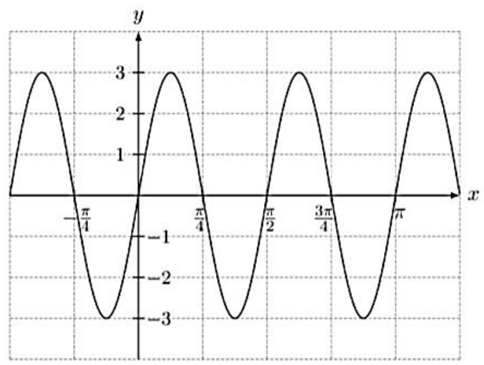
[Maximum mark: 6]

Let $f(x) = a \sin bx$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.



(a) (i) Write down the amplitude of f.

(ii) Find the value of a. [3]

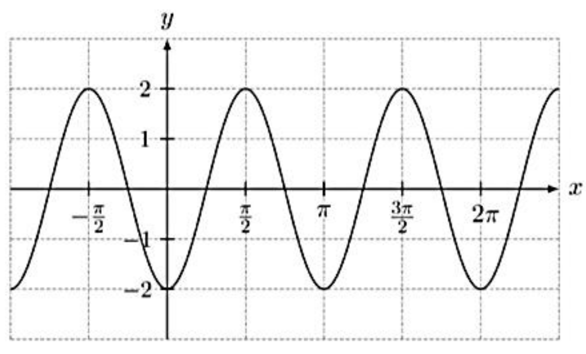
(b) (i) Write down the period of f.

(ii) Find the value of b. [3]

[Maximum mark: 6]

Let $f(x) = a \cos bx$, for $x \in \mathbb{R}$, where $b > 0$.

Part of the graph of f is shown on the diagram below.



(a) Find the value of a. [2]

(b) Find:

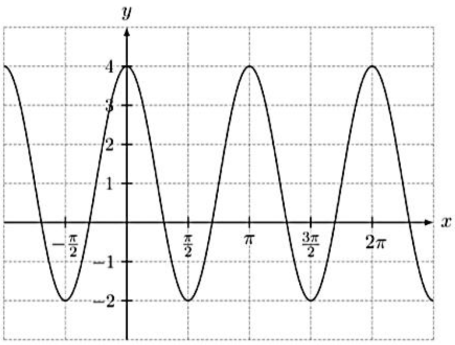
(i) the period of f;

(ii) the value of b. [4]

[Maximum mark: 7]

Let $f(x) = a\cos bx + d$, for $x \in \mathbb{R}$, where $b > 0$.

Part of the graph of f is shown on the diagram below.



(a) Find the value of a. [2]

(b) (i) Write down the period of f.

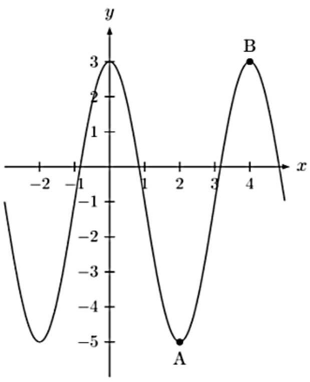
(ii) Find the value of b. [3]

(c) Find the value of d. [2]

[Maximum mark: 6]

A function is defined by $f(x) = a\cos(bx) + d$, for $x \in \mathbb{R}$, where $b > 0$.

Part of the graph of f is shown on the diagram below.



The graph of f has a minimum at A$(2, -5)$ and a maximum at B$(4, 3)$.

Find the value of:

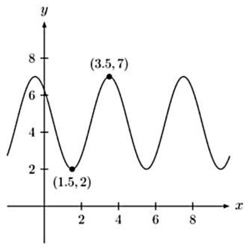
(a) a; [2]

(b) b; [2]

(c) d. [2]

[Maximum mark: 6]

The following diagram shows the curve $y = a\cos(k(x-d)) + c$ where a, k, d and c are all positive constants. The curve has a minimum point at $(1.5, 2)$ and a maximum point at $(3.5, 7)$.



(a) Write down the value of a and the value of c. [2]

(b) Find the value of k. [2]

(c) Find the smallest possible value of d, given $d > 0$. [2]

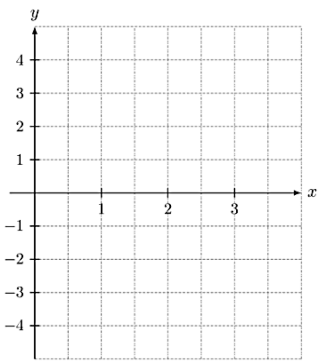
[Maximum mark: 6]

Let $f(x) = 3\sin(\pi x) + 1$.

(a) Write down the amplitude of f.

(b) Find the period of f.

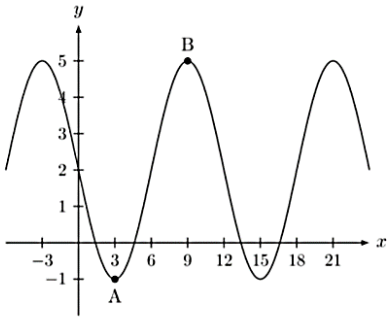
(c) On the following grid, sketch the graph of f, for $0 \le x \le 4$.



[Maximum mark: 8]

Let $f(x) = p\sin(qx) + r$, for $x \in \mathbb{R}$.

Part of the graph of f is shown on the diagram below.



The graph of f has a minimum at A$(3, -1)$ and a maximum at B$(9, 5)$.

(a) (i) Find the period of f.

(ii) Hence find the value of q. [3]

(b) Find the values of:

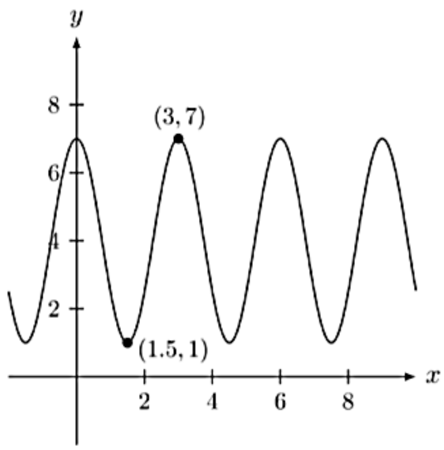
(i) p;

(ii) r. [3]

(c) Solve $f(x) = 4$, for $0 \leq x \leq 15$. [2]

[Maximum mark: 6]

The following diagram shows the curve $y = a\sin(k(x-d)) + c$ where a, k, d and c are all positive constants. The curve has a minimum point at $(1.5, 1)$ and a maximum point at $(3, 7)$.



(a) Write down the value of a and the value of c. [2]

(b) Find the value of k. [2]

(c) Find the smallest possible value of d, given $d > 0$. [2]

[Maximum mark: 7]

Let $f(x) = \sin\left(x + \frac{\pi}{6}\right) + q$. The graph of f passes through the point $\left(\frac{\pi}{3}, 5\right)$.

(a) Find the value of q. [3]

(b) Find the maximum value of f. [2]

Let $g(x) = \sin x$. The graph of g is translated to the graph of f by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

(c) Write down the values of a and b. [2]

MEDIUM

[Maximum mark: 5]

The height, h metres, of a seat on a Ferris wheel after t minutes is given by

$$h(t) = -23.5\cos(0.4t) + 25, \text{ for } t \geq 0.$$

(a) Find the initial height of the seat. [2]

Once a passenger's seat is more than 30 m above the ground, there are no trees in view and they can take unobstructed photographs of a nearby city.

(b) Given that passengers only complete one rotation on the Ferris wheel, calculate how long they can take unobstructed photographs of the nearby city. [3]

[Maximum mark: 7]

A ball on an elastic string is attached to the ceiling. It is pulled down and released. As the ball bounces up and down, the length of the elastic string, L cm, is modelled by the function $L = 80 + 30\cos(\pi t)$, where t is the time in seconds after release.

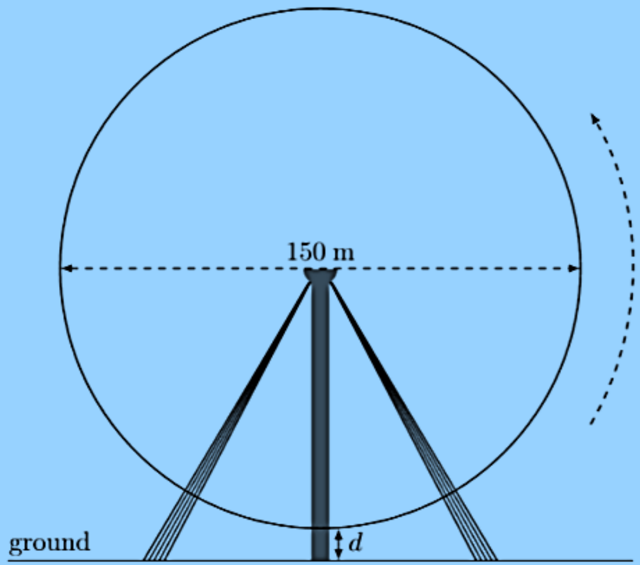
(a) Find the length of the string after 1.5 seconds. [2]

(b) Find the minimum length of the string. [2]

(c) Find the first time after release that the string is 70 cm. [3]

[Maximum mark: 7]

The Singapore Flyer is a giant observation wheel in Singapore with diameter of 150 metres. The wheel rotates at a constant speed and completes one rotation in 32 minutes. The bottom of the wheel is d metres above the ground.



A seat starts at the bottom of the wheel.

(a) After 16 minutes, the seat is 165 metres above the ground. Find d. [2]

After t minutes, the height of the seat above the ground is given by

$$h(t) = 90 + a\cos\left(\frac{\pi}{16}t\right), \quad \text{for} \quad 0 \le t \le 64.$$

(b) Find the value of a. [2]

(c) Find when the seat is 60 metres above the ground for the third time. [3]

[Maximum mark: 14]

Let $f(x) = 3\cos(2x) + 5$, for $x \in \mathbb{R}$.

The range of f is $p \leq f(x) \leq q$.

(a) Find the value of p and the value of q. [3]

Let $g(x) = 4f(3x)$, for $x \in \mathbb{R}$.

(b) Find the range of g. [3]

The function g can be written in the form $g(x) = 12\cos(kx) + c$.

(c) (i) Find the value of k and the value of c.

(ii) Find the period of g. [5]

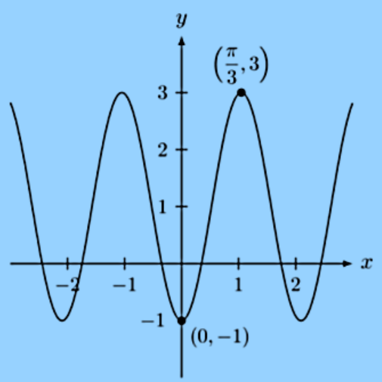
The equation $g(x) = 10$ has two solutions for $\frac{2\pi}{3} \leq x \leq \pi$.

(d) Find both solutions. [3]

[Maximum mark: 6]

A function is defined by $f(x) = a\cos(kx) + c$, for $-\pi \leq x \leq \pi$, where $k > 0$.

Part of the graph of $y = f(x)$ is shown on the diagram below.



(a) Find the value of:

(i) a;

(ii) k;

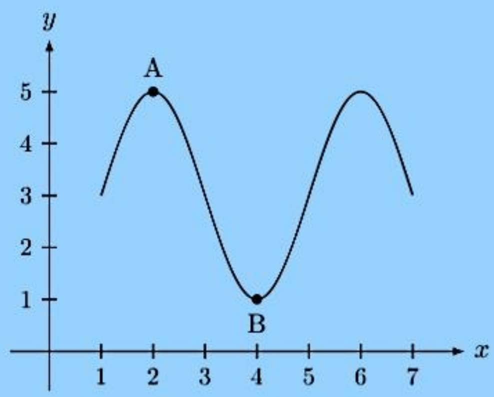
(iii) c. [4]

(b) Solve $f(x) = 2$, for $0 \leq x \leq \pi$. [2]

[Maximum mark: 9]

Let $f(x) = a\sin(k(x-d)) + c$, for $1 \leq x \leq 7$.

The graph of f is shown on the diagram below.



The graph of f has a maximum at A$(2, 5)$ and a minimum at B$(4, 1)$.

(a) Find the value of:

(i) a;

(ii) c. [3]

(b) (i) Find the value of k.

(ii) Find the smallest possible value of d, given $d > 0$. [4]

The graph of f is translated by a vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ to give the graph of a function g such that $g(x) = 3$ has only one solution in the given domain.

(c) Find the value of b. [2]

[Maximum mark: 13]

Let $f(x) = \sin\left(\frac{\pi}{3}x\right) + \cos\left(\frac{\pi}{3}x\right)$, for $0 \leq x \leq 8$.

(a) Find the values of x where f has a positive rate of change. [4]

The function f can be written in the form $f(x) = a\cos\left(\frac{\pi}{3}(x - d)\right)$ where $6 \leq d \leq 9$.

(b) Find the value of:

(i) a;

(ii) d. [5]

(c) Solve $f(x) = 1$, for $0 \leq x \leq 8$. [4]

[Maximum mark: 8]

A particle moves in a straight line with velocity $v(t) = 2\sin 2t + 1$, for $t \geq 0$, where v is in ms^{-1} and t in seconds.

(a) Find the initial velocity of the particle. [2]

(b) Find the value of t when the particle is first at rest. [3]

(c) Find the value of t when the particle first reaches its maximum velocity. [3]

DIFFICULT

[Maximum mark: 15]

Consider the function $f(x) = \sin x$, for $x \in \mathbb{R}$, where x is in radians.

(a) Write down:

(i) the maximum value of f;

(ii) the smallest positive value of x for which the maximum of f occurs. [3]

Let $g(x) = 2\sin\left(x + \frac{\pi}{4}\right)$, for $x \in \mathbb{R}$, where x is in radians.

(b) (i) Determine the two transformations the graph of f undergoes to form the graph of g.

(ii) Hence find the maximum value of g and the smallest positive value of x for which this maximum occurs. [4]

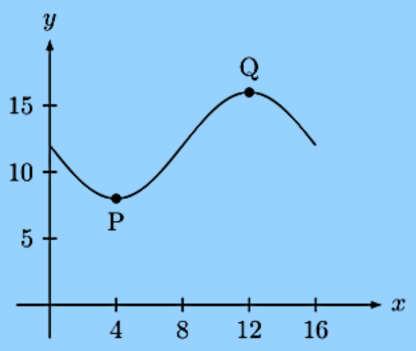
Let $h(x) = \dfrac{4}{2\sin\left(x + \frac{\pi}{4}\right) - 3}$, for $x \in \mathbb{R}$, where x is in radians.

(c) Determine if the graph of h has a vertical asymptote. Justify your answer. [4]

(d) Find the range of h. [4]

[Maximum mark: 15]

The following diagram shows the graph of $f(x) = a\sin kx + c$, for $0 \le x \le 16$.



The graph of f has a minimum at P$(4, 8)$ and a maximum at Q$(12, 16)$.

(a) (i) Find the value of c.

(ii) Show that $k = \frac{\pi}{8}$.

(iii) Find the value of a. [6]

The graph of g is obtained from the graph of f by a translation of $\begin{pmatrix} d \\ 0 \end{pmatrix}$.

The minimum point on the graph of g has coordinates $(6.5, 8)$.

(b) (i) Write down the value of d.

(ii) Find $g(x)$. [3]

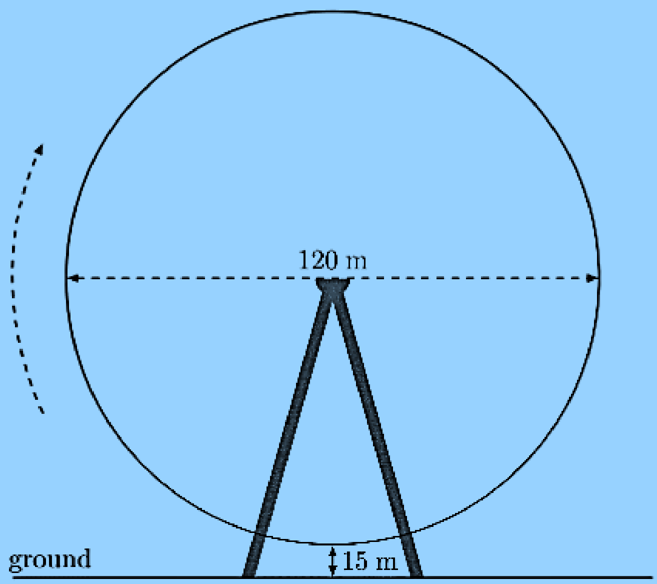
The graph of g changes from concave-up to concave-down when $x = \nu$.

(c) (i) Find ν.

(ii) Hence, or otherwise, find the maximum positive rate of change of g. [6]

[Maximum mark: 16]

The London Eye is an observation wheel in England with diameter of 120 metres. The wheel rotates at a constant speed and completes 2.5 rotations every hour. The bottom of the wheel is 15 metres above the ground.



A seat starts at the bottom of the wheel.

(a) Find the maximum height above the ground of the seat. [2]

After t minutes, the height h metres above the ground of the seat is given by

$$h(t) = 75 + a\cos(bt), \ b > 0.$$

(b) (i) Show that the period of h is 24 minutes.

(ii) Write down the **exact** value of b. [2]

(c) Find the value of a. [3]

(d) Sketch the graph of h, for $0 \leq t \leq 48$. [4]

(e) In one rotation of the wheel, find the probability that a randomly selected seat is at least 110 metres above the ground. [5]

[Maximum mark: 10]

Let $g(x) = -4x - \frac{\pi}{3}$ and $h(x) = 5\cos x - 1$, for $x \in \mathbb{R}$.

(a) Show that $(h \circ g)(x) = 5\cos\left(-4x - \frac{\pi}{3}\right) - 1$. [1]

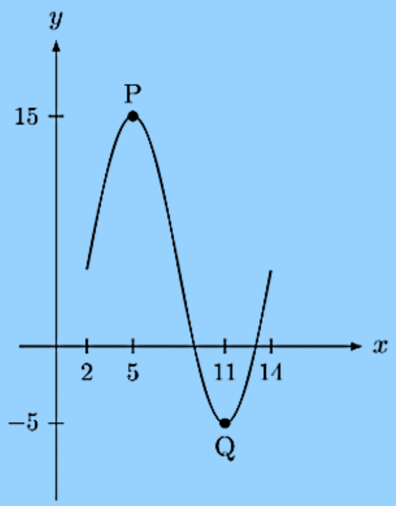
(b) Find the range of $h \circ g$. [2]

(c) Given that $(h \circ g)\left(\frac{5\pi}{12}\right) = 4$, find the next value of x, greater than $\frac{5\pi}{12}$, for which $(h \circ g)(x) = 4$. [2]

(d) The graph of $y = (h \circ g)(x)$ can be obtained by applying five transformations the graph of $y = \cos x$. State what the five transformations represent geometrically and give the order in which they are applied. [5]

[Maximum mark: 16]

The diagram below shows the graph of $f(x) = a\sin(k(x - d)) + c$, for $2 \leq x \leq 14$.



The graph of f has a maximum at $\mathrm{P}(5, 15)$ and a minimum at $\mathrm{Q}(11, -5)$.

(a) Write down the value of:

(i) a;

(ii) c. [3]

(b) (i) Show that $k = \frac{\pi}{6}$.

(ii) Find the smallest possible value of d, given $d > 0$. [4]

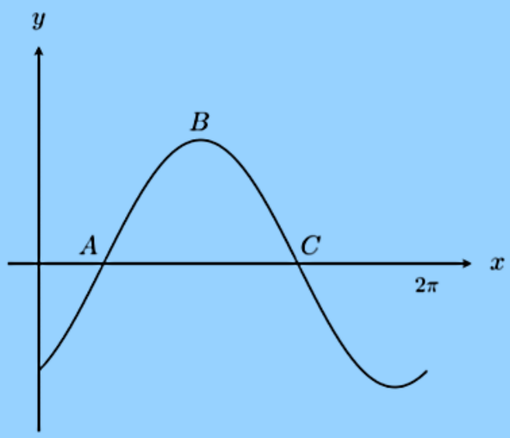
(c) Find $f'(x)$. [3]

(d) At a point R, the gradient is $-\frac{5\pi}{3}$. Find the x-coordinate of R. [6]

[Maximum mark: 14]

Let $f(x) = \sin x - \sqrt{3}\cos x$, $0 \leq x \leq 2\pi$.

The following diagram shows the graph of f.



The curve crosses the x-axis at A and C and has a maximum at point B.

(a) Find the exact coordinates of A and of C. [6]

(b) Find $f'(x)$. [2]

(c) Find the coordinates of B. [6]

[Maximum mark: 5]

Let $f(x) = \tan(x - \pi)\sin\left(x + \frac{\pi}{2}\right)$ where $0 < x < \frac{\pi}{2}$.

Express $f(x)$ in terms of $\sin x$ and/or $\cos x$.

[Maximum mark: 13]

A function, f, is defined by $f(x) = 6.2\sin\left(\frac{\pi}{9}(x - 7.5)\right) + c$, for $0 \le x \le 15$, where $c \in \mathbb{R}$.

(a) Find the period of f. [2]

The function f has a minimum at $\mathrm{A}(3, 11.8)$ and a maximum at $\mathrm{B}(12, 24.2)$.

(b) (i) Find the value of c.

(ii) Hence find the value of $f(9)$. [4]

A second function, g, is defined by $g(x) = a\sin\left(\frac{2\pi}{15}(x + 2.25)\right) + b$, for $0 \le x \le 15$, where $a, b \in \mathbb{R}$.

The function g passes through the points $\mathrm{P}(1.5, 14.5)$ and $\mathrm{Q}(14, 10.2)$.

(c) Find the value of a and the value of b. [5]

(d) Find the value of x for which the functions have the greatest difference. [2]

[Maximum mark: 13]

A function, f, is defined by $f(x) = 6.2\sin\left(\frac{\pi}{9}(x - 7.5)\right) + c$, for $0 \le x \le 15$, where $c \in \mathbb{R}$.

(a) Find the period of f. [2]

The function f has a minimum at $\mathrm{A}(3, 11.8)$ and a maximum at $\mathrm{B}(12, 24.2)$.

(b) (i) Find the value of c.

(ii) Hence find the value of $f(9)$. [4]

A second function, g, is defined by $g(x) = a\sin\left(\frac{2\pi}{15}(x + 2.25)\right) + b$, for $0 \le x \le 15$, where $a, b \in \mathbb{R}$.

The function g passes through the points $\mathrm{P}(1.5, 14.5)$ and $\mathrm{Q}(14, 10.2)$.

(c) Find the value of a and the value of b. [5]

(d) Find the value of x for which the functions have the greatest difference. [2]

Mark Scheme

(a) We have

$$\frac{\pi}{9} = \frac{2\pi}{\text{period}}$$

$$\boxed{\text{period} = 18}$$

(b) (i) We have

$$c = \frac{f_{\max} + f_{\min}}{2} = \frac{24.2 + 11.8}{2}$$

$$\boxed{= 18}$$

(ii) Evaluating $f(x)$ for $x = 9$, we get

$$f(9) = 6.2\sin((\pi/9)(9 - 7.5)) + 18$$

$$\boxed{= 21.1}$$

(c) Using the coordinates of points P and Q, we have

$$g(1.5) = 14.5$$

$$g(14) = 10.2$$

or

$$a\sin((2\pi/15)(1.5 + 2.25)) + b = 14.5 \quad (1)$$

$$a\sin((2\pi/15)(14 + 2.25)) + b = 10.2 \quad (2)$$

Hence, solving the system of linear equations (1)-(2) for a and b, we obtain

$\boxed{a = 8.6}$ and $\boxed{b = 5.9}$ [by using G.D.C.]

[Maximum mark: 15]

The first two terms of an infinite geometric sequence are $u_1 = 20$ and $u_2 = 16\sin^2\theta$, where $0 < \theta < 2\pi$, and $\theta \neq \pi$.

(a) (i) Find an expression for r in terms of θ.

(ii) Find the possible values of r. [5]

(b) Show that the sum of the infinite sequence is $\dfrac{100}{3 + 2\cos 2\theta}$. [4]

(c) Find the values of θ which give the greatest value of the sum. [6]