Paper 3 (11.12) *[27 marks]*

This question will investigate power series, as an extension to the Binomial Theorem for negative and fractional indices.

A power series in x is defined as a function of the form $f\left(x\right) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ where the $a_i \in \mathbb{R}$.

It can be considered as an infinite polynomial.

1a. Expand $\left(1 + x\right)^5$ using the Binomial Theorem. *[2 marks]*

Markscheme

$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ ***M1A1***

[2 marks]

This is an example of a power series, but is only a finite power series, since only a finite number of the a_i are non-zero.

1b. Consider the power series $1 - x + x^2 - x^3 + x^4 - \ldots$ *[4 marks]*

By considering the ratio of consecutive terms, explain why this series is equal to $\left(1 + x\right)^{-1}$ and state the values of x for which this equality is true.

Markscheme

It is an infinite GP with $a = 1$, $r = -x$ ***R1A1***

$S_\infty = \frac{1}{1-(-x)} = \frac{1}{1+x} = \left(1 + x\right)^{-1}$ ***M1A1AG***

[4 marks]

1c. Differentiate the equation obtained part (b) and hence, find the first four terms in a power series for $\left(1 + x\right)^{-2}$. *[2 marks]*

Markscheme

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$

$-1(1+x)^{-2} = -1 + 2x - 3x^2 + 4x^3 - \ldots$ ***A1***

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$ ***A1***

[2 marks]

1d. Repeat this process to find the first four terms in a power series for $(1+x)^{-3}$. *[2 marks]*

Markscheme

$-2(1+x)^{-3} = -2 + 6x - 12x^2 + 20x^3 \ldots$ ***A1***

$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 \ldots$ ***A1***

[2 marks]

1e. Hence, by recognising the pattern, deduce the first four terms in a power series for $(1+x)^{-n}$, $n \in \mathbb{Z}^+$. *[3 marks]*

Markscheme

$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 \ldots$ ***A1A1A1***

[3 marks]

We will now attempt to generalise further.

Suppose $(1+x)^q$, $q \in \mathbb{Q}$ can be written as the power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$.

1f. By substituting $x = 0$, find the value of a_0. *[1 mark]*

Markscheme

$1^q = a_0 \Rightarrow a_0 = 1$ ***A1***

[1 mark]

1g. By differentiating both sides of the expression and then substituting $x = 0$, find the value of a_1. *[2 marks]*

Markscheme

$q(1 + x)^{q-1} = a_1 + 2a_2x + 3a_3x^2 + \ldots$ ***A1***

$a_1 = q$ ***A1***

[2 marks]

1h. Repeat this procedure to find a_2 and a_3. *[4 marks]*

Markscheme

$q\left(q - 1\right)\left(1 + x\right)^{q-2} = 1 \times 2a_2 + 2 \times 3a_3x + \ldots$ ***A1***

$a_2 = \frac{q(q-1)}{2!}$ ***A1***

$q\left(q - 1\right)\left(q - 2\right)\left(1 + x\right)^{q-3} = 1 \times 2 \times 3a_3 + \ldots$ ***A1***

$a_3 = \frac{q(q-1)(q-2)}{3!}$ ***A1***

[4 marks]

1i. Hence, write down the first four terms in what is called the Extended Binomial Theorem for $\left(1 + x\right)^q$, $q \in \mathbb{Q}$. *[1 mark]*

Markscheme

$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 \ldots$ ***A1***

[1 mark]

1j. Write down the power series for $\frac{1}{1+x^2}$. *[2 marks]*

Markscheme

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ ***M1A1***

[2 marks]

1k. Hence, using integration, find the power series for $\arctan x$, giving the first four non-zero terms. *[4 marks]*

Markscheme

$\arctan x + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***M1A1***

Putting $x = 0 \Rightarrow c = 0$ ***R1***

So $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***A1***

[4 marks]