

Paper 3 (3.12) [30 marks]

A **Gaussian integer** is a complex number, z , such that $z = a + bi$ where $a, b \in \mathbb{Z}$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers, $\alpha = 3 + 4i$ and $\beta = 1 - 2i$, such that $\gamma = \alpha\beta$ for some Gaussian integer γ .

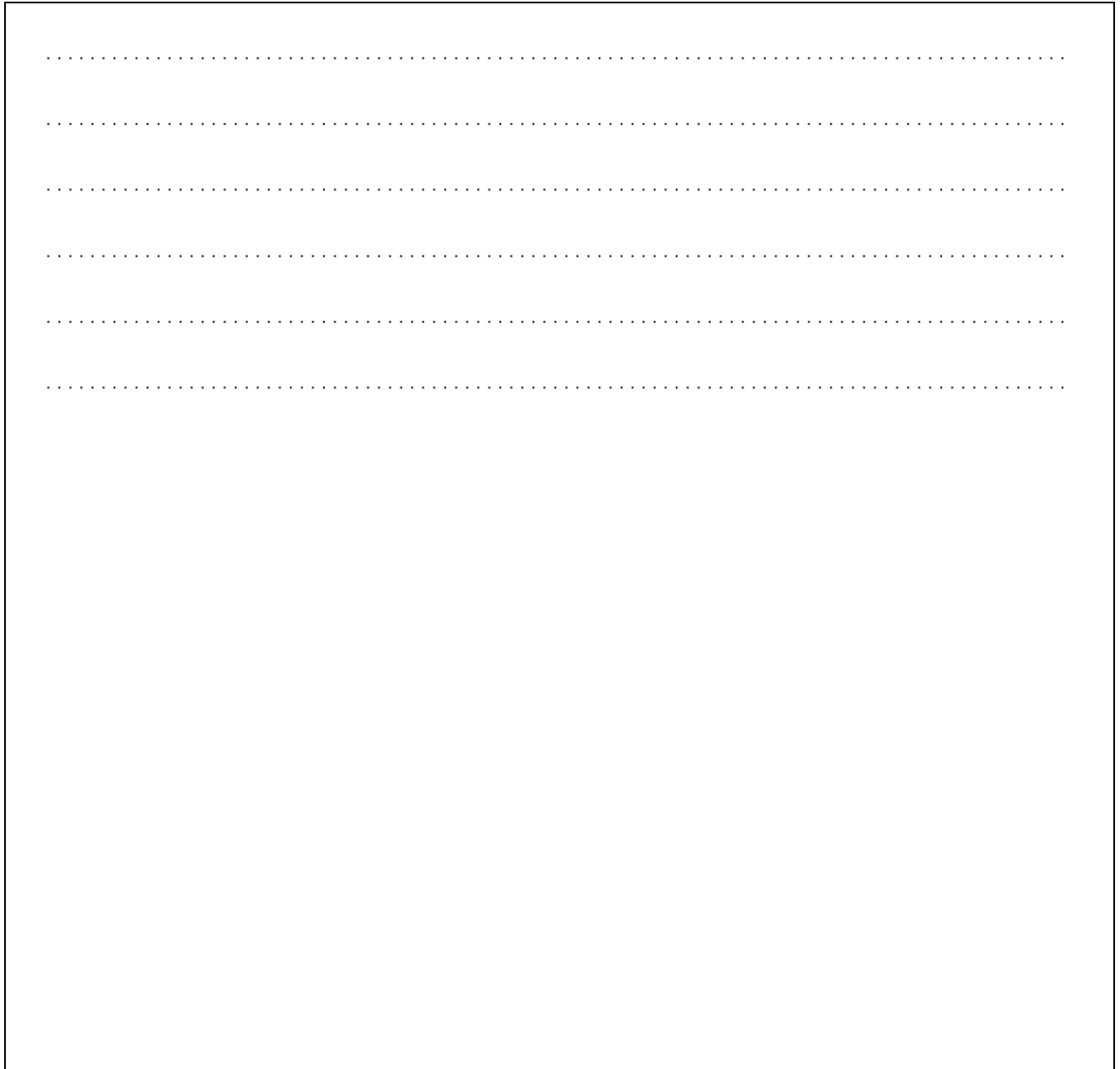
1a. Find γ .

[2 marks]

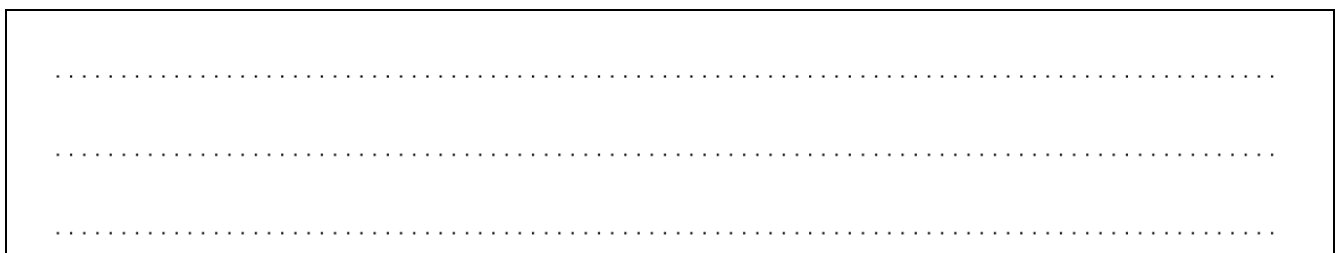
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The norm of a complex number z , denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3i$ then $N(2 + 3i) = 2^2 + 3^2 = 13$.

- 1c. On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3. [2 marks]



- 1d. Given that $\alpha = a + bi$ where $a, b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$. [1 mark]



A **Gaussian prime** is a Gaussian integer, z , that **cannot** be expressed in the form $z = \alpha\beta$ where α, β are Gaussian integers with $N(\alpha), N(\beta) > 1$.

- 1e. By expressing the positive integer $n = c^2 + d^2$ as a product of two Gaussian integers each of norm $c^2 + d^2$, show that n is not a Gaussian prime. *[3 marks]*

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The positive integer 2 is a prime number, however it is not a Gaussian prime.

- 1f. Verify that 2 is not a Gaussian prime. *[2 marks]*

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The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

1i. Hence show that $1 + 4i$ is a Gaussian prime.

[3 marks]

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1j. Use proof by contradiction to prove that a prime number, p , that is not of the form $a^2 + b^2$ is a Gaussian prime. [6 marks]

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