Paper 3 questions *[230 marks]*

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b, where $a,\ b \in \mathbb{N}$.

On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

1a. $y^2 = x^3,\ x \ge 0$ *[2 marks]*

1b. $y^2 = x^3 + 1,\ x \ge -1$ *[2 marks]*

1c. Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. *[1 mark]*

1d. By considering each curve from part (a), identify two key features that would distinguish one curve from the other. *[1 mark]*

Now, consider curves of the form $y^2 = x^3 + b$, for $x \ge -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

1e. By varying the value of b, suggest two key features common to these curves. *[2 marks]*

Next, consider the curve $y^2 = x^3 + x,\ x \ge 0$.

1f. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$, for $x > 0$. *[3 marks]*

1g. Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. *[1 mark]*

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

1h. Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3}+q}{r}}$, where $p,\ q,\ r \in \mathbb{Z}$. *[7 marks]*

$\mathrm{P}(x,\ y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $\mathrm{P}(-1,\ -1)$ lies on C.

1i. Find the equation of the tangent to C at P. *[2 marks]*

1j. Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. *[2 marks]*

1k. The point $\mathrm{S}(-1\ ,\ 1)$ also lies on C. The line $[\mathrm{QS}]$ intersects C at a further point. Determine the coordinates of this point. *[5 marks]*

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = ax + y$,

where $x,\ y,\ t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

2a. By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y$, show that $y = A\mathrm{e}^t$ where A is a constant. *[3 marks]*

2b. Show that $\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$. *[1 mark]*

2c. Solve the differential equation in part (a)(ii) to find x as a function of t. *[4 marks]*

Now consider the case where $a = -1$.

2d. By differentiating $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ with respect to t, show that $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 2\frac{\mathrm{d}y}{\mathrm{d}t}$. *[3 marks]*

2e. By substituting $Y = \frac{\mathrm{d}y}{\mathrm{d}t}$, show that $Y = B\mathrm{e}^{2t}$ where B is a constant. *[3 marks]*

2f. Hence find y as a function of t. *[2 marks]*

2g. Hence show that $x = -\frac{B}{2}\mathrm{e}^{2t} + C$, where C is a constant. *[3 marks]*

Now consider the case where $a = -4$.

2h. Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. *[3 marks]*

From previous cases, we might conjecture that a solution to this differential equation is $y = F\mathrm{e}^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

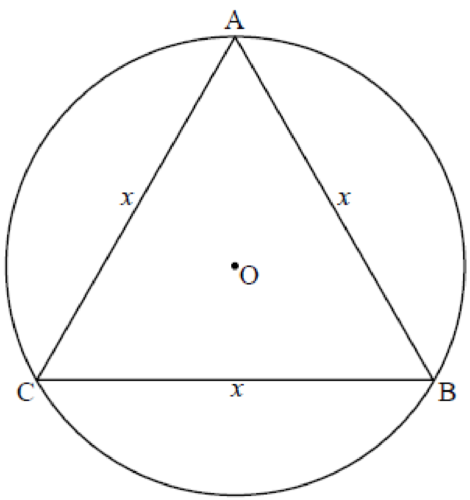
2i. Find the two values for λ that satisfy $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. *[4 marks]*

2j. Let the two values found in part (c)(ii) be λ_1 and λ_2. *[4 marks]*

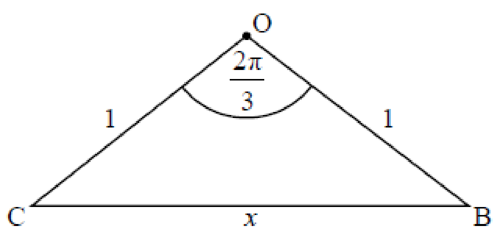
Verify that $y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$ is a solution to the differential equation in (c)(i),where G is a constant.

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

3a. Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram. *[3 marks]*



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units.

3b. Consider a square of side length, x units, inscribed in a circle of radius 1 unit. By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. *[3 marks]*

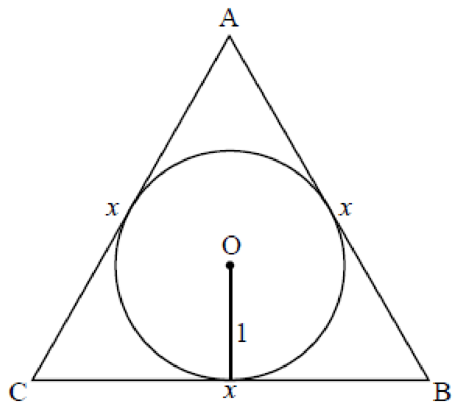
3c. Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. *[2 marks]*

Let $P_i\left(n\right)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

3d. Show that $P_i\left(n\right) = 2n\sin\left(\frac{\pi}{n}\right)$. *[3 marks]*

3e. Use an appropriate Maclaurin series expansion to find $\lim_{n \to \infty} P_i(n)$ and interpret this result geometrically. *[5 marks]*

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

3f. Show that $P_c(n) = 2n \tan\left(\frac{\pi}{n}\right)$. *[4 marks]*

3g. By writing $P_c(n)$ in the form $\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}$, find $\lim_{n \to \infty} P_c(n)$. *[5 marks]*

3h. Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. *[2 marks]*

3i. The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π. *[3 marks]*

Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π.

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

4a. On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$. *[2 marks]*

For odd values of n > 2, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

4b. local maximum points; *[3 marks]*

4c. local minimum points; *[1 mark]*

4d. On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for −1 ≤ x ≤ 1. *[2 marks]*

For even values of n > 2, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of $n$describing, in terms of n, the number of

4e. local maximum points; *[3 marks]*

4f. local minimum points. *[1 mark]*

4g. Solve the equation $f_n{'}(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and 0 < k < n. *[4 marks]*

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

4h. Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. *[2 marks]*

Consider $f_{n+1}(x) = \cos\left((n+1)\arccos x\right)$.

4i. Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos\left(n\arccos x\right)\cos\left(\arccos x\right) - \sin\left(n\arccos x\right)\sin\left(\arccos x\right)$. *[2 marks]*

4j. Hence show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n\left(x\right)$, $n \in \mathbb{Z}^+$. *[3 marks]*

4k. Hence express $f_3(x)$ as a cubic polynomial. *[2 marks]*

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \; n \in \mathbb{N}$.

5a. Find the exact values of I_0, I_1 and I_2. *[6 marks]*

5b. Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}, \; n \geqslant 2$. *[5 marks]*

5c. Explain where the condition $n \geqslant 2$ was used in your proof. *[1 mark]*

5d. Hence, find the exact values of I_3 and I_4. *[2 marks]*

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx, \; n \in \mathbb{N}$.

5e. Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. *[4 marks]*

5f. Hence, find the exact values of J_5 and J_6 *[2 marks]*

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \; n \in \mathbb{N}$.

5g. Find the exact values of T_0 and T_1. *[3 marks]*

5h. Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2}, \; n \geqslant 2$. *[3 marks]*

5i. Explain where the condition $n \geqslant 2$ was used in your proof. *[1 mark]*

5j. Hence, find the exact values of T_2 and T_3. *[2 marks]*

This question investigates some applications of differential equations to modeling population growth.

One model for population growth is to assume that the rate of change of the population is proportional to the population, i.e. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where $k \in \mathbb{R}$, t is the time (in years) and P is the population

6a. Show that the general solution of this differential equation is $P = A\mathrm{e}^{kt}$, where $A \in \mathbb{R}$. *[5 marks]*

The initial population is 1000.

Given that $k = 0.003$, use your answer from part (a) to find

6b. the population after 10 years *[2 marks]*

6c. the number of years it will take for the population to triple. *[2 marks]*

6d. $\lim\limits_{t \to \infty} P$ *[1 mark]*

Consider now the situation when k is not a constant, but a function of time.

Given that $k = 0.003 + 0.002t$, find

6e. the solution of the differential equation, giving your answer in the form $P = f(t)$. *[5 marks]*

6f. the number of years it will take for the population to triple. *[4 marks]*

Another model for population growth assumes

- there is a maximum value for the population, L.
- that k is not a constant, but is proportional to $\left(1 - \frac{P}{L}\right)$.

6g. Show that $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P(L - P)$, where $m \in \mathbb{R}$. *[2 marks]*

6h. Solve the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{m}{L}P\left(L - P\right)$, giving your answer in the form $P = g\left(t\right)$. *[10 marks]*

6i. Given that the initial population is 1000, $L = 10000$ and $m = 0.003$, find the number of years it will take for the population to triple. *[4 marks]*

This question investigates the sum of sine and cosine functions

7a. Sketch the graph $y = 3\sin x + 4\cos x$, for $-2\pi \leqslant x \leqslant 2\pi$ *[1 mark]*

7b. Write down the amplitude of this graph *[1 mark]*

7c. Write down the period of this graph *[1 mark]*

The expression $3\sin x + 4\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

7d. Use your answers from part (a) to write down the value of A, B and D. *[1 mark]*

7e. Find the value of C. *[2 marks]*

7f. Find $\arctan\frac{3}{4}$, giving the answer to 3 significant figures. *[1 mark]*

7g. Comment on your answer to part (c)(i). *[1 mark]*

The expression $5\sin x + 12\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

7h. By considering the graph of $y = 5\sin x + 12\cos x$, find the value of A, B, C and D. *[5 marks]*

In general, the expression $a\sin x + b\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $a,\ b,\ A,\ B \in \mathbb{R}^{+}$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

Conjecture an expression, in terms of a and b, for

7i. A. *[1 mark]*

7j. B. *[1 mark]*

7k. C. *[1 mark]*

7l. D. *[1 mark]*

The expression $a\sin x + b\cos x$ can also be written in the form $\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$.

Let $\frac{a}{\sqrt{a^2+b^2}} = \sin\theta$

7m. Show that $\frac{b}{\sqrt{a^2+b^2}} = \cos\theta$. *[2 marks]*

7n. Show that $\frac{a}{b} = \tan\theta$. *[1 mark]*

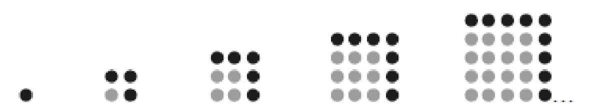
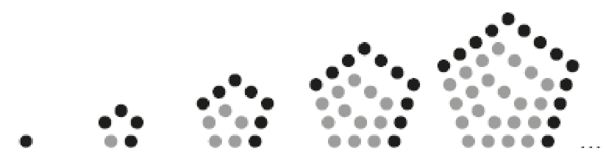
7o. Hence prove your conjectures in part (e). *[6 marks]*

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are $1,\ 3,\ 6,\ 10$ and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		1, 3, 6, 10, 15, ...
Square numbers		1, 4, 9, 16, 25, ...
Pentagonal numbers		1, 5, 12, 22, 35, ...

For an r-sided regular polygon, where $r \in \mathbb{Z}^+,\ r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}$, where $n \in \mathbb{Z}^+$.

Hence, for square numbers, $P_4(n) = \frac{(4-2)n^2 - (4-4)n}{2} = n^2$.

8a. For triangular numbers, verify that $P_3(n) = \frac{n(n+1)}{2}$. *[2 marks]*

8b. The number 351 is a triangular number. Determine which one it is. *[2 marks]*

8c. Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. *[2 marks]*

8d. State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. *[1 mark]*

8e. For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). *[1 mark]*

8f. Show that $8P_3(n)+1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. *[3 marks]*

The nth pentagonal number can be represented by the arithmetic series $P_5(n) = 1 + 4 + 7 + \ldots + (3n-2)$.

8g. Hence show that $P_5(n) = \frac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. *[3 marks]*

8h. By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. *[5 marks]*

8i. A polygonal number, $P_r(n)$, can be represented by the series *[8 marks]*

$\sum_{m=1}^{n}(1+(m-1)(r-2))$ where $r \in \mathbb{Z}^+,\ r \geq 3$.

Use mathematical induction to prove that $P_r(n) = \frac{(r-2)n^2-(r-4)n}{2}$ where $n \in \mathbb{Z}^+$.