

Mixed examination practice 15

Short questions

- ✘ 1. Express $z = 3i - \frac{2}{i + \sqrt{3}}$ in the form $x + iy$. [5 marks]
2. If z and w are complex numbers, solve the simultaneous equations:
$$3z + w = 9 + 11i$$
$$iw - z = -8 - 2i$$
 [5 marks]
3. $f(z) = z^3 + az^2 + bz + c$ where a , b and c are real constants. Two roots of $f(z) = 0$ are $z = 1$ and $z = 1 + 2i$. Find a , b and c . [6 marks]
- ✘ 4. Find the complex number z such that $3z - 5z^* = 4 - 3i$. [4 marks]
- ✘ 5. Find the exact value of $\frac{1}{(\sqrt{3} + i)^6}$. [6 marks]
6. The polynomial $z^3 + az^2 + bz - 65$ has a factor of $(z - 2 - 3i)$. Find the values of the real constants a and b . [6 marks]
7. If $w = 1 + \sqrt{3}i$ and $z = 1 + i$ show that $\operatorname{Re}\left(\frac{w + \sqrt{2}z}{w - \sqrt{2}z}\right) = 0$. [6 marks]
- ✘ 8. If $z = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ evaluate $\left(\frac{z}{w}\right)^6$ leaving your answer in a simplified form. [6 marks]
9. If $\arg((a + i)^3) = \pi$ and a is real and positive, find the exact value of a . [6 marks]
10. Let z and w be complex numbers satisfying $\frac{w + i}{w - i} = \frac{z + 1}{z - 1}$.
(a) Express w in terms of z .
(b) Show that if $\operatorname{Im}(z) = 0$ then $\operatorname{Re}(w) = 0$. [6 marks]
11. If $|z + 2i| = |z - 6i|$ find the imaginary part of z . [6 marks]
12. If $|z + 25| = 5|z + 1|$ find $|z|$. [6 marks]
13. (a) The equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots $\frac{1}{3}, \frac{2}{3}, 1, 1$ and 3 . Find the value of a .
(b) Let $1, \omega_1, \omega_2, \omega_3, \omega_4$ be the roots of the equation $z^5 = 1$. Find the value of $\omega_1 + \omega_2 + \omega_3 + \omega_4$. [5 marks]

14. (a) Show that $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$.
 (b) Let α and β be the roots of the quadratic equation $x^2 + 7x + 2 = 0$.
 Find a quadratic equation with roots α^3 and β^3 . [7 marks]
15. By considering the product $(2 + i)(3 + i)$ show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$
 [6 marks]
16. If $0 < \theta < \frac{\pi}{2}$ and $z = (\sin \theta + i(1 - \cos \theta))^2$ find in its simplest form $\arg z$. [6 marks]
17. Let z and w be complex numbers such that $w = \frac{1}{1-z}$ and $|z|^2 = 1$.
 Find the real part of w . [6 marks]
18. If $z = \text{cis } \theta$ prove that $\frac{z^2 - 1}{z^2 + 1} = i \tan \theta$. [6 marks]
19. $w = \frac{kz}{z^2 + 1}$ where $z^2 \neq -1$. If $\text{Im}(w) = \text{Im}(k) = 0$ and $\text{Im}(z) \neq 0$
 prove that $|z| = 1$. [6 marks]

Long questions

1. Let $z_1 = \frac{\sqrt{6} - i\sqrt{2}}{2}$, and $z_2 = 1 - i$.
- (a) Write z_1 and z_2 in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- (b) Show that $\frac{z_1}{z_2} = \cos \frac{\pi}{12} + i \sin \frac{\pi}{12}$.
- (c) Find the value of $\frac{z_1}{z_2}$ in the form $a + bi$, where a and b are to be determined exactly in radical (surd) form. Hence or otherwise find the exact values of $\cos \frac{\pi}{12}$ and $\sin \frac{\pi}{12}$. [8 marks]
 (© IB Organization 1999)
2. (a) Express $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ in the form $r(\cos \theta + i \sin \theta)$.
- (b) Hence show that $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^9 = ci$ where c is a real number to be found.
- (c) Find one pair of possible values of positive integers m and n such that:

$$\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^m = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^n$$
 [8 marks]

3. Let $z = \cos \theta + i \sin \theta$, for $-\frac{\pi}{4} < \theta \leq \frac{\pi}{4}$.

(a) (i) Find z^3 using the binomial theorem.

(ii) Use De Moivre's theorem to show that:

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \text{ and } \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

(b) Hence prove that $\frac{\sin 3\theta - \sin \theta}{\cos 3\theta + \cos \theta} = \tan \theta$.

(c) Given that $\sin \theta = \frac{1}{3}$, find the exact value of $\tan 3\theta$.

[10 marks]

(© IB Organization 2006)

4. If ω is a complex third root of unity and x and y are real numbers prove that:

(a) $1 + \omega + \omega^2 = 0$.

(b) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) = x^2 - xy + y^2$. [7 marks]

5. (a) A cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots x_1, x_2, x_3 .

(i) Write down the values of $x_1 + x_2 + x_3$ and $x_1 x_2 x_3$ in terms of a, b, c and d .

(ii) Show that $x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{c}{a}$.

(b) The roots α, β and γ of the equation $2x^3 + bx^2 + cx + 16 = 0$ form a geometric progression.

(i) Show that $\beta = -2$.

(ii) Show that $c = 2b$. [14 marks]

6. Let $z = \cos \theta + i \sin \theta$.

(a) Show that $2 \cos \theta = z + \frac{1}{z}$.

(b) Show that $2 \cos n\theta = z^n + \frac{1}{z^n}$.

(c) Consider the equation $3z^4 - z^3 + 2z^2 - z + 3 = 0$.

(i) Show that the equation can be written as $6 \cos 2\theta - 2 \cos \theta + 2 = 0$.

(ii) Find all four complex roots of the original equation. [7 marks]

7. Let $\omega = e^{\frac{2i\pi}{5}}$.

(a) Write ω^2, ω^3 and ω^4 in the form $e^{i\theta}$.

(b) Explain why $\omega^1 + \omega^2 + \omega^3 + \omega^4 = -1$.

(c) Show that $\omega + \omega^4 = 2 \cos\left(\frac{2\pi}{5}\right)$ and $\omega^2 + \omega^3 = 2 \cos\left(\frac{4\pi}{5}\right)$.

(d) Form a quadratic equation in $\cos\left(\frac{2\pi}{5}\right)$ and hence show that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}. \quad [10 \text{ marks}]$$

8. (a) By considering $(\cos\theta + i\sin\theta)^3$ find the expressions for $\cos(3\theta)$ and $\sin(3\theta)$.

(b) Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(c) Hence show that $\tan\left(\frac{\pi}{12}\right)$ is a root of the equation $x^3 - 3x^2 - 3x + 1 = 0$.

(d) Show that $(x-1)$ is a factor of $x^3 - 3x^2 - 3x + 1$ and hence find the exact solutions of the equation $x^3 - 3x^2 - 3x + 1 = 0$.

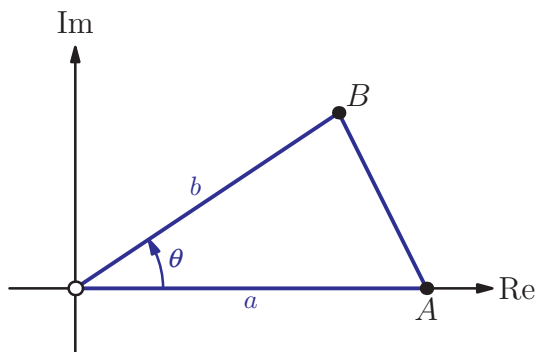
(e) By considering $\tan\left(\frac{\pi}{4}\right)$ explain why $\tan\left(\frac{\pi}{12}\right) < 1$.

(f) Hence state the exact value of $\tan\left(\frac{\pi}{12}\right)$. [14 marks]

9. (a) Points P and Q in the Argand diagram correspond to complex numbers

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2. \text{ Show that } PQ = |z_1 - z_2|.$$

(b) The diagram shows a triangle with one vertex at the origin, one at the point $A(a, 0)$ and one at the point B such that $OB = b$ and $\angle AOB = \theta$.



(i) Write down the complex number corresponding to point A .

(ii) Write down the number corresponding to point B in polar form.

(iii) Write down an expression for the length of AB in terms of a , b and θ .

(iv) Hence prove the cosine rule for the triangle AOB :

$$|AB|^2 = |OA|^2 + |OB|^2 + 2|OA||OB|\cos\theta \quad [13 \text{ marks}]$$

3. $-2, 1 + \sqrt{3}i, 1 - \sqrt{3}i$

4. $-\frac{\sqrt{2} + \sqrt{6}}{2} + \left(\frac{\sqrt{2} + \sqrt{6}}{2}\right)i,$

$\frac{\sqrt{2} - \sqrt{6}}{2} - \left(\frac{\sqrt{2} + \sqrt{6}}{2}\right)i,$

$\sqrt{2} - \sqrt{2}i$

5. $3\text{cis}\left(\frac{3\pi}{8}\right), 3\text{cis}\left(\frac{7\pi}{8}\right),$

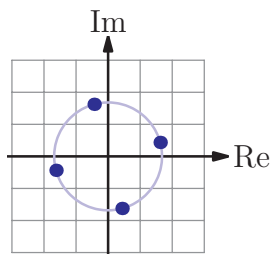
$3\text{cis}\left(\frac{-\pi}{8}\right), 3\text{cis}\left(\frac{-5\pi}{8}\right)$

6. (a) $8e^{i\frac{\pi}{3}}$

(b) $8^4 e^{i\frac{\pi}{12}}, 8^4 e^{i\frac{7\pi}{12}},$

$8^4 e^{i\frac{13\pi}{12}}, 8^4 e^{i\frac{19\pi}{12}}$

(c)



7. (a) $\sqrt{2} \pm \sqrt{2}i, -\sqrt{2} \pm \sqrt{2}i$

(b) $(z^2 + 2\sqrt{2}z + 4)$

$(z^2 - 2\sqrt{2}z + 4)$

ANSWER HINT(7)

$(1 + w_1 + w_2 + w_3 + w_4 + w_5) \times w$

9. $a^2 + b^2 - ab$

10. (a) $-1, e^{\frac{i\pi}{3}}, e^{-\frac{i\pi}{3}}$

(b) $x^3 + 6x^2 + 12x + 8$

(c) $-3, -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$

Exercise 15I

1. $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$

2. (a) $\cos^4 \theta - 6\cos^2 \theta \sin^2 \theta + \sin^4 \theta$

(b) $\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$

3. (b) $A = 2, B = 10, C = 20$

4. (b) $(z + z^{-1})^6 = z^6 + 6z^4 + 1$

$5z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$

$(z - z^{-1})^6 = z^6 - 6z^4 +$

$15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}$

5. (a) $\text{Re} : \cos^5 \theta$

$-10\cos^3 \theta$

$\text{Im} : 5\cos^4 \theta \sin \theta - 10\cos^2 \theta$

$\sin^3 \theta + \sin^5 \theta$

(c) 5

Mixed examination practice 15

Short questions

1. $-\frac{\sqrt{3}}{2} + \frac{7}{2}i$

2. $w = 5i, z = 3 + 2i$

3. $a = -3, b = 7, c = -5$

4. $-2 - \frac{3}{8}i$

5. $-\frac{1}{64}$

6. $a = -9, b = 33$

8. $64i$

9. $\frac{1}{\sqrt{3}}$

10. (a) $w = iz$

11. 2

12. 5

13. (a) -6

(b) -1

14. (b) $x^2 + 301x + 8 = 0$

16. $\tan \theta$

17. $\frac{1}{2}$

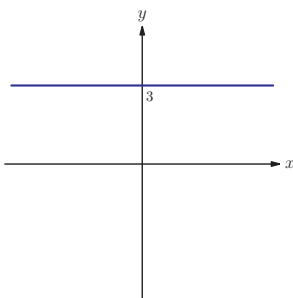
Long questions

1. (a) $z_1 = \sqrt{2} \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right)$
 $z_2 = \sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right)$
- (c) $\cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$, $\sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$
2. (a) $\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)$
 (b) $c = 1$
 (c) $m = -6$, $n = 4$
3. (a) (i) $Z^3 \cos^3 \theta - 3 \cos \theta \sin^2 \theta +$
 $(3 \cos^2 \theta \sin \theta - \sin^3 \theta) i$
 (c) $\frac{23\sqrt{2}}{20}$
5. (a) (i) $x_1 + x_2 + x_3 = -\frac{b}{a}$, $x_1 x_2 + x_2 x_3 + x_3 x_1 = -\frac{d}{a}$
6. (c) (ii) $\frac{2}{3} \pm \frac{\sqrt{5}}{3} i$, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
7. (a) $\omega^2 = e^{\frac{4\pi i}{5}}$, $\omega^3 = e^{-\frac{4\pi i}{5}}$, $\omega^4 = e^{-\frac{2\pi i}{5}}$
 (c) $4 \cos^2\left(\frac{2\pi}{5}\right) + 2 \cos\left(\frac{2\pi}{5}\right) - 1 = 0$
8. (a) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$
 $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$
 (f) $2 - \sqrt{3}$
9. (b) (i) a (ii) $b(\cos \theta + i \sin \theta)$
 (iii) $AB = \sqrt{b^2 \sin^2 \theta + (a - b \cos \theta)^2}$

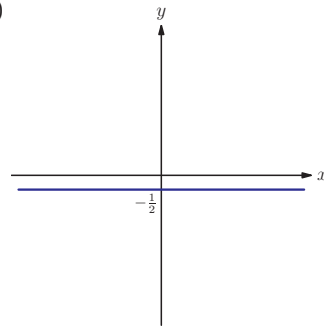
Chapter 16

Exercise 16A

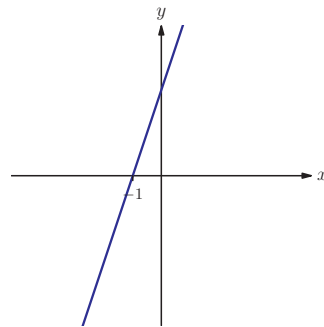
1. (a) (i)



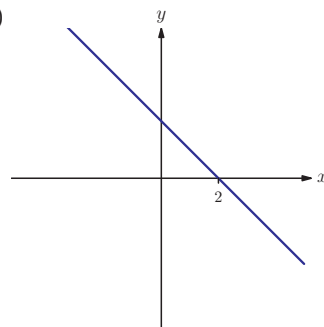
- (ii)



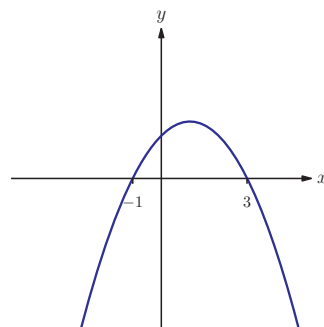
- (b) (i)



- (ii)



- (c) (i)



- (ii)

