Monday 12.12 *[55 marks]*

Let $f(x) = \frac{4x-5}{x^2-3x+2}$ $x \neq 1, x \neq 2$.

1a. Express $f(x)$ in partial fractions. *[6 marks]*

Markscheme

$f(x) = \frac{4x-5}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$ ***M1A1***

$\Rightarrow 4x - 5 \equiv A(x-2) + B(x-1)$ ***M1A1***

$x = 1 \Rightarrow A = 1 \quad x = 2 \Rightarrow B = 3$ ***A1A1***

$f(x) = \frac{1}{x-1} + \frac{3}{x-2}$

[6 marks]

1b. Use part (a) to show that $f(x)$ is always decreasing. *[3 marks]*

Markscheme

$f'(x) = -(x-1)^{-2} - 3(x-2)^{-2}$ ***M1A1***

This is always negative so function is always decreasing. ***R1AG***

[3 marks]

1c. *[4 marks]*

Use part (a) to find the exact value of $\int_{-1}^{0} f(x)dx$, giving the answer in the form $\ln q, \quad q \in \mathbb{Q}$.

Markscheme

$\int_{-1}^{0} \frac{1}{x-1} + \frac{3}{x-2}\, dx = [\ln|x-1| + 3\ln|x-2|]_{-1}^{0}$ ***M1A1***

$= (3\ln 2) - (\ln 2 + 3\ln 3) = 2\ln 2 - 3\ln 3 = \ln\frac{4}{27}$ ***A1A1***

[4 marks]

Consider the series $\ln x + p\ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

2a. Show that $p = \pm\frac{1}{\sqrt{3}}$. *[2 marks]*

Markscheme

EITHER

attempt to use a ratio from consecutive terms ***M1***

$\frac{p \ln x}{\ln x} = \frac{\frac{1}{3}\ln x}{p \ln x}$ OR $\frac{1}{3}\ln x = (\ln x) r^2$ OR $p \ln x = \ln x \left(\frac{1}{3p}\right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} \ldots$ and consider the powers of x in geometric sequence

Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm\frac{1}{\sqrt{3}}$ ***A1***

$p = \pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

[2 marks]

2b. Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. *[3 marks]*

Markscheme

$\frac{\ln x}{1-\frac{1}{\sqrt{3}}}\ \left(=3+\sqrt{3}\right)$ ***(A1)***

$\ln x=3-\frac{3}{\sqrt{3}}+\sqrt{3}-\frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x=3-\sqrt{3}+\sqrt{3}-1\ \left(\Rightarrow \ln x=2\right)$ ***A1***

$x=\mathrm{e}^2$ ***A1***

[3 marks]

Now consider the case where the series is arithmetic with common difference d.

2c. Show that $p=\frac{2}{3}$. *[3 marks]*

Markscheme

METHOD 1

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p\ln x - \ln x = \frac{1}{3}\ln x - p\ln x$ OR $\frac{1}{3}\ln x = \ln x + 2(p\ln x - \ln x)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.

Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$

$2p\ln x = \frac{4}{3}\ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1+u_3}{2}$ ***M1***

$p\ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ ***A1***

$2p\ln x = \frac{4}{3}\ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3}\ln x = \ln x + 2d \ \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p\ln x - \ln x = -\frac{1}{3}\ln x$ ***A1***

$p\ln x = \frac{2}{3}\ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

[3 marks]

2d. Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. *[1 mark]*

Markscheme

$d = -\frac{1}{3} \ln x$ ***A1***

[1 mark]

2e. The sum of the first n terms of the series is $-3 \ln x$. *[6 marks]*

Find the value of n.

Markscheme

METHOD 1

$S_n = \frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right]$

attempt to substitute into S_n and equate to $-3\ln x$ ***(M1)***

$\frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right] = -3\ln x$

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2\ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n\ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right)\ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ or equivalent

Note: Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ ***(M1)***

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2) = 0$

$n = 9$ ***A1***

METHOD 2

listing the first 7 terms of the sequence ***(A1)***

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + \ldots$

recognizing first 7 terms sum to 0 ***M1***

8^{th} term is $-\frac{4}{3}\ln x$ ***(A1)***

9^{th} term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8^{th} and 9^{th} term $= -3\ln x$ ***(A1)***

$n = 9$ ***A1***

[6 marks]

3. Consider the graphs of $y = \frac{x^2}{x-3}$ and $y = m\left(x+3\right)$, $m \in \mathbb{R}$. *[5 marks]*

Find the set of values for m such that the two graphs have no intersection points.

Markscheme

METHOD 1

sketching the graph of $y = \frac{x^2}{x-3}$ $(y = x + 3 + \frac{9}{x-3})$ ***M1***

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining (−3, 0) and (0, 0) ***M1***

so m > 0 ***R1***

hence 0 < m ≤ 1 ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2 = m\left(x^2 - 9\right)$

$\Rightarrow \left(m-1\right)x^2 - 9m = 0$ ***A1***

EITHER

attempting to solve $-4\left(m-1\right)\left(-9m\right) < 0$ for m ***M1***

OR

attempting to solve x^2 < 0 *ie* $\frac{9m}{m-1} < 0$ $\left(m \neq 1\right)$ for m ***M1***

THEN

$\Rightarrow 0 < m < 1$ ***A1***

a valid reason to explain why $m = 1$ gives no solutions *eg* if $m = 1$,

$\left(m-1\right)x^2 - 9m = 0 \Rightarrow -9 = 0$ and so 0 < m ≤ 1 ***R1***

[5 marks]

The function f is defined by $f(x) = \frac{4x+1}{x+4}$, where $x \in \mathbb{R},\ x \neq -4$.

For the graph of f

4a. write down the equation of the vertical asymptote. *[1 mark]*

Markscheme

$x = -4$ ***A1***

[1 mark]

4b. find the equation of the horizontal asymptote. *[2 marks]*

Markscheme

attempt to substitute into $y = \frac{a}{c}$ OR table with large values of x OR sketch of f showing asymptotic behaviour ***(M1)***

$y = 4$ ***A1***

[2 marks]

4c. Find $f^{-1}(x)$. *[4 marks]*

Markscheme

$y = \frac{4x+1}{x+4}$

attempt to interchange x and y (seen anywhere) ***M1***

$xy + 4y = 4x + 1$ OR $xy + 4x = 4y + 1$ ***(A1)***

$xy - 4x = 1 - 4y$ OR $xy - 4y = 1 - 4x$ ***(A1)***

$f^{-1}(x) = \frac{1-4x}{x-4}$ (accept $y = \frac{1-4x}{x-4}$) ***A1***

[4 marks]

4d. Using an algebraic approach, show that the graph of f^{-1} is obtained by a reflection of the graph of f in the y-axis followed by a reflection in the x-axis. *[4 marks]*

Markscheme

reflection in y-axis given by $f(-x)$ ***(M1)***

$f(-x) = \frac{-4x+1}{-x+4}$ ***(A1)***

reflection of their $f(-x)$ in x-axis given by $-f(-x)$ accept "now $-f(x)$" ***M1***

$(-f(-x) =)\ -\frac{-4x+1}{-x+4}$

$= \frac{-4x+1}{x-4}$ OR $\frac{4x-1}{-x+4}$ ***A1***

$= \frac{1-4x}{x-4}\ \left(= f^{-1}(x)\right)$ ***AG***

Note: If the candidate attempts to show the result using a particular coordinate on the graph of f rather than a general coordinate on the graph of f, where appropriate, award marks as follows:
M0A0 **for eg** $(2, 3) \to (-2, 3)$
M0A0 **for** $(-2, 3) \to (-2, -3)$

[4 marks]

The graphs of f and f^{-1} intersect at $x = p$ and $x = q$, where $p < q$.

4e. Find the value of p and the value of q. *[2 marks]*

Markscheme

attempt to solve $f(x) = f^{-1}(x)$ using graph or algebraically ***(M1)***

$p = -1$ AND $q = 1$ ***A1***

Note: Award ***(M1)A0*** if only one correct value seen.

[2 marks]

4f. Hence, find the area enclosed by the graph of f and the graph of f^{-1}. *[3 marks]*

Markscheme

attempt to set up an integral to find area between f and f^{-1} ***(M1)***

$\int_{-1}^{1}\left(\frac{4x+1}{x+4} - \frac{1-4x}{x-4}\right)\mathrm{d}\,x$ ***(A1)***

$= 0.675231\ldots$

$= 0.675$ ***A1***

[3 marks]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

5a. Find the probability that a bag selected at random is rejected. *[2 marks]*

Markscheme

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let $X =$ mass of a bag of sugar

evidence of identifying the correct area ***(M1)***

$\mathrm{P}(X < 995) = 0.0765637\ldots$

$= 0.0766$ ***A1***

[2 marks]

5b. Estimate the number of bags which will be rejected from a random sample of 100 bags. *[1 mark]*

Markscheme

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let $X=$ mass of a bag of sugar

0.0766×100

≈ 8 ***A1***

Note: Accept 7.66.

[1 mark]

5c. Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. *[3 marks]*

Markscheme

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let $X=$ mass of a bag of sugar

recognition that $\mathrm{P}\left(X>1005 \mid X\geq 995\right)$ is required ***(M1)***

$\frac{\mathrm{P}\left(X\geq 995\cap X>1005\right)}{\mathrm{P}\left(X\geq 995\right)}$

$\frac{\mathrm{P}\left(X>1005\right)}{\mathrm{P}\left(X\geq 995\right)}$ ***(A1)***

$\frac{0.0765637\ldots}{1-0.0765637\ldots}\left(=\frac{0.0765637\ldots}{0.923436\ldots}\right)$

$=0.0829$ ***A1***

[3 marks]