vectors and planes *[96 marks]*

1. The plane *Π* has the Cartesian equation $2x + y + 2z = 3$ *[7 marks]*

   The line *L* has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line *L* and the plane *Π* is 30°.

   Find the possible values of p.

The lines l_1 and l_2 have the following vector equations where $\lambda,\ \mu \in \mathbb{R}$ and $m \in \mathbb{R}$.

$$l_1 : r_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \quad l_2 : r_2 = \begin{pmatrix} -1 \\ -4 \\ -2m \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$$

2a. Show that l_1 and l_2 are never perpendicular to each other. *[3 marks]*

The plane Π has Cartesian equation $x + 4y - z = p$ where $p \in \mathbb{R}$.

Given that l_1 and Π have no points in common, find

2b. the value of m. *[2 marks]*

2c. the condition on the value of p. *[2 marks]*

The points $\mathrm{A}(5,\ -2,\ 5)$, $\mathrm{B}(5,\ 4,\ -1)$, $\mathrm{C}(-1,\ -2,\ -1)$ and $\mathrm{D}(7,\ -4,\ -3)$ are the vertices of a right-pyramid.

3a. Find the vectors $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$. *[2 marks]*

3b. Use a vector method to show that $\mathrm{B\hat{A}C} = 60^\circ$. *[3 marks]*

3c. Show that the Cartesian equation of the plane Π that contains the triangle ABC is $-x+y+z=-2$. *[3 marks]*

The line L passes through the point D and is perpendicular to Π.

3d. Find a vector equation of the line L. *[1 mark]*

3e. Hence determine the minimum distance, $d_{\min}$, from D to Π. *[4 marks]*

3f. Find the volume of right-pyramid ABCD. *[4 marks]*

Consider the three planes

$\prod_1 :\ 2x-y+z=4$

$\prod_2 :\ x-2y+3z=5$

$\prod_3 : -9x+3y-2z=32$

4a. Show that the three planes do not intersect. *[4 marks]*

4b. Verify that the point $\mathrm{P}(1,\ -2,\ 0)$ lies on both $\prod_1$ and $\prod_2$. *[1 mark]*

4c. Find a vector equation of L, the line of intersection of $\prod_1$ and $\prod_2$. *[4 marks]*

4d. Find the distance between L and $\prod_3$. *[6 marks]*

Consider the vectors a and b such that $a=\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|b|=15$.

5a. Find the possible range of values for $|a+b|$. *[2 marks]*

Consider the vector p such that $p=a+b$.

5b. Given that $|a+b|$ is a minimum, find p. *[2 marks]*

Consider the vector q such that $q = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x,\ y \in \mathbb{R}^+$.

5c. Find q such that $|q| = |b|$ and q is perpendicular to a. *[5 marks]*

Two airplanes, A and B, have position vectors with respect to an origin O given respectively by

$$r_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \le t \le 2.5$.

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

6a. Find the three-figure bearing on which airplane B is travelling. *[2 marks]*

6b. Show that airplane A travels at a greater speed than airplane B. *[2 marks]*

6c. Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. *[4 marks]*

The two airplanes' lines of flight cross at point P.

6d. Find the coordinates of P. *[5 marks]*

6e. Determine the length of time between the first airplane arriving at P and the second airplane arriving at P. *[2 marks]*

6f. Let $D(t)$ represent the distance between airplane A and airplane B for $0 \le t \le 2.5$. *[5 marks]*

Find the minimum value of $D(t)$.

Three points $\mathrm{A}(3,\ 0,\ 0),\ \mathrm{B}(0,\ -2,\ 0)$ and $\mathrm{C}(1,\ 1,\ -7)$ lie on the plane Π_1.

7a. Find the vector $\overrightarrow{\mathrm{AB}}$ and the vector $\overrightarrow{\mathrm{AC}}$. *[2 marks]*

7b. Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a,\ b,\ c,\ d \in \mathbb{Z}$. *[5 marks]*

Plane Π_2 has equation $3x - y + 2z = 2$.

7c. The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. *[2 marks]*

The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

7d. Show that at the point $\mathrm{P},\ \lambda = \frac{3}{4}$. *[2 marks]*

7e. Hence find the coordinates of P. *[1 mark]*

The point $\mathrm{B}(0,\ -2,\ 0)$ lies on L.

7f. Find the reflection of the point B in the plane Π_3. *[7 marks]*

7g. Hence find the vector equation of the line formed when L is reflected in the plane Π_3. *[2 marks]*