Paper 3 (3.12) *[30 marks]*

A **Gaussian integer** is a complex number, z, such that $z = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\beta = 1 - 2\mathrm{i}$, such that $\gamma = \alpha\beta$ for some Gaussian integer γ.

1a. Find γ. *[2 marks]*

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Now consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\gamma = 11 + 2\mathrm{i}$.

1b. Determine whether $\frac{\gamma}{\alpha}$ is a Gaussian integer. *[3 marks]*

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The norm of a complex number z, denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3\mathrm{i}$ then $N(2 + 3\mathrm{i}) = 2^2 + 3^2 = 13$.

1c. On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3. *[2 marks]*

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1d. Given that $\alpha = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$. *[1 mark]*

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A **Gaussian prime** is a Gaussian integer, z, that **cannot** be expressed in the form $z = \alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers with $N(\alpha),\ N(\beta) > 1$.

1e. By expressing the positive integer $n = c^2 + d^2$ as a product of two Gaussian integers each of norm $c^2 + d^2$, show that n is not a Gaussian prime. *[3 marks]*

The positive integer 2 is a prime number, however it is not a Gaussian prime.

1f. Verify that 2 is not a Gaussian prime. *[2 marks]*

1g. Write down another prime number of the form $c^2 + d^2$ that is not a Gaussian prime and express it as a product of two Gaussian integers. *[2 marks]*

Let $\alpha,\ \beta$ be Gaussian integers.

1h. Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. *[6 marks]*

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

1i. Hence show that $1 + 4\mathrm{i}$ is a Gaussian prime. *[3 marks]*

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1j. Use proof by contradiction to prove that a prime number, p, that is not of the form $a^2 + b^2$ is a Gaussian prime. *[6 marks]*