Tuesday 13.12 *[51 marks]*

The flight times, T minutes, between two cities can be modelled by a normal distribution with a mean of 75 minutes and a standard deviation of σ minutes.

1a. Given that 2% of the flight times are longer than 82 minutes, find the value of σ. *[3 marks]*

Markscheme

use of inverse normal to find z-score ***(M1)***

$z = 2.0537\ldots$

$2.0537\ldots = \frac{82-75}{\sigma}$ ***(A1)***

$\sigma = 3.408401\ldots$

$\sigma = 3.41$ ***A1***

[3 marks]

1b. Find the probability that a randomly selected flight will have a flight time of more than 80 minutes. *[2 marks]*

Markscheme

evidence of identifying the correct area under the normal curve ***(M1)***

$\mathrm{P}(T > 80) = 0.071193\ldots$

$\mathrm{P}(T > 80) = 0.0712$ ***A1***

[2 marks]

1c. Given that a flight between the two cities takes longer than 80 minutes, find the probability that it takes less than 82 minutes. *[4 marks]*

Markscheme

recognition that $\mathrm{P}(80 < T < 82)$ is required ***(M1)***

$\mathrm{P}\left(T < 82 \,\middle|\, T > 80\right) = \frac{\mathrm{P}(80<T<82)}{\mathrm{P}(T>80)} = \left(\frac{0.051193\ldots}{0.071193\ldots}\right)$ ***(M1)(A1)***

$= 0.719075\ldots$

$= 0.719$ ***A1***

[4 marks]

On a particular day, there are 64 flights scheduled between these two cities.

1d. Find the expected number of flights that will have a flight time of more than 80 minutes. *[3 marks]*

Markscheme

recognition of binomial probability ***(M1)***

$X \sim \mathrm{B}(64,\ 0.071193\ldots)$ or $\mathrm{E}(X) = 64 \times 0.071193\ldots$ ***(A1)***

$\mathrm{E}(X) = 4.556353\ldots$

$\mathrm{E}(X) = 4.56$ (flights) ***A1***

[3 marks]

1e. Find the probability that more than 6 of the flights on this particular day will have a flight time of more than 80 minutes. *[3 marks]*

Markscheme

$\mathrm{P}(X > 6) = \mathrm{P}(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 1 - 0.83088\ldots$ ***(A1)***

$= 0.1691196\ldots$

$= 0.169$ ***A1***

[3 marks]

The curve C has equation $\mathrm{e}^{2y} = x^3 + y$.

2a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y}-1}$. *[3 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts implicit differentiation on both sides of the equation **M1**

$2\mathrm{e}^{2y}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$\left(2\mathrm{e}^{2y} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ **A1**

so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y}-1}$ **AG**

[3 marks]

2b. The tangent to C at the point P is parallel to the y-axis. *[4 marks]*

Find the x-coordinate of P.

Markscheme

attempts to solve $2\mathrm{e}^{2y} - 1 = 0$ for y **(M1)**

$y = -0.346\ldots \left(= \frac{1}{2}\ln\frac{1}{2}\right)$ **A1**

attempts to solve $\mathrm{e}^{2y} = x^3 + y$ for x given their value of y **(M1)**

$x = 0.946\left(= \left(\frac{1}{2}\left(1 - \ln\frac{1}{2}\right)\right)^{\frac{1}{3}}\right)$ **A1**

[4 marks]

Consider the functions $f\left(x\right) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

3a. Find $(f \circ g)(x)$. *[2 marks]*

Markscheme

$(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

3b. Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. *[5 marks]*

Markscheme

$\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognising to use tan or cot ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle)

(A1)

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) = \right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.
Award ***A1A0*** for correct answers in degrees.
Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

Consider the three planes

$\prod_1 : \ 2x - y + z = 4$

$\prod_2 : \ x - 2y + 3z = 5$

$\prod_3 : -9x + 3y - 2z = 32$

4a. Show that the three planes do not intersect. *[4 marks]*

Markscheme

METHOD 1

attempt to eliminate a variable ***M1***

obtain a pair of equations in two variables

EITHER

$-3x + z = -3$ and ***A1***

$-3x + z = 44$ ***A1***

OR

$-5x + y = -7$ and ***A1***

$-5x + y = 40$ ***A1***

OR

$3x - z = 3$ and ***A1***

$3x - z = -\frac{79}{5}$ ***A1***

THEN

the two lines are parallel ($-3 \neq 44$ or $-7 \neq 40$ or $3 \neq -\frac{79}{5}$) ***R1***

Note: There are other possible pairs of equations in two variables.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

METHOD 2

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in $\prod_3$ ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)= 32$

$-15 = 32$, a contradiction ***R1***

hence the three planes do not intersect ***AG***

METHOD 3

attempt to eliminate a variable ***M1***

$-3y + 5z = 6$ ***A1***

$-3y + 5z = 100$ ***A1***

$0 = 94$, a contradiction ***R1***

Note: Accept other equivalent alternatives. Accept other valid methods.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

4b. Verify that the point $\mathrm{P}(1,\ -2,\ 0)$ lies on both $\prod_1$ and $\prod_2$. *[1 mark]*

Markscheme

$\prod_1 : 2+2+0=4$ and $\prod_2 : 1+4+0=5$ ***A1***

[1 mark]

4c. Find a vector equation of L, the line of intersection of $\prod_1$ and $\prod_2$. *[4 marks]*

Markscheme

METHOD 1

attempt to find the vector product of the two normals ***M1***

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$ ***A1***

$$r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if “$r =$” is missing.
Accept any multiple of the direction vector.
Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of “$r =$” only once.

METHOD 2

attempt to eliminate a variable from $\prod_1$ and $\prod_2$ ***M1***

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$

Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form)
A1

$$r = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if “$r =$” is missing.
Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes $\prod_1$ and $\prod_2$.

[4 marks]

4d. Find the distance between L and $\prod_3$. *[6 marks]*

Markscheme

METHOD 1

the line connecting L and $\prod_3$ is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$s = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$ ***A1***

substitute $(1 - 9t, -2 + 3t, -2t)$ in $\prod_3$ ***M1***

$-9(1 - 9t) + 3(-2 + 3t) - 2(-2t) = 32$

$94t = 47 \Rightarrow t = \frac{1}{2}$ ***A1***

attempt to find distance between $(1, -2, 0)$ and their point $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$ ***(M1)***

$= \left| \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$

$= \frac{\sqrt{94}}{2}$ ***A1***

METHOD 2

unit normal vector equation of $\prod_3$ is given by $\frac{\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$= \frac{32}{\sqrt{94}}$ ***A1***

let $\prod_4$ be the plane parallel to $\prod_3$ and passing through P,
then the normal vector equation of $\prod_4$ is given by

$\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = -15$ ***M1***

unit normal vector equation of $\prod_4$ is given by

$$\frac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}=\frac{-15}{\sqrt{94}}$$ ***A1***

distance between the planes is $\frac{32}{\sqrt{94}}-\frac{-15}{\sqrt{94}}$ ***(M1)***

$=\frac{47}{\sqrt{94}}\left(=\frac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

5a. Find the number of ways these five people can be seated in this row. *[3 marks]*

Markscheme

$6\times5!$ ***(A1)(A1)***

$=720$ (accept $6!$) ***A1***

[3 marks]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

5b. Find the number of ways these five people can now be seated in this row. *[4 marks]*

Markscheme

METHOD 1

(Peter apart from girls, in an end seat) $^{8}P_{4}(=1680)$ OR

(Peter apart from girls, not in end seat) $^{7}P_{4}(=840)$ ***(A1)***

case 1: Peter at either end

$2\times{}^{8}P_{4}(=3360)$ OR $2\times{}^{8}C_{4}\times4!(=3360)$ ***(A1)***

case 2: Peter not at the end

$8\times{}^{7}P_{4}(=6720)$ OR $8\times{}^{7}C_{4}\times4!(=6720)$ ***(A1)***

Total number of ways $=3360+6720$

$=10080$ ***A1***

METHOD 2

(Peter next to girl, in an end seat) $4\times{}^{8}P_{3}(=1344)$ OR

(Peter next to one girl, not in end seat) $2\times4\times{}^{7}P_{3}(=1680)$ OR

(Peter next to two girls, not in end seat) $4\times3\times{}^{7}P_{2}(=504)$ ***(A1)***

case 1: Peter at either end

$2\times4\times{}^{8}P_{3}(=2688)$ ***(A1)***

case 2: Peter not at the end

$8\left(2\times4\times{}^{7}P_{3}+4\times3\times{}^{7}P_{2}\right)(=17472)$ ***(A1)***

Total number of ways $={}^{10}P_{5}-(2688+17472)$

$=10080$ ***A1***

[4 marks]