Paper 3 (3.12) *[30 marks]*

A **Gaussian integer** is a complex number, z, such that $z = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\beta = 1 - 2\mathrm{i}$, such that $\gamma = \alpha\beta$ for some Gaussian integer γ.

1a. Find γ. *[2 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(3 + 4\mathrm{i})(1 - 2\mathrm{i}) = 11 - 2\mathrm{i}$ ***(M1)A1***

[2 marks]

Now consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\gamma = 11 + 2\mathrm{i}$.

1b. Determine whether $\frac{\gamma}{\alpha}$ is a Gaussian integer. *[3 marks]*

Markscheme

$\frac{\gamma}{\alpha} = \frac{41}{25} - \frac{38}{25}\mathrm{i}$ ***(M1)A1***

(Since $\mathrm{Re}\frac{\gamma}{\alpha}\left(= \frac{41}{25}\right)$ and/or $\mathrm{Im}\frac{\gamma}{\alpha}\left(= -\frac{38}{25}\right)$ are not integers)

$\frac{\gamma}{\alpha}$ is not a Gaussian integer ***R1***

Note: Award ***R1*** for correct conclusion from their answer.

[3 marks]

The norm of a complex number z, denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3\mathrm{i}$ then $N(2 + 3\mathrm{i}) = 2^2 + 3^2 = 13$.

1c. On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3. *[2 marks]*

Markscheme

$\pm 1,\ \pm \mathrm{i},\ 0$ plotted and labelled ***A1***

$1 \pm \mathrm{i},\ -1 \pm \mathrm{i}$ plotted and labelled ***A1***

Note: Award ***A1A0*** if extra points to the above are plotted and labelled.

[2 marks]

1d. Given that $\alpha = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$. *[1 mark]*

Markscheme

$|z| = \sqrt{a^2 + b^2}$ (and as $N(z) = |z|^2$) ***A1***

then $N(\alpha) = a^2 + b^2$ ***AG***

[1 mark]

A **Gaussian prime** is a Gaussian integer, z, that **cannot** be expressed in the form $z = \alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers with $N(\alpha),\ N(\beta) > 1$.

1e. By expressing the positive integer $n = c^2 + d^2$ as a product of two Gaussian integers each of norm $c^2 + d^2$, show that n is not a Gaussian prime. *[3 marks]*

Markscheme

$c^2 + d^2 = (c + d\mathrm{i})(c - d\mathrm{i})$ ***A1***

and $N(c + d\mathrm{i}) = N(c - d\mathrm{i}) = c^2 + d^2$ ***R1***

$N(c + d\mathrm{i}),\ N(c - d\mathrm{i}) > 1$ (since $c,\ d$ are positive) ***R1***

so $c^2 + d^2$ is not a Gaussian prime, by definition ***AG***

[3 marks]

The positive integer 2 is a prime number, however it is not a Gaussian prime.

1f. Verify that 2 is not a Gaussian prime. *[2 marks]*

Markscheme

$2\left(= 1^2 + 1^2\right) = (1 + \mathrm{i})(1 - \mathrm{i})$ ***(A1)***

$N(1 + \mathrm{i}) = N(1 - \mathrm{i}) = 2$ ***A1***

so 2 is not a Gaussian prime ***AG***

[2 marks]

1g. Write down another prime number of the form $c^2 + d^2$ that is not a Gaussian prime and express it as a product of two Gaussian integers. *[2 marks]*

Markscheme

For example, $5\left(= 1^2 + 2^2\right) = (1 + 2\mathrm{i})(1 - 2\mathrm{i})$ ***(M1)A1***

[2 marks]

Let $\alpha,\ \beta$ be Gaussian integers.

1h. Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. *[6 marks]*

Markscheme

METHOD 1

Let $\alpha = m + n\mathrm{i}$ and $\beta = p + q\mathrm{i}$

LHS:

$\alpha\beta = (mp - nq) + (mq + np)\mathrm{i}$ ***M1***

$N(\alpha\beta) = (mp - nq)^2 + (mq + np)^2$ ***A1***

$(mp)^2 - 2mnpq + (nq)^2 + (mq)^2 + 2mnpq + (np)^2$ ***A1***

$(mp)^2 + (nq)^2 + (mq)^2 + (np)^2$ ***A1***

RHS:

$N(\alpha)N(\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$(mp)^2 + (mq)^2 + (np)^2 + (nq)^2$ ***A1***

LHS = RHS and so $N(\alpha\beta) = N(\alpha)N(\beta)$ ***AG***

METHOD 2

Let $\alpha = m + n\mathrm{i}$ and $\beta = p + q\mathrm{i}$

LHS

$N(\alpha\beta) = \left(m^2 + n^2\right)\left(p^2 + q^2\right)$ ***M1***

$= (m + n\mathrm{i})(m - n\mathrm{i})(p + q\mathrm{i})(p - q\mathrm{i})$ ***A1***

$= (m + n\mathrm{i})(p + q\mathrm{i})(m - n\mathrm{i})(p - q\mathrm{i})$

$= ((mp - nq) + (mq + np)\mathrm{i})((mp - nq) - (mq + np)\mathrm{i})$ ***M1A1***

$= (mp - nq)^2 + (mq + np)^2$ ***A1***

$N = ((mp - nq) + (mq + np)\mathrm{i})$ ***A1***

$= N(\alpha)N(\beta)$ (= RHS) ***AG***

[6 marks]

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

1i. Hence show that $1 + 4\mathrm{i}$ is a Gaussian prime. *[3 marks]*

Markscheme

$N(1+4\mathrm{i})=17$ which is a prime (in $\mathbb{Z}$) ***R1***

if $1+4\mathrm{i}=\alpha\beta$ then $17=N(\alpha\beta)=N(\alpha)N(\beta)$ ***R1***

we cannot have $N(\alpha),\ N(\beta)>1$ ***R1***

Note: Award ***R1*** for stating that $1+4\mathrm{i}$ is not the product of Gaussian integers of smaller norm because no such norms divide 17

so $1+4\mathrm{i}$ is a Gaussian prime ***AG***

[3 marks]

1j. Use proof by contradiction to prove that a prime number, p, that is not of the form a^2+b^2 is a Gaussian prime. *[6 marks]*

Markscheme

Assume p is not a Gaussian prime

$\Rightarrow p=\alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers and $N(\alpha),\ N(\beta)>1$ ***M1***

$\Rightarrow N(p)=N(\alpha)N(\beta)$ ***M1***

$p^2=N(\alpha)N(\beta)$ ***A1***

It cannot be $N(\alpha)=1,\ N(\beta)=p^2$ from definition of Gaussian prime ***R1***

hence $N(\alpha)=p,\ N(\beta)=p$ ***R1***

If $\alpha=a+b\mathrm{i}$ then $N(\alpha)=a^2+b^2=p$ which is a contradiction ***R1***

hence a prime number, p, that is not of the form a^2+b^2 is a Gaussian prime ***AG***

[6 marks]