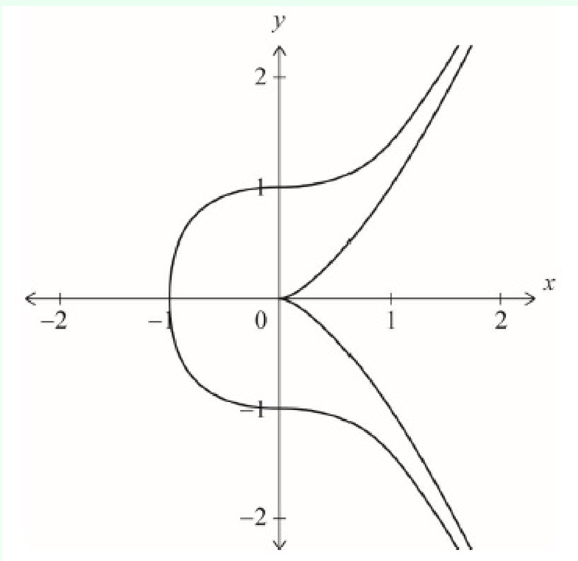
Paper 3 questions *[230 marks]*

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b, where $a,\ b \in \mathbb{N}$.

On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

1a. $y^2 = x^3,\ x \ge 0$ *[2 marks]*

Markscheme



approximately symmetric about the x-axis graph of $y^2 = x^3$ ***A1***

including cusp/sharp point at $(0,\ 0)$ ***A1***

[2 marks]

Note: Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct. Award ***A1A0A0A0*** if only one correct branch of both curves are seen.

Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0***.

1b. $y^2 = x^3 + 1,\ x \geq -1$ *[2 marks]*

Markscheme

approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts ***A1***
some indication of position of intersections at $x = -1$, $y = \pm 1$ ***A1***

[2 marks]

Note: Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct.
Award ***A1A0A0A0*** if only one correct branch of both curves are seen.

Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0***.

1c. Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. *[1 mark]*

Markscheme

$(0,\ 1)$ and $(0,\ -1)$ ***A1***

[1 mark]

1d. By considering each curve from part (a), identify two key features that would distinguish one curve from the other. *[1 mark]*

Markscheme

Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different x-axis intercepts (one goes through the origin, and the other does not)

graphs have different y-axis intercepts ***A1***

Note: Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent.

[1 mark]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \geq -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

1e. By varying the value of b, suggest two key features common to these curves. *[2 marks]*

Markscheme

Any **two** from:

as , $x \to \infty,\ y \to \pm\infty$

as $x \to \infty,\ y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have x intercepts at $x = -\sqrt[3]{b}$

they have y intercepts at $y =(\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or x-axis) is a line of symmetry

they all have the same line of symmetry $(y = 0)$

they have one x-axis intercept

they have two y-axis intercepts

they have two points of inflexion

at x-axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at x-axis intercepts ***A1A1***

Note: The last example is the only valid answer for things “not” present. Do not credit an answer of “they are all symmetrical” without some reference to the line of symmetry.

Note: Do not allow same/ similar shape or equivalent.

Note: In accordance with marking rules, mark their first two responses and ignore any subsequent.

[2 marks]

Next, consider the curve $y^2 = x^3 + x,\ x \geq 0$.

1f. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$, for $x > 0$. *[3 marks]*

Markscheme

METHOD 1

attempt to differentiate implicitly ***M1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+1}{2y}$ OR $(\pm)2\sqrt{x^3+x}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

METHOD 2

attempt to use chain rule $y = (\pm)\sqrt{x^3+x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)\frac{1}{2}\left(x^3+x\right)^{-\frac{1}{2}}\left(3x^2+1\right)$ ***A1A1***

Note: Award ***A1*** for $(\pm)\frac{1}{2}\left(x^3+x\right)^{-\frac{1}{2}}$, ***A1*** for $\left(3x^2+1\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

[3 marks]

1g. Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. *[1 mark]*

Markscheme

EITHER

local minima/maxima occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$1 + 3x^2 = 0$ has no (real) solutions (or equivalent) ***R1***

OR

$\left(x^2 \geq 0 \Rightarrow\right) 3x^2 + 1 > 0$, so $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ ***R1***

THEN

so, no local minima/maxima exist ***AG***

[1 mark]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

1h. Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3}+q}{r}}$, where $p,\ q,\ r \in \mathbb{Z}$. *[7 marks]*

Markscheme

EITHER

attempt to use quotient rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\pm)\frac{12x\sqrt{x+x^3} - \left(1+3x^2\right)\left(x+x^3\right)^{-\frac{1}{2}}\left(1+3x^2\right)}{4\left(x+x^3\right)}$ ***A1A1***

Note: Award ***A1*** for correct $12x\sqrt{x+x^3}$ and correct denominator, ***A1*** for correct $-\left(1+3x^2\right)\left(x+x^3\right)^{-\frac{1}{2}}\left(1+3x^2\right)$.

Note: Future ***A*** marks may be awarded if the denominator is missing or incorrect.

stating or using $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ (may be seen anywhere) ***(M1)***

$12x\sqrt{x+x^3} = \left(1+3x^2\right)\left(x+x^3\right)^{-\frac{1}{2}}\left(1+3x^2\right)$

OR

attempt to use product rule to find $\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2}$ ***M1***

$\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = \frac{1}{2}\left(3x^2+1\right)\left(-\frac{1}{2}\right)\left(3x^2+1\right)\left(x^3+x\right)^{-\frac{3}{2}} + 3x\left(x^3+x\right)^{-\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct first term, ***A1*** for correct second term.

setting $\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = 0$ ***(M1)***

OR

attempts implicit differentiation on $2y\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 3x^2+1$ ***M1***

$2\left(\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\right)^2 + 2y\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = 6x$ ***A1***

recognizes that $\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = 0$ ***(M1)***

$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \pm\sqrt{3x}$

$(\pm)\frac{3x^2+1}{2\sqrt{x^3+x}} = (\pm)\sqrt{3x}$ ***(A1)***

THEN

$12x\left(x+x^3\right) = \left(1+3x^2\right)^2$

$12x^2 + 12x^4 = 9x^4 + 6x^2 + 1$

$3x^4 + 6x^2 - 1 = 0$ ***A1***

attempt to use quadratic formula or equivalent ***(M1)***

$x^2 = \frac{-6\pm\sqrt{48}}{6}$

$(x > 0 \Rightarrow) x = \sqrt{\frac{2\sqrt{3}-3}{3}}\ (p=2,\ q=-3,\ r=3)$ ***A1***

Note: Accept any integer multiple of $p,\ q$ and r (e.g. $4,\ -6$ and 6).

[7 marks]

$\mathrm{P}(x,\ y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $\mathrm{P}(-1,\ -1)$ lies on C.

1i. Find the equation of the tangent to C at P. *[2 marks]*

Markscheme

attempt to find tangent line through $(-1,\ -1)$ ***(M1)***

$y + 1 = -\frac{3}{2}(x + 1)$ OR $y = -1.5x - 2.5$ ***A1***

[2 marks]

1j. Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. *[2 marks]*

Markscheme

attempt to solve simultaneously with $y^2 = x^3 + 2$ ***(M1)***

Note: The ***M1*** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. $(4.25,\ -8.875)$).

obtain $\left(\frac{17}{4},\ -\frac{71}{8}\right)$ ***A1***

[2 marks]

1k. The point $\mathrm{S}(-1\ ,\ 1)$ also lies on C. The line $[\mathrm{QS}]$ intersects C at a further point. Determine the coordinates of this point. *[5 marks]*

Markscheme

attempt to find equation of $[\mathrm{QS}]$ ***(M1)***

$\frac{y-1}{x+1} = -\frac{79}{42}(= -1.88095\ldots)$ ***(A1)***

solve simultaneously with $y^2 = x^3 + 2$ ***(M1)***

$x = 0.28798\ldots(= \frac{127}{441})$ ***A1***

$y = -1.4226\ldots(= \frac{13175}{9261})$ ***A1***

$(0.228,\ -1.42)$

OR

attempt to find vector equation of $[\mathrm{QS}]$ ***(M1)***

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{21}{4} \\ -\frac{79}{8} \end{pmatrix}$ ***(A1)***

$x = -1 + \frac{21}{4}\lambda$

$y = 1 - \frac{79}{8}\lambda$

attempt to solve $\left(1 - \frac{79}{8}\lambda\right)^2 = \left(-1 + \frac{21}{4}\lambda\right)^3 + 2$ ***(M1)***

$\lambda = 0.2453\ldots$

$x = 0.28798\ldots(= \frac{127}{441})$ ***A1***

$y = -1.4226\ldots(= \frac{13175}{9261})$ ***A1***

$(0.228,\ -1.42)$

[5 marks]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = ax + y$,

where $x,\ y,\ t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

2a. By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y$, show that $y = A\mathrm{e}^t$ where A is a constant. *[3 marks]*

Markscheme

METHOD 1

$\frac{\mathrm{d}y}{\mathrm{d}t} = y$

$\int \frac{\mathrm{d}y}{y} = \int \mathrm{d\,t}$ ***(M1)***

$\ln y = t + c$ OR $\ln |y| = t + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ and ***A1*** for t and c.

$y = A\mathrm{e}^t$ ***AG***

METHOD 2

rearranging to $\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$ AND multiplying by integrating factor e^{-t} ***M1***

$\mathrm{y}\mathrm{e}^{-t} = A$ ***A1A1***

$y = A\mathrm{e}^t$ ***AG***

[3 marks]

2b. Show that $\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$. *[1 mark]*

Markscheme

substituting $y = A\mathrm{e}^t$ into differential equation in x ***M1***

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - A\mathrm{e}^t$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$ ***AG***

[1 mark]

2c. Solve the differential equation in part (a)(ii) to find x as a function of t. *[4 marks]*

Markscheme

integrating factor (IF) is $\mathrm{e}^{\int -1\,\mathrm{d}t}$ ***(M1)***

$= \mathrm{e}^{-t}$ ***(A1)***

$\mathrm{e}^{-t}\frac{\mathrm{d}x}{\mathrm{d}t} - x\mathrm{e}^{-t} = -A$

$x\mathrm{e}^{-t} = -At + D$ ***(A1)***

$x = (-At + D)\mathrm{e}^t$ ***A1***

Note: The first constant must be A, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[4 marks]

Now consider the case where $a = -1$.

2d. By differentiating $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ with respect to t, show that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2\frac{\mathrm{d}y}{\mathrm{d}t}$. *[3 marks]*

Markscheme

$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

EITHER

$= -x + y + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$= \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

OR

$= -x + y + (-x + y)$ ***(M1)***

$= 2(-x + y)$ ***A1***

THEN

$= 2\frac{\mathrm{d}y}{\mathrm{d}t}$ ***AG***

[3 marks]

2e. By substituting $Y = \frac{\mathrm{d}y}{\mathrm{d}t}$, show that $Y = B\mathrm{e}^{2t}$ where B is a constant. *[3 marks]*

Markscheme

$\frac{\mathrm{d}Y}{\mathrm{d}t} = 2Y$ ***A1***

$\int \frac{\mathrm{d}Y}{Y} = \int 2\,\mathrm{d}t$ ***M1***

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ ***A1***

$Y = B\mathrm{e}^{2t}$ ***AG***

[3 marks]

2f. Hence find y as a function of t. *[2 marks]*

Markscheme

$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = B\mathrm{e}^{2t}$

$y = \int B\mathrm{e}^{2t}\,\mathrm{d}\,t$ ***M1***

$y = \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

Note: The first constant must be B, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[2 marks]

2g. Hence show that $x = -\frac{B}{2}\mathrm{e}^{2t} + C$, where C is a constant. *[3 marks]*

Markscheme

METHOD 1

substituting $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$ and their (iii) into $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ ***M1(M1)***

$B\mathrm{e}^{2t} = -x + \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

Note: Follow through from incorrect part (iii) cannot be awarded if it does not lead to the ***AG***.

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -\frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}\left(x\mathrm{e}^{-t}\right)}{\mathrm{d}t} = -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}$ ***M1***

$x\mathrm{e}^{-t} = \int -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}\,\mathrm{d}t$

$x\mathrm{e}^{-t} = -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t} + D$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C + D\mathrm{e}^{t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y \Rightarrow B\mathrm{e}^{2t} = \frac{B}{2}\mathrm{e}^{2t} - C - D\mathrm{e}^{t} + \frac{B}{2}\mathrm{e}^{2t} + C \Rightarrow D = 0$ ***M1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

[3 marks]

Now consider the case where $a = -4$.

2h. Show that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. *[3 marks]*

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}t} = -4x + y$

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ seen anywhere ***M1***

METHOD 1

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4(x - y) + \frac{\mathrm{d}y}{\mathrm{d}t}$

attempt to eliminate x ***M1***

$= -4\left(\frac{1}{4}\left(y - \frac{\mathrm{d}y}{\mathrm{d}t}\right) - y\right) + \frac{\mathrm{d}y}{\mathrm{d}t}$

$= 2\frac{\mathrm{d}y}{\mathrm{d}t} + 3y$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$ ***AG***

METHOD 2

rewriting LHS in terms of x and y ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ ***A1***

$= 0$ ***AG***

[3 marks]

From previous cases, we might conjecture that a solution to this differential equation is $y = F\mathrm{e}^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

2i. Find the two values for λ that satisfy $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. *[4 marks]*

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda \mathrm{e}^{\lambda t},\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F\lambda^2 \mathrm{e}^{\lambda t}$ ***(A1)***

$F\lambda^2 \mathrm{e}^{\lambda t} - 2F\lambda \mathrm{e}^{\lambda t} - 3F\mathrm{e}^{\lambda t} = 0$ ***(M1)***

$\lambda^2 - 2\lambda - 3 = 0$ (since $\mathrm{e}^{\lambda t} \neq 0$) ***A1***

λ_1 and λ_2 are 3 and -1 (either order) ***A1***

[4 marks]

2j. Let the two values found in part (c)(ii) be λ_1 and λ_2. *[4 marks]*

Verify that $y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$ is a solution to the differential equation in (c)(i),where G is a constant.

Markscheme

METHOD 1

$y = F\mathrm{e}^{3t} + G\mathrm{e}^{-t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = 3F\mathrm{e}^{3t} - G\mathrm{e}^{-t},\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 9F\mathrm{e}^{3t} - G\mathrm{e}^{-t}$ ***(A1)(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t} - 2\left(3F\mathrm{e}^{3t} - G\mathrm{e}^{-t}\right) - 3\left(F\mathrm{e}^{3t} - G\mathrm{e}^{-t}\right)$
M1

$= 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t} - 6F\mathrm{e}^{3t} + 2G\mathrm{e}^{-t} - 3F\mathrm{e}^{3t} - 3G\mathrm{e}^{-t}$ ***A1***

$= 0$ ***AG***

METHOD 2

$y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda_1\mathrm{e}^{\lambda_1 t} + G\lambda_2\mathrm{e}^{\lambda_2 t},\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F{\lambda_1}^2\mathrm{e}^{\lambda_1 t} + G{\lambda_2}^2\mathrm{e}^{\lambda_2 t}$ ***(A1)(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = F{\lambda_1}^2\mathrm{e}^{\lambda_1 t} + G{\lambda_2}^2\mathrm{e}^{\lambda_2 t} - 2\left(F\lambda_1\mathrm{e}^{\lambda_1 t} + G\lambda_2\mathrm{e}^{\lambda_2 t}\right) - 3\left(F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}\right.$
M1

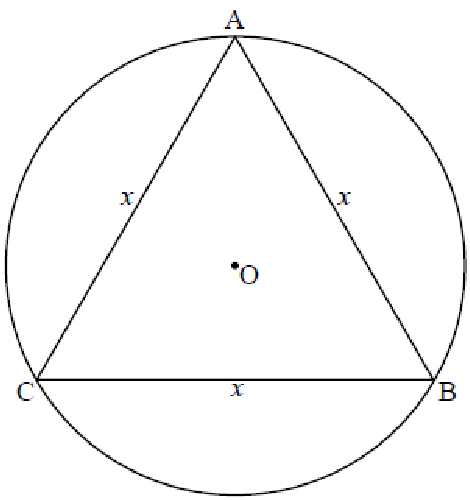
$= F\mathrm{e}^{\lambda_1 t}\left(\lambda^2 - 2\lambda - 3\right) + G\mathrm{e}^{\lambda_2 t}\left(\lambda^2 - 2\lambda - 3\right)$ ***A1***

$= 0$ ***AG***

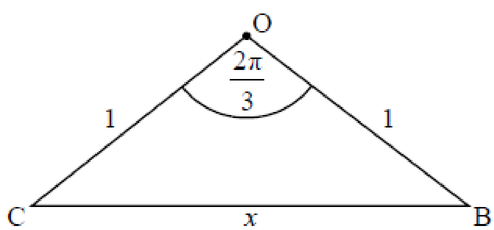
[4 marks]

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

3a. Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram. *[3 marks]*



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units.

Markscheme

METHOD 1

consider right-angled triangle OCX where CX $= \frac{x}{2}$

$\sin\frac{\pi}{3} = \frac{\frac{x}{2}}{1}$ ***M1A1***

$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2\left(1\right)\left(1\right)\cos\frac{2\pi}{3}$ ***M1A1***

$x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

Note: Accept use of sine rule.

[3 marks]

3b. Consider a square of side length, x units, inscribed in a circle of radius 1 unit. By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. *[3 marks]*

Markscheme

$\sin\frac{\pi}{4} = \frac{1}{x}$ where x = side of square ***M1***

$x = \sqrt{2}$ ***A1***

$P_i = 4\sqrt{2}$ ***A1***

[3 marks]

3c. Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. *[2 marks]*

Markscheme

6 equilateral triangles ⇒x = 1 ***A1***

$P_i = 6$ ***A1***

[2 marks]

Let $P_i\left(n\right)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

3d. Show that $P_i\left(n\right) = 2n\sin\left(\frac{\pi}{n}\right)$. *[3 marks]*

Markscheme

in right-angled triangle $\sin\left(\frac{\pi}{n}\right) = \frac{\frac{x}{2}}{1}$ ***M1***

$\Rightarrow x = 2\sin\left(\frac{\pi}{n}\right)$ ***A1***

$P_i = n \times x$

$P_i = n \times 2\sin\left(\frac{\pi}{n}\right)$ ***M1***

$P_i = 2n\sin\left(\frac{\pi}{n}\right)$ ***AG***

[3 marks]

3e. Use an appropriate Maclaurin series expansion to find $\lim\limits_{n\to\infty} P_i\left(n\right)$ and interpret this result geometrically. *[5 marks]*

Markscheme

consider $\lim_{n \to \infty} 2n \sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ ***M1***

$2n \sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \ldots\right)$ ***(A1)***

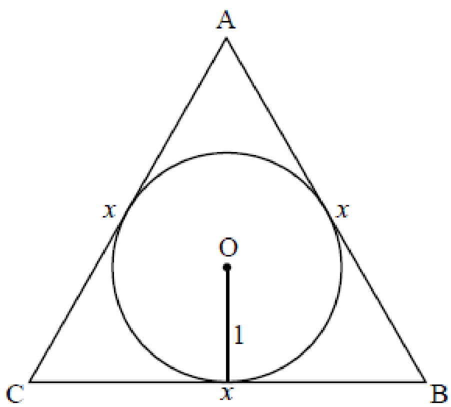
$= 2\left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \ldots\right)$ ***A1***

$\Rightarrow \lim_{n \to \infty} 2n \sin\left(\frac{\pi}{n}\right) = 2\pi$ ***A1***

as $n \to \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$ ***R1***

[5 marks]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

3f. Show that $P_c(n) = 2n \tan\left(\frac{\pi}{n}\right)$. *[4 marks]*

Markscheme

consider an n-sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre ***M1A1***

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2\tan\left(\frac{\pi}{n}\right)$ ***M1A1***

Perimeter $P_c = 2n\tan\left(\frac{\pi}{n}\right)$ ***AG***

[4 marks]

3g. By writing $P_c(n)$ in the form $\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}$, find $\lim_{n\to\infty} P_c(n)$. *[5 marks]*

Markscheme

consider $\lim_{n\to\infty} 2n\tan\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right) = \frac{0}{0}$ ***R1***

attempt to use L'Hopital's rule ***M1***

$= \lim_{n\to\infty}\left(\frac{-\frac{2\pi}{n^2}\sec^2\left(\frac{\pi}{n}\right)}{-\frac{1}{n^2}}\right)$ ***A1A1***

$= 2\pi$ ***A1***

[5 marks]

3h. Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. *[2 marks]*

Markscheme

$P_i < 2\pi < P_c$

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$ ***M1***

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$ ***A1***

[2 marks]

3i. The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π. *[3 marks]*

Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π.

Markscheme

attempt to find the lower bound and upper bound approximations within 0.005 of π ***(M1)***

n = 46 ***A2***

[3 marks]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, −1 ≤ x ≤ 1 and $n \in \mathbb{Z}^+$.

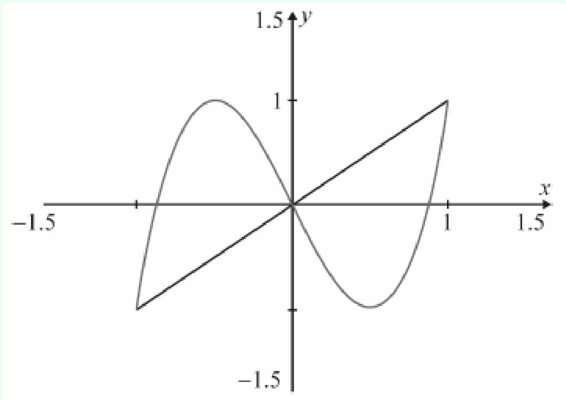
Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

4a. On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for −1 ≤ x ≤ 1. *[2 marks]*

Markscheme

correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

For odd values of n > 2, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

4b. local maximum points; *[3 marks]*

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 3, 1 local maximum point and 1 local minimum point

n = 5, 2 local maximum points and 2 local minimum points

n = 7, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

[3 marks]

4c. local minimum points; *[1 mark]*

Markscheme

$\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

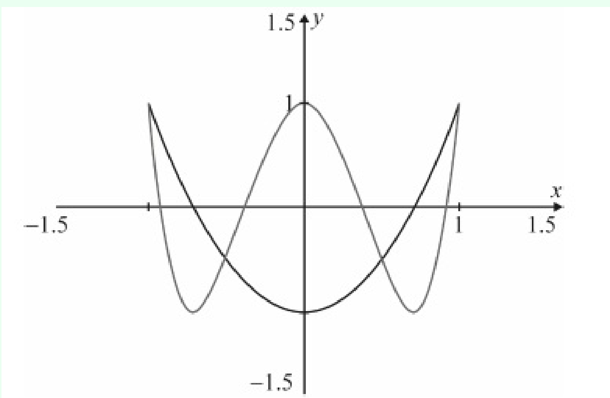
[1 mark]

4d. On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$. *[2 marks]*

Markscheme

correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

For even values of n > 2, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of $n$describing, in terms of n, the number of

4e. local maximum points; *[3 marks]*

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 2, 0 local maximum point and 1 local minimum point

n = 4, 1 local maximum points and 2 local minimum points

n = 6, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

[3 marks]

4f. local minimum points. *[1 mark]*

Markscheme

$\frac{n}{2}$ local minimum points ***A1***

[1 mark]

4g. Solve the equation $f_n{}'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and 0 < k < n. *[4 marks]*

Markscheme

$f_n(x) = \cos\left(n \arccos\left(x\right)\right)$

$f_n{}'(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f_n{}'(x) = 0 \Rightarrow n \sin\left(n \arccos\left(x\right)\right) = 0$ ***M1***

$n \arccos\left(x\right) = k\pi \ \ \left(k \in \mathbb{Z}^+\right)$ ***A1***

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and 0 < k < n) ***AG***

[4 marks]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

4h. Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. *[2 marks]*

Markscheme

$f_2(x) = \cos(2\arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

4i. Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$. *[2 marks]*

Markscheme

$f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A+B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

4j. Hence show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x)$, $n \in \mathbb{Z}^+$. *[3 marks]*

Markscheme

$f_{n-1}(x) = \cos((n-1)\arccos x)$ ***A1***

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ ***A1***

$= 2xf_n(x)$ ***AG***

[3 marks]

4k. Hence express $f_3(x)$ as a cubic polynomial. *[2 marks]*

Markscheme

$f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$ ***A1***

[2 marks]

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$, $n \in \mathbb{N}$.

5a. Find the exact values of I_0, I_1 and I_2. *[6 marks]*

Markscheme

$I_0 = \int_0^{\frac{\pi}{2}} 1\,dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$ ***M1A1***

$I_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$ ***M1A1***

$I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ ***M1A1***

[6 marks]

5b. Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2},\ n \geqslant 2$. *[5 marks]*

Markscheme

$u = \sin^{n-1} x$ $\qquad v = -\cos x$

$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$ $\qquad \frac{dv}{dx} = \sin x$

$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos^2 x\,dx$ ***M1A1A1***

$= 0 + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x\left(1 - \sin^2 x\right)\,dx = (n-1)\left(I_{n-2} - I_n\right)$ ***M1A1***

$\Rightarrow nI_n = (n-1)\,I_{n-2} \Rightarrow I_n = \frac{(n-1)}{n} I_{n-2}$ ***AG***

[6 marks]

5c. Explain where the condition $n \geqslant 2$ was used in your proof. *[1 mark]*

Markscheme

need $n \geqslant 2$ so that $\sin^{n-1}\frac{\pi}{2} = 0$ in $\left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}}$ ***R1***

[1 mark]

5d. Hence, find the exact values of I_3 and I_4. *[2 marks]*

Markscheme

$I_3 = \frac{2}{3}I_1 = \frac{2}{3}$ $\quad I_4 = \frac{3}{4}I_2 = \frac{3\pi}{16}$ ***A1A1***

[2 marks]

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx,\ n \in \mathbb{N}$.

5e. Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. *[4 marks]*

Markscheme

$x = \frac{\pi}{2} - u \Rightarrow \frac{dx}{du} = -1$ ***A1***

$J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_{\frac{\pi}{2}}^{0} -\cos^n\left(\frac{\pi}{2} - u\right) du = -\int_{\frac{\pi}{2}}^{0} \sin^n u\, du = \int_0^{\frac{\pi}{2}} \sin^n u\, du = I_n$

M1A1A1AG

[4 marks]

5f. Hence, find the exact values of J_5 and J_6 *[2 marks]*

Markscheme

$J_5 = I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$ $J_6 = I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3\pi}{16} = \frac{5\pi}{32}$ ***A1A1***

[2 marks]

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, $n \in \mathbb{N}$.

5g. Find the exact values of T_0 and T_1. *[3 marks]*

Markscheme

$T_0 = \int_0^{\frac{\pi}{4}} 1 \, dx = [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$ ***A1***

$T_1 = \int_0^{\frac{\pi}{4}} \tan dx = [-\ln|\cos x|]_0^{\frac{\pi}{4}} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$ ***M1A1***

[3 marks]

5h. Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2}$, $n \geqslant 2$. *[3 marks]*

Markscheme

$T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\sec^2 x - 1\right) \, dx$

M1

$\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - T_{n-2} = \frac{1}{n-1} - T_{n-2}$

A1A1AG

[3 marks]

5i. Explain where the condition $n \geqslant 2$ was used in your proof. *[1 mark]*

Markscheme

need $n \geqslant 2$ so that the powers of tan in $\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$ are not negative ***R1***

[1 mark]

5j. Hence, find the exact values of T_2 and T_3. *[2 marks]*

Markscheme

$T_2 = 1 - T_0 = 1 - \frac{\pi}{4}$ ***A1***

$T_3 = \frac{1}{2} - T_1 = \frac{1}{2} - \ln\sqrt{2}$ ***A1***

[2 marks]

This question investigates some applications of differential equations to modeling population growth.

One model for population growth is to assume that the rate of change of the population is proportional to the population, i.e. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where $k \in \mathbb{R}$, t is the time (in years) and P is the population

6a. Show that the general solution of this differential equation is $P = A\mathrm{e}^{kt}$, where $A \in \mathbb{R}$. *[5 marks]*

Markscheme

$\int \frac{1}{P}\mathrm{d}P = \int k\mathrm{d}t$ ***M1A1***

$\ln P = kt + c$ ***A1A1***

$P = e^{kt+c}$ ***A1***

$P = Ae^{kt}$, where $A = e^c$ ***AG***

[5 marks]

The initial population is 1000.

Given that $k = 0.003$, use your answer from part (a) to find

6b. the population after 10 years *[2 marks]*

Markscheme

when $t = 0,\ P = 1000$

$\Rightarrow A = 1000$ ***A1***

$P\left(10\right) = 1000e^{0.003(10)} = 1030$ ***A1***

[2 marks]

6c. the number of years it will take for the population to triple. *[2 marks]*

Markscheme

$3000 = 1000e^{0.003t}$ ***M1***

$t = \frac{\ln 3}{0.003} = 366$ years ***A1***

[2 marks]

6d. $\lim\limits_{t \to \infty} P$ *[1 mark]*

Markscheme

$\lim\limits_{t \to \infty} P = \infty$ ***A1***

[1 mark]

Consider now the situation when k is not a constant, but a function of time.

Given that $k = 0.003 + 0.002t$, find

6e. the solution of the differential equation, giving your answer in the form $P = f(t)$. *[5 marks]*

Markscheme

$\int \frac{1}{P} \mathrm{d}P = \int (0.003 + 0.002t)\, \mathrm{d}t$ ***M1***

$\ln P = 0.003t + 0.001t^2 + c$ ***A1A1***

$P = e^{0.003t + 0.001t^2 + c}$ ***A1***

when $t = 0,\ P = 1000$

$\Rightarrow e^c = 1000$ ***M1***

$P = 1000e^{0.003t + 0.001t^2}$

[5 marks]

6f. the number of years it will take for the population to triple. *[4 marks]*

Markscheme

$3000 = 1000e^{0.003t+0.001t^2}$ ***M1***

$\ln 3 = 0.003t + 0.001t^2$ ***A1***

Use of quadratic formula or GDC graph or GDC polysmlt ***M1***

$t = 31.7$ years ***A1***

[4 marks]

Another model for population growth assumes

- there is a maximum value for the population, L.
- that k is not a constant, but is proportional to $\left(1 - \frac{P}{L}\right)$.

6g. Show that $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P\left(L - P\right)$, where $m \in \mathbb{R}$. *[2 marks]*

Markscheme

$k = m\left(1 - \frac{P}{L}\right)$, where m is the constant of proportionality ***A1***

So $\frac{\mathrm{d}P}{\mathrm{d}t} = m\left(1 - \frac{P}{L}\right)P$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P\left(L - P\right)$ ***AG***

[2 marks]

6h. Solve the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P\left(L - P\right)$, giving your answer in the form $P = g\left(t\right)$. *[10 marks]*

Markscheme

$\int \frac{1}{P(L-P)}\mathrm{d}P = \int \frac{m}{L}\mathrm{d}t$ ***M1***

$\frac{1}{P(L-P)} = \frac{A}{P} + \frac{B}{L-P}$ ***M1***

$1 \equiv A(L-P) + BP$ ***A1***

$A = \frac{1}{L},\ B = \frac{1}{L}$ ***A1***

$\frac{1}{L}\int\left(\frac{1}{P} + \frac{1}{L-P}\right)\mathrm{d}P = \int \frac{m}{L}\mathrm{d}t$

$\frac{1}{L}(\ln P - \ln(L-P)) = \frac{m}{L}t + c$ ***A1A1***

$\ln\left(\frac{P}{L-P}\right) = mt + d$, where $d = cL$ ***M1***

$\frac{P}{L-P} = Ce^{mt}$, where $C = e^{d}$ ***A1***

$P\left(1 + Ce^{mt}\right) = CLe^{mt}$ ***M1***

$P = \frac{CLe^{mt}}{(1+Ce^{mt})}\ \left(= \frac{L}{(De^{-mt}+1)},\ \text{where } D = \frac{1}{C}\right)$ ***A1***

[10 marks]

6i. Given that the initial population is 1000, $L = 10000$ and $m = 0.003$, find the number of years it will take for the population to triple. *[4 marks]*

Markscheme

$1000 = \frac{10000}{D+1}$ ***M1***

$D = 9$ ***A1***

$3000 = \frac{10000}{9e^{-0.003t}+1}$ ***M1***

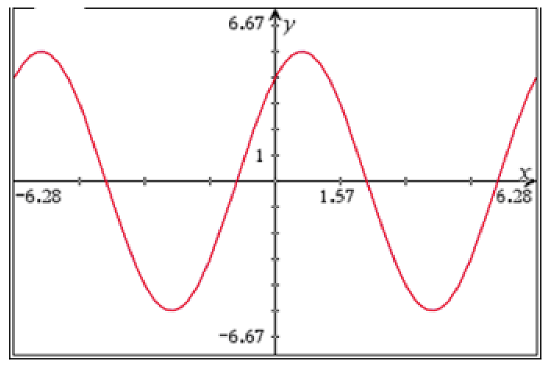
$t = 450$ years ***A1***

[4 marks]

This question investigates the sum of sine and cosine functions

7a. Sketch the graph $y = 3\sin x + 4\cos x$, for $-2\pi \leqslant x \leqslant 2\pi$ *[1 mark]*

Markscheme



A1

[1 mark]

7b. Write down the amplitude of this graph *[1 mark]*

Markscheme

5 ***A1***

[1 mark]

7c. Write down the period of this graph *[1 mark]*

Markscheme

2π ***A1***

[1 mark]

The expression $3\sin x + 4\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

7d. Use your answers from part (a) to write down the value of A, B and D. *[1 mark]*

Markscheme

$A = 5$, $B = 1$, $D = 0$ ***A1***

[1 mark]

7e. Find the value of C. *[2 marks]*

Markscheme

maximum at $x = 0.644$ ***M1***

So $C = -0.644$ ***A1***

[2 marks]

7f. Find $\arctan\frac{3}{4}$, giving the answer to 3 significant figures. *[1 mark]*

Markscheme

0.644 ***A1***

[1 mark]

7g. Comment on your answer to part (c)(i). *[1 mark]*

Markscheme

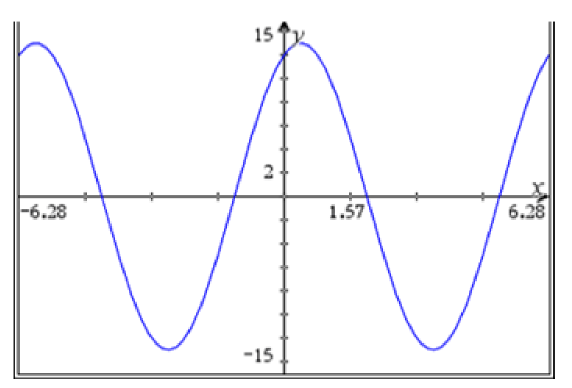
it appears that $C = -\arctan\frac{3}{4}$ ***A1***

[1 mark]

The expression $5\sin x + 12\cos x$ can be written in the form $A\cos(Bx + C) + D$, where A, $B \in \mathbb{R}^+$ and C, $D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

7h. By considering the graph of $y = 5\sin x + 12\cos x$, find the value of A, B, C and D. *[5 marks]*

Markscheme



M1

$A = 13$ ***A1***

$B = 1$ and $D = 0$ ***A1***

maximum at $x = 0.395$ ***M1***

So C = −0.395 $\left(= -\arctan\frac{5}{12}\right)$ ***A1***

[5 marks]

In general, the expression $a\sin x + b\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $a,\ b,\ A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

Conjecture an expression, in terms of a and b, for

7i. A. *[1 mark]*

Markscheme

$A = \sqrt{a^2 + b^2}$ ***A1***

[1 mark]

7j. B. *[1 mark]*

Markscheme

$B = 1$ ***A1***

[1 mark]

7k. C. *[1 mark]*

Markscheme

$C = -\arctan \frac{a}{b}$ ***A1***

[1 mark]

7l. D. *[1 mark]*

Markscheme

$D = 0$ ***A1***

[1 mark]

The expression $a \sin x + b \cos x$ can also be written in the form $\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$.

Let $\frac{a}{\sqrt{a^2+b^2}} = \sin \theta$

7m. Show that $\frac{b}{\sqrt{a^2+b^2}} = \cos \theta$. *[2 marks]*

Markscheme

EITHER

use of a right triangle and Pythgoras' to show the missing side length is b ***M1A1***

OR

Use of $\sin^2\theta + \cos^2\theta = 1$, leading to the required result ***M1A1***

[2 marks]

7n. Show that $\frac{a}{b} = \tan \theta$. *[1 mark]*

Markscheme

EITHER

use of a right triangle, leading to the required result. ***M1***

OR

Use of $\tan\theta = \frac{\sin\theta}{\cos\theta}$, leading to the required result. ***M1***

[1 mark]

7o. Hence prove your conjectures in part (e). *[6 marks]*

Markscheme

$a\sin x + b\cos x = \sqrt{a^2+b^2}\left(\sin\theta\sin x + \cos\theta\cos x\right)$ ***M1***

$a\sin x + b\cos x = \sqrt{a^2+b^2}\left(\cos\left(x-\theta\right)\right)$ ***M1A1***

So $A = \sqrt{a^2+b^2}$, $B = 1$ and $D = 0$ ***A1***

And $C = -\theta$ ***M1***

So $C = -\arctan\frac{a}{b}$ ***A1***

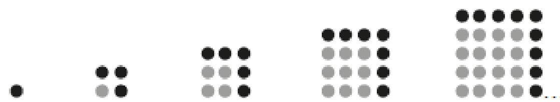
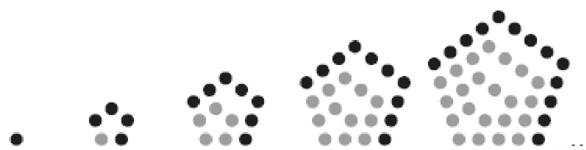
[6 marks]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are $1,\ 3,\ 6,\ 10$ and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		1, 3, 6, 10, 15, ...
Square numbers		1, 4, 9, 16, 25, ...
Pentagonal numbers		1, 5, 12, 22, 35, ...

For an r-sided regular polygon, where $r \in \mathbb{Z}^+,\ r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

Hence, for square numbers, $P_4(n) = \frac{(4-2)n^2 - (4-4)n}{2} = n^2$.

8a. For triangular numbers, verify that $P_3(n) = \frac{n(n+1)}{2}$. *[2 marks]*

Markscheme

$P_3(n) = \frac{(3-2)n^2 - (3-4)n}{2}$ OR $P_3(n) = \frac{n^2 - (-n)}{2}$ ***A1***

$P_3(n) = \frac{n^2+n}{2}$ ***A1***

Note: Award ***A0A1*** if $P_3(n) = \frac{n^2+n}{2}$ only is seen.

Do not award any marks for numerical verification.

so for triangular numbers, $P_3(n) = \frac{n(n+1)}{2}$ ***AG***

[2 marks]

8b. The number 351 is a triangular number. Determine which one it is. *[2 marks]*

Markscheme

METHOD 1

uses a table of values to find a positive integer that satisfies $P_3(n) = 351$ ***(M1)***

for example, a list showing at least 3 consecutive terms $(\ldots 325,\ 351,\ 378 \ldots)$

Note: Award ***(M1)*** for use of a GDC's numerical solve or graph feature.

$n = 26$ (26th triangular number) ***A1***

Note: Award ***A0*** for $n = -27, 26$. Award ***A0*** if additional solutions besides $n = 26$ are given.

METHOD 2

attempts to solve $\frac{n(n+1)}{2} = 351$ $\left(n^2 + n - 702 = 0\right)$ for n ***(M1)***

$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-702)}}{2}$ OR $(n - 26)(n + 27) = 0$

$n = 26$ (26th triangular number) ***A1***

Note: Award ***A0*** for $n = -27, 26$. Award ***A0*** if additional solutions besides $n = 26$ are given.

[2 marks]

8c. Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. *[2 marks]*

Markscheme

attempts to form an expression for $P_3(n)+P_3(n+1)$ in terms of n ***M1***

EITHER

$P_3(n)+P_3(n+1)\equiv \frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}$

$\equiv \frac{(n+1)(2n+2)}{2}\ \left(\equiv \frac{2(n+1)(n+1)}{2}\right)$ ***A1***

OR

$P_3(n)+P_3(n+1)\equiv\left(\frac{n^2}{2}+\frac{n}{2}\right)+\left(\frac{(n+1)^2}{2}+\frac{n+1}{2}\right)$

$\equiv\left(\frac{n^2+n}{2}\right)+\left(\frac{n^2+2n+1+n+1}{2}\right)\ \left(\equiv n^2+2n+1\right)$ ***A1***

THEN

$\equiv (n+1)^2$ ***AG***

[2 marks]

8d. State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. *[1 mark]*

Markscheme

the sum of the nth and $(n+1)$th triangular numbers

is the $(n+1)$th square number ***A1***

[1 mark]

8e. For $n=4$, sketch a diagram clearly showing your answer to part (b)(ii). *[1 mark]*

Markscheme

X	X	X	X	X
O	X	X	X	X
O	O	X	X	X
O	O	O	X	X
O	O	O	O	X

A1

Note: Accept equivalent single diagrams, such as the one above, where the 4th and 5th triangular numbers and the 5th square number are clearly shown. Award ***A1*** for a diagram that show $P_3(4)$ (a triangle with 10 dots) and $P_3(5)$ (a triangle with 15 dots) and $P_4(5)$ (a square with 25 dots).

[1 mark]

8f. Show that $8P_3(n)+1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. *[3 marks]*

Markscheme

METHOD 1

$8P_3(n)+1 = 8\left(\frac{n(n+1)}{2}\right)+1\ (= 4n(n+1)+1)$ ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$= 4n^2 + 4n + 1$

$= (2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

METHOD 2

$8P_3(n)+1 = 8\left((n+1)^2 - P_3(n+1)\right)+1\left(= 8\left((n+1)^2 - \frac{(n+1)(n+2)}{2}\right)+1\right)$

A1

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$8\left(n^2+2n+1\right)-4\left(n^2+3n+2\right)+1\ \left(= 4n^2+4n+1\right)$

$= (2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

Method 3

$8P_3(n)+1 = 8\left(\frac{n(n+1)}{2}\right)+1\ \left(= (An+B)^2\right)$ (where $A, B \in \mathbb{Z}^+$) ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$4n^2+4n+1\ \left(= A^2n^2 + 2ABn + B^2\right)$

now equates coefficients and obtains $B = 1$ and $A = 2$

$= (2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

[3 marks]

The nth pentagonal number can be represented by the arithmetic series $P_5(n) = 1 + 4 + 7 + \ldots + (3n - 2)$.

8g. Hence show that $P_5(n) = \frac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. *[3 marks]*

Markscheme

EITHER

$u_1 = 1$ and $d = 3$ ***(A1)***

substitutes their u_1 and their d into $P_5(n) = \frac{n}{2}(2u_1 + (n-1)d)$ ***M1***

$P_5(n) = \frac{n}{2}(2 + 3(n-1))$ $\left(= \frac{n}{2}(2 + 3n - 3)\right)$ ***A1***

OR

$u_1 = 1$ and $u_n = 3n - 2$ ***(A1)***

substitutes their u_1 and their u_n into $P_5(n) = \frac{n}{2}(u_1 + u_n)$ ***M1***

$P_5(n) = \frac{n}{2}(1 + 3n - 2)$ ***A1***

OR

$P_5(n) = (3(1) - 2) + (3(2) - 2) + (3(3) - 2) + \ldots 3n - 2$

$P_5(n) = (3(1) + 3(2) + 3(3) + \ldots + 3n) - 2n$ $(= 3(1 + 2 + 3 + \ldots + n) - 2n)$ ***(A1)***

substitutes $\frac{n(n+1)}{2}$ into their expression for $P_5(n)$ ***M1***

$P_5(n) = 3\left(\frac{n(n+1)}{2}\right) - 2n$

$P_5(n) = \frac{n}{2}(3(n+1) - 4)$ ***A1***

OR

attempts to find the arithmetic mean of n terms ***(M1)***

$= \frac{1 + (3n-2)}{2}$ ***A1***

multiplies the above expression by the number of terms n

$P_5(n) = \frac{n}{2}(1 + 3n - 2)$ ***A1***

THEN

so $P_5(n) = \frac{n(3n-1)}{2}$ ***AG***

[3 marks]

8h. By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. *[5 marks]*

Markscheme

METHOD 1

forms a table of $P_3(n)$ values that includes some values for $n > 5$ ***(M1)***

forms a table of $P_5(m)$ values that includes some values for $m > 5$ ***(M1)***

Note: Award ***(M1)*** if at least one $P_3(n)$ value is correct. Award ***(M1)*** if at least one $P_5(m)$ value is correct. Accept as above for $\left(n^2 + n\right)$ values and $\left(3m^2 - m\right)$ values.

$n = 20$ for triangular numbers ***(A1)***

$m = 12$ for pentagonal numbers ***(A1)***

Note: Award ***(A1)*** if $n = 20$ is seen in or out of a table. Award ***(A1)*** if $m = 12$ is seen in or out of a table. Condone the use of the same parameter for triangular numbers and pentagonal numbers, for example, $n = 20$ for triangular numbers and $n = 12$ for pentagonal numbers.

210 (is a triangular number and a pentagonal number) ***A1***

Note: Award all five marks for 210 seen anywhere with or without working shown.

METHOD 2

EITHER

attempts to express $P_3(n) = P_5(m)$ as a quadratic in n ***(M1)***

$n^2 + n + \left(m - 3m^2\right)(= 0)$ (or equivalent)

attempts to solve their quadratic in n ***(M1)***

$$n = \frac{-1 \pm \sqrt{12m^2 - 4m + 1}}{2} \left(= \frac{-1 \pm \sqrt{1^2 - 4\left(m - 3m^2\right)}}{2}\right)$$

OR

attempts to express $P_3(n) = P_5(m)$ as a quadratic in m ***(M1)***

$3m^2 - m - (n^2 + n)(= 0)$ (or equivalent)

attempts to solve their quadratic in m ***(M1)***

$m = \frac{1 \pm \sqrt{12n^2 - 12n + 1}}{6} \left(= \frac{1 \pm \sqrt{(-1)^2 + 12(n^2 + n)}}{6} \right)$

THEN

$n = 20$ for triangular numbers ***(A1)***

$m = 12$ for pentagonal numbers ***(A1)***

210 (is a triangular number and a pentagonal number) ***A1***

METHOD 3

$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2}$

let $n = m + k$ $(n > m)$ and so $3m^2 - m = (m + k)(m + k + 1)$ ***M1***

$2m^2 - 2(k + 1)m - (k^2 + k) = 0$ ***A1***

attempts to find the discriminant of their quadratic

and recognises that this must be a perfect square ***M1***

$\Delta = 4(k + 1)^2 + 8(k^2 + k)$

$N^2 = 4(k + 1)^2 + 8(k^2 + k)$ $(= 4(k + 1)(3k + 1))$

determines that $k = 8$ leading to $2m^2 - 18m - 72 = 0 \Rightarrow m = -3, 12$ and so $m = 12$ ***A1***

210 (is a triangular number and a pentagonal number) ***A1***

METHOD 4

$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2}$

let $m = n - k$ $(m < n)$ and so $n^2 + n = (n - k)(3(n - k) - 1)$ ***M1***

$2n^2 - 2(3k + 1)n + (3k^2 + k) = 0$ ***A1***

attempts to find the discriminant of their quadratic

and recognises that this must be a perfect square ***M1***

$\Delta = 4(3k + 1)^2 - 8(3k^2 + k)$

$N^2 = 4(3k + 1)^2 - 8(3k^2 + k)$ $(= 4(k + 1)(3k + 1))$

determines that $k=8$ leading to $2n^2-50n+200=0 \Rightarrow n=5, 20$ and so $n=20$ ***A1***

210 (is a triangular number and a pentagonal number) ***A1***

[5 marks]

8i. A polygonal number, $P_r(n)$, can be represented by the series *[8 marks]*

$\sum_{m=1}^{n}(1+(m-1)(r-2))$ where $r \in \mathbb{Z}^+,\ r \geq 3$.

Use mathematical induction to prove that $P_r(n)=\frac{(r-2)n^2-(r-4)n}{2}$ where $n \in \mathbb{Z}^+$.

Markscheme

Note: Award a maximum of ***R1M0M0A1M1A1A1R0*** for a 'correct' proof using n and $n+1$.

consider $n=1$: $P_r(1)=1+(1-1)(r-2)=1$ and

$P_r(1)=\frac{(r-2)(1^2)-(r-4)(1)}{2}=1$

so true for $n=1$ ***R1***

Note: Accept $P_r(1)=1$ and $P_r(1)=\frac{(r-2)(1^2)-(r-4)(1)}{2}=1$.
Do not accept one-sided considerations such as '$P_r(1)=1$ and so true for $n=1$'.
Subsequent marks after this ***R1*** are independent of this mark can be awarded.

Assume true for $n=k$, *ie.* $P_r(k)=\frac{(r-2)k^2-(r-4)k}{2}$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$". The assumption of truth must be clear.
Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Consider $n=k+1$:

($P_r(k+1)$ can be represented by the sum

$\sum_{m=1}^{k+1}(1+(m-1)(r-2))=\sum_{m=1}^{k}(1+(m-1)(r-2))+(1+k(r-2))$ and so

$P_r(k+1)=\frac{(r-2)k^2-(r-4)k}{2}+(1+k(r-2))$ $(P_r(k+1)=P_r(k)+(1+k(r-2)))$ ***M1***

$=\frac{(r-2)k^2-(r-4)k+2+2k(r-2)}{2}$ ***A1***

$=\frac{(r-2)\left(k^2+2k\right)-(r-4)k+2}{2}$

$=\frac{(r-2)\left(k^2+2k+1\right)-(r-2)-(r-4)k+2}{2}$ ***M1***

$=\frac{(r-2)(k+1)^2-(r-4)k-(r-4)}{2}$ ***(A1)***

$=\frac{(r-2)(k+1)^2-(r-4)(k+1)}{2}$ ***A1***

hence true for $n=1$ and $n=k$ true $\Rightarrow n=k+1$ true ***R1***

therefore true for all $n\in\mathbb{Z}^+$

Note: Only award the final ***R1*** if the first five marks have been awarded.
Award marks as appropriate for solutions that expand both the LHS and (given) RHS of the equation.

[8 marks]