Paper 3 (11.12) *[27 marks]*

This question will investigate power series, as an extension to the Binomial Theorem for negative and fractional indices.

A power series in x is defined as a function of the form $f\left(x\right) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ where the $a_i \in \mathbb{R}$.

It can be considered as an infinite polynomial.

1a. Expand $\left(1 + x\right)^5$ using the Binomial Theorem. *[2 marks]*

....

....

....

....

....

....

This is an example of a power series, but is only a finite power series, since only a finite number of the a_i are non-zero.

1b. Consider the power series $1 - x + x^2 - x^3 + x^4 - \ldots$ *[4 marks]*

By considering the ratio of consecutive terms, explain why this series is equal to $(1+x)^{-1}$ and state the values of x for which this equality is true.

. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .

1c. Differentiate the equation obtained part (b) and hence, find the first four terms in a power series for $(1+x)^{-2}$. *[2 marks]*

1d. Repeat this process to find the first four terms in a power series for $(1+x)^{-3}$. *[2 marks]*

1e. Hence, by recognising the pattern, deduce the first four terms in a power series for $(1+x)^{-n}$, $n \in \mathbb{Z}^{+}$. *[3 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

We will now attempt to generalise further.

Suppose $(1+x)^{q}$, $q \in \mathbb{Q}$ can be written as the power series $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$.

1f. By substituting $x = 0$, find the value of a_0. *[1 mark]*

. . . .

. . . .

. . . .

1g. By differentiating both sides of the expression and then substituting $x = 0$, find the value of a_1. *[2 marks]*

1h. Repeat this procedure to find a_2 and a_3. *[4 marks]*

1i. Hence, write down the first four terms in what is called the Extended Binomial Theorem for $(1+x)^q$, $q \in \mathbb{Q}$. *[1 mark]*

....

....

....

1j. Write down the power series for $\frac{1}{1+x^2}$. *[2 marks]*

....

....

....

....

....

....

1k. Hence, using integration, find the power series for $\arctan x$, giving the first four non-zero terms. *[4 marks]*