecognition that p or b is a negative multiple of a(M1)
$$p = -2\widehat{a}$$
 OR  $b = -\frac{15}{13}a\left(=-\frac{15}{13}\begin{pmatrix}12\\-5\end{pmatrix}\right)$  $p = -\frac{2}{13}\begin{pmatrix}12\\-5\end{pmatrix}\left(=\begin{pmatrix}-1.85\\0.769\end{pmatrix}\right)$ A1[2 marks]

Consider the vector 
$$q$$
 such that  $q=inom{x}{y}$  , where  $x,\;y\in\mathbb{R}^+.$ 

5c. Find q such that ert q ert = ert b ert and q is perpendicular to a.

[5 marks]

$$q \text{ is perpendicular to} \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

$$\Rightarrow q \text{ is in the direction} \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad (M1)$$

$$q = k \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad (A1)$$

$$(|q|=)\sqrt{(5k)^2 + (12k)^2} = 15 \quad (M1)$$

$$k = \frac{15}{13} \quad (A1)$$

$$q = \frac{15}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix} \left( = \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix} = \begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix} \right) \quad A1$$

#### METHOD 2

q is perpendicular to  $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ 

attempt to set scalar product q. a = 0 OR product of gradients = -1 (M1)

$$12x-5y=0$$
 (A1) $(|q|=)\sqrt{x^2+y^2}=15$ 

attempt to solve simultaneously to find a quadratic in x or y (M1)

$$x^{2} + \left(rac{12x}{5}
ight)^{2} = 15^{2} \text{ OR } \left(rac{5y}{12}
ight)^{2} + y^{2} = 15^{2}$$
 $q = \left(rac{75}{13}}{rac{180}{13}}
ight) \left(= \left(rac{5.77}{13.8}
ight)
ight)$  A1A1

**Note:** Award **A1** independently for each value. Accept values given as  $x = \frac{75}{13}$  and  $y = \frac{180}{13}$  or equivalent.

#### [5 marks]

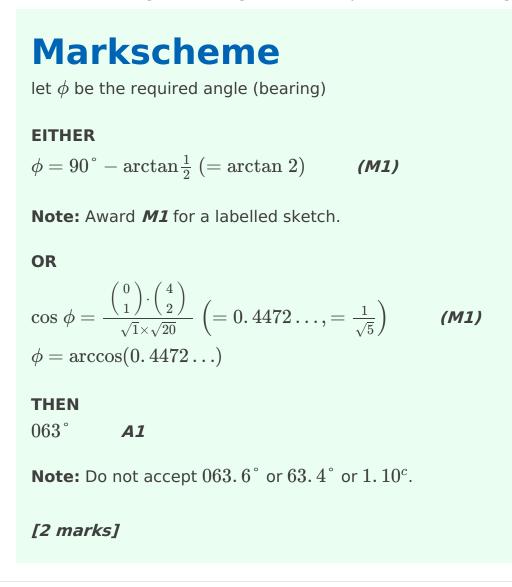
Two airplanes,  $\boldsymbol{A}$  and  $\boldsymbol{B},$  have position vectors with respect to an origin  $\boldsymbol{O}$  given respectively by

$$r_A = egin{pmatrix} 19 \ -1 \ 1 \end{pmatrix} + t egin{pmatrix} -6 \ 2 \ 4 \end{pmatrix} r_B = egin{pmatrix} 1 \ 0 \ 12 \end{pmatrix} + t egin{pmatrix} 4 \ 2 \ -2 \end{pmatrix}$$

where t represents the time in minutes and  $0 \le t \le 2.5$ .

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

6a. Find the three-figure bearing on which airplane B is travelling. [2 marks]



[2 marks]

#### METHOD 1

let  $|b_A|$  be the speed of A and let  $|b_B|$  be the speed of Battempts to find the speed of one of A or B (M1)  $|b_A| = \sqrt{(-6)^2 + 2^2 + 4^2}$  or  $|b_B| = \sqrt{4^2 + 2^2 + (-2)^2}$ 

Note: Award *MO* for  $|b_A| = \sqrt{19^2 + (-1)^2 + 1^2}$  and  $|b_B| = \sqrt{1^2 + 0^2 + 12^2}$ .

$$egin{aligned} |b_A| &= 7.48\dots \left( = \sqrt{56} 
ight) \mbox{ (km min^{-1}) and } |b_B| &= 4.89\dots \left( = \sqrt{24} 
ight) \mbox{ (km min^{-1})} \ & \mathbf{A1} \ |b_A| &> |b_B| \mbox{ so } A \mbox{ travels at a greater speed than } B \ & \mathbf{AG} \end{aligned}$$

#### **METHOD 2**

attempts to use speed =  $\frac{\text{distance}}{\text{time}}$ speed<sub>A</sub> =  $\frac{|r_A(t_2) - r_A(t_1)|}{t_2 - t_1}$  and speed<sub>B</sub> =  $\frac{|r_B(t_2) - r_B(t_1)|}{t_2 - t_1}$  (M1) for example: speed<sub>A</sub> =  $\frac{|r_A(1) - r_A(0)|}{1}$  and speed<sub>B</sub> =  $\frac{|r_B(1) - r_B(0)|}{1}$ speed<sub>A</sub> =  $\frac{\sqrt{(-6)^2 + 2^2 + 4^2}}{1}$  and speed<sub>B</sub> =  $\frac{\sqrt{4^2 + 2^2 + 2^2}}{1}$ speed<sub>A</sub> = 7.48...( $2\sqrt{14}$ ) and speed<sub>B</sub> = 4.89...( $\sqrt{24}$ ) A1 speed<sub>A</sub> > speed<sub>B</sub> so A travels at a greater speed than B AG [2 marks]

6c. Find the acute angle between the two airplanes' lines of flight. Give your [4 marks] answer in degrees.

# **Markscheme** attempts to use the angle between two direction vectors formula (M1) $\cos \theta = \frac{(-6)(4) + (2)(2) + (4)(-2)}{\sqrt{(-6)^2 + 2^2 + 4^2}\sqrt{4^2 + 2^2 + (-2)^2}}$ (A1) $\cos \theta = -0.7637 \dots \left( = -\frac{7}{\sqrt{84}} \right) \text{ or }$ $\theta = \arccos(-0.7637 \dots) (= 2.4399 \dots)$ attempts to find the acute angle $180^\circ - \theta$ using their value of $\theta$ (M1) $= 40.2^\circ$ A1 [4 marks]

The two airplanes' lines of flight cross at point  $\boldsymbol{P}.$ 

6d. Find the coordinates of  $\boldsymbol{P}.$ 

[5 marks]

for example, sets  $r_A(t_1)=r_B(t_2)$  and forms at least two equations (M1)  $19-6t_1=1+4t_2$  $-1+2t_1=2t_2$  $1+4t_1=12-2t_2$ 

**Note:** Award MO for equations involving t only.

#### EITHER

attempts to solve the system of equations for one of  $t_1$  or  $t_2$  (M1)

$$t_1=2$$
 or  $t_2=rac{3}{2}$  Al

#### OR

attempts to solve the system of equations for  $t_1$  and  $t_2$  (M1)

$$t_1=2$$
 or  $t_2=rac{3}{2}$  **A1**

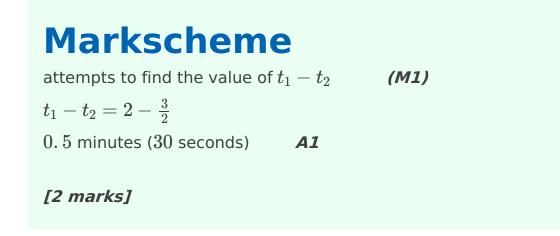
#### THEN

substitutes their  $t_1$  or  $t_2$  value into the corresponding  $r_A$  or  $r_B$  (M1)  ${\rm P}(7,3,9)$  A1

**Note:** Accept  $\overrightarrow{OP} = \begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}$ . Accept 7 km east of O, 3 km north of O and 9 km above sea level.

[5 marks]

6e. Determine the length of time between the first airplane arriving at P and <code>[2 marks]</code> the second airplane arriving at P.



6f. Let D(t) represent the distance between airplane A and airplane B for [5 marks]  $0 \le t \le 2.5$ .

Find the minimum value of D(t).

#### EITHER

attempts to find  $r_B - r_A$  (M1)  $r_B - r_A = \begin{pmatrix} -18\\1\\11 \end{pmatrix} + t \begin{pmatrix} 10\\0\\-6 \end{pmatrix}$ 

attempts to find their D(t) (M1)

$$D(t) = \sqrt{(10t - 18)^2 + 1 + (11 - 6t)^2}$$
 **A1**

#### OR

attempts to find  $r_A - r_B$  (M1)

$$r_A - r_B = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t \begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their D(t) (M1)

$$D(t) = \sqrt{(18 - 10t)^2 + (-1)^2 + (-11 + 6t)^2}$$
 A1

**Note:** Award *MOMOAO* for expressions using two different time parameters.

#### THEN

either attempts to find the local minimum point of D(t) or attempts to find the value of t such that D'(t) = 0 (or equivalent) (M1)

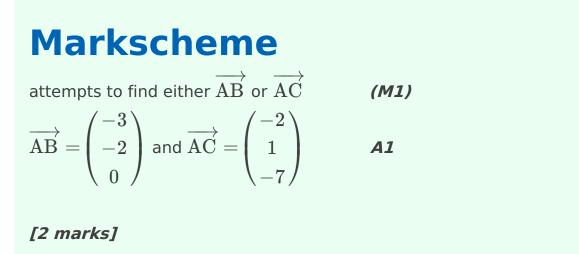
 $t = 1.8088... \left(=\frac{123}{68}
ight)$ D(t) = 1.01459...minimum value of D(t) is  $1.01 \left(=\frac{\sqrt{1190}}{34}
ight)$  (km) **A1** 

Note: Award *MO* for attempts at the shortest distance between two lines.

#### [5 marks]

Three points A(3, 0, 0), B(0, -2, 0) and C(1, 1, -7) lie on the plane  $\Pi_1$ .

<sup>7a.</sup> Find the vector  $\overrightarrow{AB}$  and the vector  $\overrightarrow{AC}.$ 



7b. Hence find the equation of  $\Pi_1$ , expressing your answer in the form ax + by + cz = d, where  $a, b, c, d \in \mathbb{Z}$ .

[5 marks]

[2 marks]

# Markscheme

METHOD 1

attempts to find 
$$\overrightarrow{AB} \times \overrightarrow{AC}$$
 (M1)  
 $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$  A1

#### EITHER

equation of plane is of the form 14x - 21y - 7z = d (2x - 3y - z = d) (A1)

substitutes a valid point e.g (3, 0, 0) to obtain a value of d **M1**  $d = 42 \ (d = 6)$ 

#### OR

attempts to use  $r \cdot n = a \cdot n$  (M1)

$$r \cdot \begin{pmatrix} 14\\ -21\\ -7 \end{pmatrix} = \begin{pmatrix} 3\\ 0\\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14\\ -21\\ -7 \end{pmatrix} \quad \left( r \cdot \begin{pmatrix} 14\\ -21\\ -7 \end{pmatrix} = 42 \right) \qquad \qquad \textbf{A1}$$

$$r \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \quad \left( r \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6 \right)$$

#### THEN

$$14x - 21y - 7z = 42 \ (2x - 3y - z = 6)$$
 **A1**

#### **METHOD 2**

equation of plane is of the form 
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$$
  
A1  
attempts to form equations for  $x, y, z$  in terms of their parameters  
(M1)  
 $x = 3 - 3s - 2t, y = -2s + t, z = -7t$  A1  
eliminates at least one of their parameters (M1)  
for example,  $2x - 3y = 6 - 7t (\Rightarrow 2x - 3y = 6 + z)$   
 $2x - 3y - z = 6$  A1  
[5 marks]

Plane  $\Pi_2$  has equation 3x - y + 2z = 2.

7c. The line L is the intersection of  $\Pi_1$  and  $\Pi_2$ . Verify that the vector [2 marks] equation of L can be written as  $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ .

**Markscheme**  
METHOD 1  
substitutes 
$$r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$
 into their  $\Pi_1$  and  $\Pi_2$  (given) (M1)  
 $\Pi_1: 2\lambda - 3(-2 + \lambda) - (-\lambda) = 6$  and  $\Pi_2: 3\lambda - 3(-2 + \lambda) + 2(-\lambda) = 2$   
A1

**Note:** Award **(M1)A0** for correct verification using a specific value of  $\lambda$ .

so the vector equation of L can be written as  $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ 

AG

METHOD 2

EITHER

attempts to find 
$$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$
 *M1*

 $= \begin{pmatrix} -7\\ -7\\ 7 \end{pmatrix}$ 

OR

$$\begin{pmatrix} 2\\-3\\-1 \end{pmatrix} \cdot \begin{pmatrix} 1\\1\\-1 \end{pmatrix} = (2-3+1) = 0 \text{ and } \begin{pmatrix} 3\\-1\\2 \end{pmatrix} \cdot \begin{pmatrix} 1\\1\\-1 \end{pmatrix} = (3-1-2) = 0$$

$$\textbf{M1}$$

#### THEN

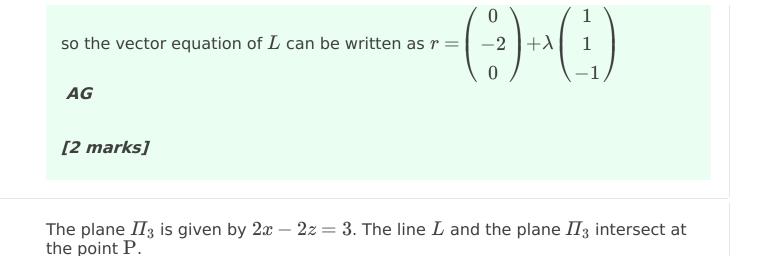
substitutes (0, -2, 0) into  $\Pi_1$  and  $\Pi_2$  $\Pi_1: 2(0)-3(-2)-(0)=6$  and  $\Pi_2: 3(0)-(-2)+2(0)=2$  **A1** so the vector equation of L can be written as  $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ 

#### AG

#### METHOD 3

attempts to solve 2x - 3y - z = 6 and 3x - y + 2z = 2 (M1) for example,  $x = -\lambda, \ y = -2 - \lambda, \ z = \lambda$  A1

**Note:** Award **A1** for substituting x = 0 (or y = -2 or z = 0) into  $\Pi_1$  and  $\Pi_2$  and solving simultaneously. For example, solving -3y - z = 6 and -y + 2z = 2 to obtain y = -2 and z = 0.



7d. Show that at the point  $P,\;\lambda=rac{3}{4}.$ 

[2 marks]

[1 mark]

**Markscheme** substitutes the equation of *L* into the equation of  $\Pi_3$  (M1)  $2\lambda + 2\lambda = 3 \Rightarrow 4\lambda = 3$  A1  $\lambda = \frac{3}{4}$  AG

7e. Hence find the coordinates of  $\boldsymbol{P}.$ 

MarkschemeP has coordinates  $(\frac{3}{4}, -\frac{5}{4}, -\frac{3}{4})$ A1[1 mark]

The point  ${
m B}(0,\ -2,\ 0)$  lies on L.

7f. Find the reflection of the point B in the plane  $\varPi_3$ .

[7 marks]

# Markscheme

normal to 
$$\Pi_3$$
 is  $n = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$  (A1)

**Note:** May be seen or used anywhere.

considers the line normal to 
$$\Pi_3$$
 passing through  $B(0, -2, 0)$  (M1)  
 $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$  A1

#### **EITHER**

r

finding the point on the normal line that intersects  $\Pi_3$  attempts to solve simultaneously with plane 2x-2z=3(M1)

$$4\mu + 4\mu = 3$$

$$\mu=rac{3}{8}$$
 Al

point is  $\left(\frac{3}{4}, -2, -\frac{3}{4}\right)$ 

#### OR

$$\begin{pmatrix} \begin{pmatrix} 2\mu \\ -2 \\ -2\mu \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 0$$

$$4\mu - \frac{3}{2} + 4\mu - \frac{3}{2} = 0$$

$$\mu = \frac{3}{8}$$

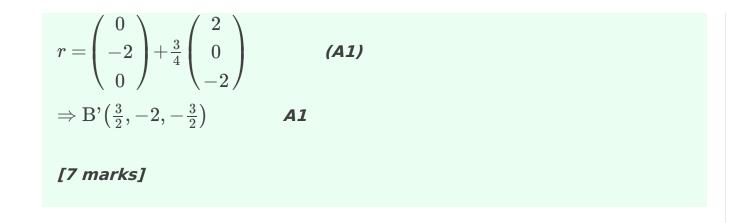
$$A1$$

#### OR

attempts to find the equation of the plane parallel to  $\Pi_3$  containing  ${
m B'}~(x-z=3)$  and solve simultaneously with L(M1)  $2\mu' + 2\mu' = 3$  $\mu'=rac{3}{4}$  Al

#### THEN

so, another point on the reflected line is given by



7g. Hence find the vector equation of the line formed when L is reflected in <code>[2 marks]</code> the plane  $\varPi_3.$ 

#### EITHER

attempts to find the direction vector of the reflected line using their P and  $B^{\prime}$  (M1)

$$\overrightarrow{\text{PB'}} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

#### OR

attempts to find their direction vector of the reflected line using a vector approach *(M1)* 

$$\overrightarrow{\text{PB'}} = \overrightarrow{\text{PB}} + \overrightarrow{\text{BB'}} = -\frac{3}{4} \begin{pmatrix} 1\\1\\-1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1\\0\\-1 \end{pmatrix}$$

#### THEN

$$r = \begin{pmatrix} \frac{3}{2} \\ -2 \\ -\frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$
 (or equivalent) **A1**

**Note:** Award **A***O* for either 'r =' or ' $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  =' not stated. Award **A***O* for 'L' ='

[2 marks]

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