vectors and planes *[96 marks]*

1. The plane Π has the Cartesian equation $2x + y + 2z = 3$ *[7 marks]*

   The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

   Find the possible values of p.

Markscheme

recognition that the angle between the normal and the line is 60° (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}\right|}{\sqrt{9} \times \sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4|p|$

attempt to square both sides ***M1***

$9(5+p^2) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

[7 marks]

The lines l_1 and l_2 have the following vector equations where $\lambda,\ \mu \in \mathbb{R}$ and $m \in \mathbb{R}$.

$$l_1 : r_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} l_2 : r_2 = \begin{pmatrix} -1 \\ -4 \\ -2m \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$$

2a. Show that l_1 and l_2 are never perpendicular to each other. *[3 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to calculate $\begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$ **(M1)**

$= -1 - m^2$ **A1**

since $m^2 \geq 0,\ -1 - m^2 < 0$ for $m \in \mathbb{R}$ **R1**

so l_1 and l_2 are never perpendicular to each other **AG**

[3 marks]

The plane Π has Cartesian equation $x + 4y - z = p$ where $p \in \mathbb{R}$.

Given that l_1 and Π have no points in common, find

2b. the value of m. *[2 marks]*

Markscheme

(since l_1 is parallel to Π, l_1 is perpendicular to the normal of Π and so)

$\begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = 0$ **R1**

$2 + 4 - m = 0$

$m = 6$ **A1**

[2 marks]

2c. the condition on the value of p. *[2 marks]*

Markscheme

since there are no points in common, $(3,\ -2,\ 0)$ does not lie in Π

EITHER

substitutes $(3,\ -2,\ 0)$ into $x + 4y - z (\neq p)$ **(M1)**

OR

$\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} (\neq p)$ **(M1)**

THEN

$p \neq -5$ **A1**

[2 marks]

The points $\mathrm{A}(5,\ -2,\ 5)$, $\mathrm{B}(5,\ 4,\ -1)$, $\mathrm{C}(-1,\ -2,\ -1)$ and $\mathrm{D}(7,\ -4,\ -3)$ are the vertices of a right-pyramid.

3a. Find the vectors $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$. *[2 marks]*

Markscheme

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$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \left(= 6 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right)$ **A1**

$\overrightarrow{AC} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \left(= 6 \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right)$ **A1**

[2 marks]

3b. Use a vector method to show that $\mathrm{B\hat{A}C} = 60^\circ$. *[3 marks]*

Markscheme

attempts to use $\cos \mathrm{B\hat{A}C} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right| \left|\overrightarrow{AC}\right|}$ **(M1)**

$= \frac{\begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix}}{\sqrt{72} \times \sqrt{72}}$ **A1**

$= \frac{1}{2}$ **A1**

so $\mathrm{B\hat{A}C} = 60^\circ$ **AG**

[3 marks]

3c. Show that the Cartesian equation of the plane Π that contains the triangle ABC is $-x + y + z = -2$. *[3 marks]*

Markscheme

attempts to find a vector normal to Π **M1**

for example, $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{pmatrix} -36 \\ 36 \\ 36 \end{pmatrix} \left(= 36 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right)$ leading to **A1**

a vector normal to Π is $n = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

EITHER

substitutes $(5,\ -2,\ -5)$ (or $(5,\ 4,\ -1)$ or $(-1,\ -2,\ -1)$) into $-x + y + z = d$ and attempts to find the value of d

for example, $d = -5 - 2 + 5 (= -2)$ **M1**

OR

attempts to use $r \cdot n = a \cdot n$ **M1**

for example, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

THEN

leading to the Cartesian equation of Π as $-x + y + z = -2$ **AG**

[3 marks]

The line L passes through the point D and is perpendicular to Π.

3d. Find a vector equation of the line L. *[1 mark]*

Markscheme

$r = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} (\lambda \in \mathbb{R})$ **A1**

[1 mark]

3e. Hence determine the minimum distance, d_{min}, from D to $\mathit{\Pi}$. *[4 marks]*

Markscheme

substitutes $x=7-\lambda, y=-4+\lambda, z=-3+\lambda$ into $-x+y+z=-2$ **(M1)**

$-(7-\lambda)+(-4+\lambda)+(-3+\lambda)=-2(3\lambda=12)$

$\lambda=4$ **A1**

shows a correct calculation for finding d_{min}, for example, attempts to find

$$\left|4\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|$$ **M1**

$d_{\text{min}}=4\sqrt{3}(=6.93)$ **A1**

[4 marks]

3f. Find the volume of right-pyramid ABCD. *[4 marks]*

Markscheme

let the area of triangle ABC be A

EITHER

attempts to find $A = \frac{1}{2}\left|\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}\right|$, for example **M1**

$$A = \frac{1}{2}\left|\begin{pmatrix} -36 \\ 36 \\ 36 \end{pmatrix}\right|$$

OR

attempts to find $\frac{1}{2}\left|\overrightarrow{\mathrm{AB}}\right|\left|\overrightarrow{\mathrm{AC}}\right|\sin\theta$, for example **M1**

$A = \frac{1}{2} \times 6\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$ (where $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$)

THEN

$A = 18\sqrt{3} (= 31.2)$ **A1**

uses $V = \frac{1}{3}Ah$ where A is the area of triangle ABC and $h = d_{\min}$ **M1**

$V = \frac{1}{3} \times 18\sqrt{3} \times 4\sqrt{3}$

$= 72$ **A1**

[4 marks]

Consider the three planes

$\prod_1 : \ 2x - y + z = 4$

$\prod_2 : \ x - 2y + 3z = 5$

$\prod_3 : -9x + 3y - 2z = 32$

4a. Show that the three planes do not intersect. *[4 marks]*

Markscheme

METHOD 1

attempt to eliminate a variable ***M1***

obtain a pair of equations in two variables

EITHER

$-3x+z=-3$ and ***A1***

$-3x+z=44$ ***A1***

OR

$-5x+y=-7$ and ***A1***

$-5x+y=40$ ***A1***

OR

$3x-z=3$ and ***A1***

$3x-z=-\frac{79}{5}$ ***A1***

THEN

the two lines are parallel ($-3\neq 44$ or $-7\neq 40$ or $3\neq -\frac{79}{5}$) ***R1***

Note: There are other possible pairs of equations in two variables.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

METHOD 2

vector product of the two normals $=\begin{pmatrix}-1\\-5\\-3\end{pmatrix}$ (or equivalent) ***A1***

$r=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\5\\3\end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r=$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in $\prod_3$ ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)=32$

$-15=32$, a contradiction ***R1***

hence the three planes do not intersect ***AG***

METHOD 3

attempt to eliminate a variable ***M1***

$-3y + 5z = 6$ ***A1***

$-3y + 5z = 100$ ***A1***

$0 = 94$, a contradiction ***R1***

Note: Accept other equivalent alternatives. Accept other valid methods.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

4b. Verify that the point $\mathrm{P}(1,\ -2,\ 0)$ lies on both $\prod_1$ and $\prod_2$. *[1 mark]*

Markscheme

$\prod_1 : 2 + 2 + 0 = 4$ and $\prod_2 : 1 + 4 + 0 = 5$ ***A1***

[1 mark]

4c. Find a vector equation of L, the line of intersection of $\prod_1$ and $\prod_2$. *[4 marks]*

Markscheme

METHOD 1

attempt to find the vector product of the two normals ***M1***

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$ ***A1***

$$r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector.
Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "$r =$" only once.

METHOD 2

attempt to eliminate a variable from $\prod_1$ and $\prod_2$ ***M1***

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$

Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form)
A1

$$r = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes $\prod_1$ and $\prod_2$.

[4 marks]

4d. Find the distance between L and $\prod_3$. *[6 marks]*

Markscheme

METHOD 1

the line connecting L and $\prod_3$ is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$s = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$ ***A1***

substitute $(1 - 9t, -2 + 3t, -2t)$ in $\prod_3$ ***M1***

$-9(1 - 9t) + 3(-2 + 3t) - 2(-2t) = 32$

$94t = 47 \Rightarrow t = \frac{1}{2}$ ***A1***

attempt to find distance between $(1, -2, 0)$ and their point $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$ ***(M1)***

$= \left|\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$

$= \frac{\sqrt{94}}{2}$ ***A1***

METHOD 2

unit normal vector equation of $\prod_3$ is given by $\frac{\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$= \frac{32}{\sqrt{94}}$ ***A1***

let $\prod_4$ be the plane parallel to $\prod_3$ and passing through P,
then the normal vector equation of $\prod_4$ is given by

$\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = -15$ ***M1***

unit normal vector equation of $\prod_4$ is given by

$$\frac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}=\frac{-15}{\sqrt{94}}$$ ***A1***

distance between the planes is $\frac{32}{\sqrt{94}}-\frac{-15}{\sqrt{94}}$ ***(M1)***

$=\frac{47}{\sqrt{94}}\left(=\frac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

Consider the vectors a and b such that $a=\begin{pmatrix}12\\-5\end{pmatrix}$ and $|b|=15$.

5a. Find the possible range of values for $|a+b|$. *[2 marks]*

Markscheme

$|a|=\sqrt{12^2+(-5)^2}(=13)$ ***(A1)***

$2\le|a+b|\le 28$ (accept min 2 and max 28) ***A1***

Note: Award ***(A1)A0*** for 2 and 28 seen with no indication that they are the endpoints of an interval.

[2 marks]

Consider the vector p such that $p=a+b$.

5b. Given that $|a+b|$ is a minimum, find p. *[2 marks]*

Markscheme

recognition that p or b is a negative multiple of a ***(M1)***

$p = -2\hat{a}$ OR $b = -\frac{15}{13}a\left(= -\frac{15}{13}\begin{pmatrix}12\\-5\end{pmatrix}\right)$

$p = -\frac{2}{13}\begin{pmatrix}12\\-5\end{pmatrix}\left(=\begin{pmatrix}-1.85\\0.769\end{pmatrix}\right)$ ***A1***

[2 marks]

Consider the vector q such that $q = \begin{pmatrix}x\\y\end{pmatrix}$, where $x,\ y \in \mathbb{R}^{+}$.

5c. Find q such that $|q| = |b|$ and q is perpendicular to a. *[5 marks]*

Markscheme

METHOD 1

q is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

$\Rightarrow q$ is in the direction $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ ***(M1)***

$q = k\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ ***(A1)***

$(|q|=)\sqrt{(5k)^2 + (12k)^2} = 15$ ***(M1)***

$k = \frac{15}{13}$ ***(A1)***

$q = \frac{15}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix}\left(=\begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix}=\begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix}\right)$ ***A1***

METHOD 2

q is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

attempt to set scalar product $q.a = 0$ OR product of gradients $= -1$ ***(M1)***

$12x - 5y = 0$ ***(A1)***

$(|q|=)\sqrt{x^2 + y^2} = 15$

attempt to solve simultaneously to find a quadratic in x or y ***(M1)***

$x^2 + \left(\frac{12x}{5}\right)^2 = 15^2$ OR $\left(\frac{5y}{12}\right)^2 + y^2 = 15^2$

$q = \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix}\left(=\begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix}\right)$ ***A1A1***

Note: Award ***A1*** independently for each value. Accept values given as $x = \frac{75}{13}$ and $y = \frac{180}{13}$ or equivalent.

[5 marks]

Two airplanes, A and B, have position vectors with respect to an origin O given respectively by

$$r_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \leq t \leq 2.5$.

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

6a. Find the three-figure bearing on which airplane B is travelling. *[2 marks]*

Markscheme

let ϕ be the required angle (bearing)

EITHER

$\phi = 90^\circ - \arctan\frac{1}{2}\ (= \arctan\ 2)$ ***(M1)***

Note: Award ***M1*** for a labelled sketch.

OR

$\cos\ \phi = \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix}}{\sqrt{1} \times \sqrt{20}}\ \left(= 0.4472\ldots, = \frac{1}{\sqrt{5}}\right)$ ***(M1)***

$\phi = \arccos(0.4472\ldots)$

THEN

063° ***A1***

Note: Do not accept 063.6° or 63.4° or 1.10^c.

[2 marks]

6b. Show that airplane A travels at a greater speed than airplane B. *[2 marks]*

Markscheme

METHOD 1

let $|b_A|$ be the speed of A and let $|b_B|$ be the speed of B

attempts to find the speed of one of A or B ***(M1)***

$|b_A|= \sqrt{(-6)^2 + 2^2 + 4^2}$ or $|b_B|= \sqrt{4^2 + 2^2 + (-2)^2}$

Note: Award ***M0*** for $|b_A|= \sqrt{19^2 + (-1)^2 + 1^2}$ and $|b_B|= \sqrt{1^2 + 0^2 + 12^2}$.

$|b_A|= 7.48\ldots \left(= \sqrt{56}\right)$ (km min^{-1}) and $|b_B|= 4.89\ldots \left(= \sqrt{24}\right)$ (km min^{-1}) ***A1***

$|b_A|>|b_B|$ so A travels at a greater speed than B ***AG***

METHOD 2

attempts to use speed $= \frac{\text{distance}}{\text{time}}$

$\text{speed}_A = \frac{|r_A(t_2)-r_A(t_1)|}{t_2-t_1}$ and $\text{speed}_B = \frac{|r_B(t_2)-r_B(t_1)|}{t_2-t_1}$ ***(M1)***

for example:

$\text{speed}_A = \frac{|r_A(1)-r_A(0)|}{1}$ and $\text{speed}_B = \frac{|r_B(1)-r_B(0)|}{1}$

$\text{speed}_A = \frac{\sqrt{(-6)^2+2^2+4^2}}{1}$ and $\text{speed}_B = \frac{\sqrt{4^2+2^2+2^2}}{1}$

$\text{speed}_A = 7.48\ldots\left(2\sqrt{14}\right)$ and $\text{speed}_B = 4.89\ldots\left(\sqrt{24}\right)$ ***A1***

$\text{speed}_A > \text{speed}_B$ so A travels at a greater speed than B ***AG***

[2 marks]

6c. Find the acute angle between the two airplanes’ lines of flight. Give your answer in degrees. *[4 marks]*

Markscheme

attempts to use the angle between two direction vectors formula ***(M1)***

$\cos\theta = \frac{(-6)(4)+(2)(2)+(4)(-2)}{\sqrt{(-6)^2+2^2+4^2}\sqrt{4^2+2^2+(-2)^2}}$ ***(A1)***

$\cos\theta = -0.7637\ldots \left(= -\frac{7}{\sqrt{84}}\right)$ or

$\theta = \arccos(-0.7637\ldots)\ (= 2.4399\ldots)$

attempts to find the acute angle $180^\circ - \theta$ using their value of θ ***(M1)***

$= 40.2^\circ$ ***A1***

[4 marks]

The two airplanes' lines of flight cross at point P.

6d. Find the coordinates of P. *[5 marks]*

Markscheme

for example, sets $r_A(t_1) = r_B(t_2)$ and forms at least two equations ***(M1)***

$19 - 6t_1 = 1 + 4t_2$

$-1 + 2t_1 = 2t_2$

$1 + 4t_1 = 12 - 2t_2$

Note: Award ***M0*** for equations involving t only.

EITHER

attempts to solve the system of equations for one of t_1 or t_2 ***(M1)***

$t_1 = 2$ or $t_2 = \frac{3}{2}$ ***A1***

OR

attempts to solve the system of equations for t_1 and t_2 ***(M1)***

$t_1 = 2$ or $t_2 = \frac{3}{2}$ ***A1***

THEN

substitutes their t_1 or t_2 value into the corresponding r_A or r_B ***(M1)***

$\mathrm{P}(7, 3, 9)$ ***A1***

Note: Accept $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}$. Accept 7 km east of O, 3 km north of O and 9 km above sea level.

[5 marks]

6e. Determine the length of time between the first airplane arriving at P and the second airplane arriving at P. *[2 marks]*

Markscheme

attempts to find the value of $t_1 - t_2$ ***(M1)***

$t_1 - t_2 = 2 - \frac{3}{2}$

0.5 minutes (30 seconds) ***A1***

[2 marks]

6f. Let $D(t)$ represent the distance between airplane A and airplane B for $0 \leq t \leq 2.5$. *[5 marks]*

Find the minimum value of $D(t)$.

Markscheme

EITHER

attempts to find $r_B - r_A$ ***(M1)***

$$r_B - r_A = \begin{pmatrix} -18 \\ 1 \\ 11 \end{pmatrix} + t\begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$$D\left(t\right) = \sqrt{\left(10t - 18\right)^2 + 1 + \left(11 - 6t\right)^2}$$ ***A1***

OR

attempts to find $r_A - r_B$ ***(M1)***

$$r_A - r_B = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t\begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$$D\left(t\right) = \sqrt{\left(18 - 10t\right)^2 + \left(-1\right)^2 + \left(-11 + 6t\right)^2}$$ ***A1***

Note: Award ***M0M0A0*** for expressions using two different time parameters.

THEN

either attempts to find the local minimum point of $D(t)$ or attempts to find the value of t such that $D'(t) = 0$ (or equivalent) ***(M1)***

$t = 1.8088\ldots \left(= \frac{123}{68}\right)$

$D(t) = 1.01459\ldots$

minimum value of $D(t)$ is $1.01 \left(= \frac{\sqrt{1190}}{34}\right)$ (km) ***A1***

Note: Award ***M0*** for attempts at the shortest distance between two lines.

[5 marks]

Three points $\mathrm{A}(3,\ 0,\ 0),\ \mathrm{B}(0,\ -2,\ 0)$ and $\mathrm{C}(1,\ 1,\ -7)$ lie on the plane Π_1.

7a. Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$. *[2 marks]*

Markscheme

attempts to find either $\overrightarrow{\mathrm{AB}}$ or $\overrightarrow{\mathrm{AC}}$ ***(M1)***

$\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-3\\-2\\0\end{pmatrix}$ and $\overrightarrow{\mathrm{AC}}=\begin{pmatrix}-2\\1\\-7\end{pmatrix}$ ***A1***

[2 marks]

7b. Hence find the equation of Π_1, expressing your answer in the form $ax+by+cz=d$, where $a,\ b,\ c,\ d\in\mathbb{Z}$. *[5 marks]*

Markscheme

METHOD 1

attempts to find $\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}$ ***(M1)***

$\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}=\begin{pmatrix}14\\-21\\-7\end{pmatrix}$ ***A1***

EITHER

equation of plane is of the form $14x-21y-7z=d\ \ (2x-3y-z=d)$ ***(A1)***

substitutes a valid point e.g $(3,\ 0,\ 0)$ to obtain a value of d ***M1***

$d=42\ \ (d=6)$

OR

attempts to use $r\cdot n=a\cdot n$ ***(M1)***

$r\cdot\begin{pmatrix}14\\-21\\-7\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}14\\-21\\-7\end{pmatrix}\ \left(r\cdot\begin{pmatrix}14\\-21\\-7\end{pmatrix}=42\right)$ ***A1***

$$r \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \quad \left(r \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6 \right)$$

THEN

$14x - 21y - 7z = 42 \ \ (2x - 3y - z = 6)$ ***A1***

METHOD 2

equation of plane is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$
A1

attempts to form equations for $x, \ y, \ z$ in terms of their parameters
(M1)

$x = 3 - 3s - 2t \ , \ y = -2s + t \ , \ z = -7t$ ***A1***

eliminates at least one of their parameters ***(M1)***

for example, $2x - 3y = 6 - 7t (\Rightarrow 2x - 3y = 6 + z)$

$2x - 3y - z = 6$ ***A1***

[5 marks]

Plane Π_2 has equation $3x - y + 2z = 2$.

7c. The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. *[2 marks]*

Markscheme

METHOD 1

substitutes $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ into their Π_1 and Π_2 (given) ***(M1)***

$\Pi_1 : \ 2\lambda - 3(-2 + \lambda) - (-\lambda) = 6$ and $\Pi_2 : \ 3\lambda - 3(-2 + \lambda) + 2(-\lambda) = 2$
A1

Note: Award ***(M1)A0*** for correct verification using a specific value of λ.

so the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

AG

METHOD 2

EITHER

attempts to find $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} -7 \\ -7 \\ 7 \end{pmatrix}$

OR

$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (2 - 3 + 1) = 0$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (3 - 1 - 2) = 0$

M1

THEN

substitutes $(0, -2, 0)$ into Π_1 and Π_2

$\Pi_1 :\ 2(0) - 3(-2) - (0) = 6$ and $\Pi_2 :\ 3(0) - (-2) + 2(0) = 2$ ***A1***

so the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

AG

METHOD 3

attempts to solve $2x - 3y - z = 6$ and $3x - y + 2z = 2$ ***(M1)***

for example, $x = -\lambda,\ y = -2 - \lambda,\ z = \lambda$ ***A1***

Note: Award **A1** for substituting $x = 0$ (or $y = -2$ or $z = 0$) into Π_1 and Π_2 and solving simultaneously. For example, solving $-3y - z = 6$ and $-y + 2z = 2$ to obtain $y = -2$ and $z = 0$.

so the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

AG

[2 marks]

The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

7d. Show that at the point $\mathrm{P},\ \lambda = \frac{3}{4}$. *[2 marks]*

Markscheme

substitutes the equation of L into the equation of Π_3 ***(M1)***

$2\lambda + 2\lambda = 3 \Rightarrow 4\lambda = 3$ ***A1***

$\lambda = \frac{3}{4}$ ***AG***

[2 marks]

7e. Hence find the coordinates of P. *[1 mark]*

Markscheme

P has coordinates $\left(\frac{3}{4}, -\frac{5}{4}, -\frac{3}{4}\right)$ ***A1***

[1 mark]

The point $\mathrm{B}(0,\ -2,\ 0)$ lies on L.

7f. Find the reflection of the point B in the plane Π_3. *[7 marks]*

Markscheme

normal to Π_3 is $n = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ***(A1)***

Note: May be seen or used anywhere.

considers the line normal to Π_3 passing through $\mathrm{B}(0, -2, 0)$ ***(M1)***

$r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ***A1***

EITHER

finding the point on the normal line that intersects Π_3
attempts to solve simultaneously with plane $2x - 2z = 3$ ***(M1)***

$4\mu + 4\mu = 3$

$\mu = \frac{3}{8}$ ***A1***

point is $\left(\frac{3}{4}, \ -2, -\frac{3}{4}\right)$

OR

$\left(\begin{pmatrix} 2\mu \\ -2 \\ -2\mu \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 0$ ***(M1)***

$4\mu - \frac{3}{2} + 4\mu - \frac{3}{2} = 0$

$\mu = \frac{3}{8}$ ***A1***

OR

attempts to find the equation of the plane parallel to Π_3 containing
$\mathrm{B}'\ (x - z = 3)$ and solve simultaneously with L ***(M1)***

$2\mu' + 2\mu' = 3$

$\mu' = \frac{3}{4}$ ***A1***

THEN

so, another point on the reflected line is given by

$$r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$ ***(A1)***

$\Rightarrow \text{B'}\left(\frac{3}{2}, -2, -\frac{3}{2}\right)$ ***A1***

[7 marks]

7g. Hence find the vector equation of the line formed when L is reflected in the plane Π_3. *[2 marks]*

Markscheme

EITHER

attempts to find the direction vector of the reflected line using their P and $\mathrm{B'}$ ***(M1)***

$$\overrightarrow{\mathrm{PB'}} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

OR

attempts to find their direction vector of the reflected line using a vector approach ***(M1)***

$$\overrightarrow{\mathrm{PB'}} = \overrightarrow{\mathrm{PB}} + \overrightarrow{\mathrm{BB'}} = -\frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

THEN

$$r = \begin{pmatrix} \frac{3}{2} \\ -2 \\ -\frac{3}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$ (or equivalent) ***A1***

Note: Award ***A0*** for either '$r =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'

[2 marks]

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