Combinatorics *[46 marks]*

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

1a. Find the number of ways these five people can be seated in this row. *[3 marks]*

Markscheme

$6 \times 5!$ ***(A1)(A1)***

$= 720$ (accept $6!$) ***A1***

[3 marks]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

1b. Find the number of ways these five people can now be seated in this row. *[4 marks]*

Markscheme

METHOD 1

(Peter apart from girls, in an end seat) $^{8}P_{4}(= 1680)$ OR

(Peter apart from girls, not in end seat) $^{7}P_{4}(= 840)$ ***(A1)***

case 1: Peter at either end

$2 \times {}^{8}P_{4}(= 3360)$ OR $2 \times {}^{8}C_{4} \times 4!(= 3360)$ ***(A1)***

case 2: Peter not at the end

$8 \times {}^{7}P_{4}(= 6720)$ OR $8 \times {}^{7}C_{4} \times 4!(= 6720)$ ***(A1)***

Total number of ways $= 3360 + 6720$

$= 10080$ ***A1***

METHOD 2

(Peter next to girl, in an end seat) $4 \times {}^{8}P_{3}(= 1344)$ OR

(Peter next to one girl, not in end seat) $2 \times 4 \times {}^{7}P_{3}(= 1680)$ OR

(Peter next to two girls, not in end seat) $4 \times 3 \times {}^{7}P_{2}(= 504)$ ***(A1)***

case 1: Peter at either end

$2 \times 4 \times {}^{8}P_{3}(= 2688)$ ***(A1)***

case 2: Peter not at the end

$8\left(2 \times 4 \times {}^{7}P_{3} + 4 \times 3 \times {}^{7}P_{2}\right)(= 17472)$ ***(A1)***

Total number of ways $= {}^{10}P_{5} - (2688 + 17472)$

$= 10080$ ***A1***

[4 marks]

Consider the set of six-digit positive integers that can be formed from the digits $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$ and 9.

Find the total number of six-digit positive integers that can be formed such that

2a. the digits are distinct. *[2 marks]*

Markscheme

$9 \times 9 \times 8 \times 7 \times 6 \times 5 \; (= 9 \times {}_9P_5)$ ***(M1)***

$= 136080 \; \left(= 9 \times \frac{9!}{4!}\right)$ ***A1***

Note: Award ***M1A0*** for $10 \times 9 \times 8 \times 7 \times 6 \times 5 \; \left(= {}_{10}P_6 = 151200 = \frac{10!}{4!}\right)$.

Note: Award ***M1A0*** for ${}_9P_6 = 60480$

[2 marks]

2b. the digits are distinct and are in increasing order. *[2 marks]*

Markscheme

METHOD 1

EITHER

every unordered subset of 6 digits from the set of 9 non-zero digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{6}(\times 1)$ ***A1***

THEN

$= 84$ ***A1***

METHOD 2

EITHER

removes 3 digits from the set of 9 non-zero digits and these 6 remaining digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{3}(\times 1)$ ***A1***

THEN

$= 84$ ***A1***

[2 marks]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.

Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

3a. Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. *[4 marks]*

Markscheme

METHOD 1

B has one less pen to select ***(M1)***

EITHER

A and B can be placed in 6×5 ways ***(A1)***

C, D, E have 6 choices each ***(A1)***

OR

A (or B), C, D, E have 6 choices each ***(A1)***

B (or A) has only 5 choices ***(A1)***

THEN

$5 \times 6^4 (= 6480)$ ***A1***

METHOD 2

total number of ways $= 6^5$ ***(A1)***

number of ways with Amber and Brownie together $= 6^4$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

$6^5 - 6^4$

$= 5 \times 6^4 (= 6480)$ ***A1***

[4 marks]

3b. Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. *[4 marks]*

Markscheme

METHOD 1

total number of ways $= 6!(= 720)$ ***(A1)***

number of ways with Amber and Brownie sharing a boundary

$= 2 \times 7 \times 4!(= 336)$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

$720 - 336 = 384$ ***A1***

METHOD 2

case 1: number of ways of placing A in corner pen

$3 \times 4 \times 3 \times 2 \times 1$

Four corners total no of ways is $4 \times (3 \times 4 \times 3 \times 2 \times 1) = 12 \times 4!(= 288)$ ***(A1)***

case 2: number of ways of placing A in the middle pen

$2 \times 4 \times 3 \times 2 \times 1$

two middle pens so $2 \times (2 \times 4 \times 3 \times 2 \times 1) = 4 \times 4!(= 96)$ ***(A1)***

attempt to add (may be seen in words) ***(M1)***

total no of ways $= 288 + 96$

$= 16 \times 4!(= 384)$ ***A1***

[4 marks]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

4a. in the position immediately after Andrea. *[2 marks]*

Markscheme

Jack and Andrea finish in that order (as a unit) so we are considering the arrangement of 7 objects ***(M1)***

$7!(= 5040)$ ways ***A1***

[2 marks]

4b. in any position after Andrea. *[3 marks]*

Markscheme

METHOD 1

the number of ways that Andrea finishes in front of Jack is equal to the number of ways that Jack finishes in front of Andrea ***(M1)***

total number of ways is 8! ***(A1)***

$\frac{8!}{2}(= 20160)$ ways ***A1***

METHOD 2

the other six runners can finish in $6!(= 720)$ ways ***(A1)***

when Andrea finishes first, Jack can finish in 7 different positions

when Andrea finishes second, Jack can finish in 6 different positions etc

$7 + 6 + 5 + 4 + 3 + 2 + 1(= 28)$ ways ***(A1)***

hence there are $(7 + 6 + 5 + 4 + 3 + 2 + 1) \times 6!$ ways

$28 \times 6!(= 20160)$ ways ***A1***

[3 marks]

A team of four is to be chosen from a group of four boys and four girls.

5a. Find the number of different possible teams that could be chosen. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$\binom{8}{4}$ ***(A1)***

$= \frac{8!}{4!4!} = \frac{8\times7\times6\times5}{4\times3\times2\times1} = 7 \times 2 \times 5$ ***(M1)***

= 70 ***A1***

METHOD 2

recognition that they need to count the teams with 0 boys, 1 boy… 4 boys ***M1***

$1 + \binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} + 1$

$= 1 + (4 \times 4) + (6 \times 6) + (4 \times 4) + 1$ ***(A1)***

= 70 ***A1***

[3 marks]

5b. Find the number of different possible teams that could be chosen, given that the team must include at least one girl and at least one boy. *[2 marks]*

Markscheme

EITHER

recognition that the answer is the total number of teams minus the number of teams with all girls or all boys ***(M1)***

70 − 2

OR

recognition that the answer is the total of the number of teams with 1 boy,

2 boys, 3 boys ***(M1)***

$$\binom{4}{1}\times\binom{4}{3}+\binom{4}{2}\times\binom{4}{2}+\binom{4}{1}\times\binom{4}{3}=(4\times4)+(6\times6)+(4\times4)$$

THEN

= 68 ***A1***

[2 marks]

6a. Express the binomial coefficient $\binom{3n+1}{3n-2}$ as a polynomial in n. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\binom{3n+1}{3n-2}=\frac{(3n+1)!}{(3n-2)!3!}$ ***(M1)***

$=\frac{(3n+1)3n(3n-1)}{3!}$ **A1**

$=\frac{9}{2}n^3-\frac{1}{2}n$ or equivalent ***A1***

[3 marks]

6b. Hence find the least value of n for which $\binom{3n+1}{3n-2}>10^6$. *[3 marks]*

Markscheme

attempt to solve $= \frac{9}{2}n^3 - \frac{1}{2}n > 10^6$ ***(M1)***

$n > 60.57\ldots$ ***(A1)***

Note: Allow equality.

$\Rightarrow n = 61$ ***A1***

[3 marks]

7. Three girls and four boys are seated randomly on a straight bench. Find the probability that the girls sit together and the boys sit together. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

total number of arrangements 7! ***(A1)***

number of ways for girls and boys to sit together $= 3! \times 4! \times 2$ ***(M1)(A1)***

Note: Award ***M1A0*** if the 2 is missing.

probability $\frac{3! \times 4! \times 2}{7!}$ ***M1***

Note: Award ***M1*** for attempting to write as a probability.

$\frac{2 \times 3 \times 4! \times 2}{7 \times 6 \times 5 \times 4!}$

$= \frac{2}{35}$ ***A1***

Note: Award ***A0*** if not fully simplified.

METHOD 2

$\frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}$ ***(M1)A1A1***

Note: Accept $\frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} \times 2$ or $\frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times 2$.

$= \frac{2}{35}$ ***(M1)A1***

Note: Award ***A0*** if not fully simplified.

[5 marks]

8. In a trial examination session a candidate at a school has to take 18 examination papers including the physics paper, the chemistry paper and the biology paper. No two of these three papers may be taken consecutively. There is no restriction on the order in which the other examination papers may be taken. *[6 marks]*

Find the number of different orders in which these 18 examination papers may be taken.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

consideration of all papers

all papers may be sat in 18! ways ***A1***

number of ways of positioning "pairs" of science subjects $= 3! \times 17!$ ***A1***

but this includes two copies of each "triple" ***(R1)***

number of ways of positioning "triplets" of science subjects $= 3! \times 16!$ ***A1***

hence number of arrangements is $18! - 3! \times 17! + 3! \times 16!$ ***M1A1***

$(= 4.39 \times 10^{15})$

METHOD 2

consideration of all the non-science papers ***(M1)***

hence all non-science papers can be sat in 15! ways ***A1***

there are $16 \times 15 \times 14\ (= 3360)$ ways of positioning the three science papers ***(M1)A1***

hence the number of arrangements is $16 \times 15 \times 14 \times 15!\ (= 4.39 \times 10^{15})$ ***(M1)A1***

METHOD 3

consideration of all papers

all papers may be sat in 18! ways ***A1***

number of ways of positioning exactly two science subjects $= 3! \times 15! \times 16 \times 15$ ***M1A1***

number of ways of positioning "triplets" of science subjects $= 3! \times 16!$ ***A1***

hence number of arrangements is $18! - 3! \times 16! - 3! \times 15! \times 16 \times 15$ ***M1A1***

$(= 4.39 \times 10^{15})$

[6 marks]