1 Counting principles

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations which enable us to model real-world situations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn…

- how to find the number of ways of choosing an option from list A and an option from list B
- how to find the number of ways of choosing an option from list A or an option from list B
- how to find the number of permutations of n items
- how to find the number of ways of choosing r items from a list of n items, both when the order does not matter and when the order does matter.

CONCEPTS

The following concepts will be addressed in this chapter:

- Formulas are a **generalization** made on the basis of specific examples, which can then be extended to new examples.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Without a calculator, evaluate:

a $5!$

b ${}^{7}C_{3}$

2 A fair dice is rolled and a fair coin is flipped. Find the probability of:

a rolling a 6 on the dice and flipping heads on the coin

b rolling a 6 on the dice or flipping heads on the coin or both.

Figure 1.1 How many different combinations are there?

It may seem strange to only start talking about 'counting principles' at this stage of your mathematics education! However, while simple counting is one of the very first things we learn to do, counting arrangements and selections of items in certain situations can be rather complicated. It is important to have some basic strategies and to work systematically to make sure we neither miss anything out nor count the same thing more than once.

Starter Activity

Look at the pictures in Figure 1.1. Discuss why being able to count the number of ways certain events can occur is important.

Now look at this problem:

- **a** Write down all possible arrangements of the letters A, B, C.
- **b** Write down all possible selections of three letters from A, B, C, D, E. Note that ABC, BCA, and so on, count as the same selection.
- **c** Hence, without writing them all out, determine the number of possible arrangements of three letters chosen from A, B, C, D, E.

LEARNER PROFILE – Inquirers

Is mathematics just about answering other people's questions? Before you can do this, you need to get used to questioning other people's mathematics – asking questions like 'When does this work?', 'What assumptions are being made here?' or 'How does this link to what I already know?' are all second nature to mathematicians.

1A Basic techniques

The AND rule and the OR rule

If you want to choose one option from list A *and* one option from list B, then you can find the number of possible ways of doing this by multiplying the number of options in list A, $n(A)$, by the number of options in list B, $n(B)$.

KEY POINT 1.1

The AND rule:

$$n(A \text{ AND } B) = n(A) \times n(B)$$

Similarly, for the number of ways of choosing one option from list A *or* one option from list B, you add the number of options in each list. However, you need to make sure that the lists are mutually exclusive.

KEY POINT 1.2

The OR rule:

If A and B are mutually exclusive, then

$$n(A \text{ OR } B) = n(A) + n(B)$$

You saw very similar rules for probability in Chapter 7 of the Mathematics: analysis and approaches SL book.

WORKED EXAMPLE 1.1

Rohan has four jackets and seven ties in his wardrobe.

Calculate the number of different ways he can choose to dress if he wears:

a a jacket and a tie

b a jacket or a tie.

Use $n(A \text{ AND } B) = n(A) \times n(B)$

a Number of ways $= 4 \times 7$
$= 28$

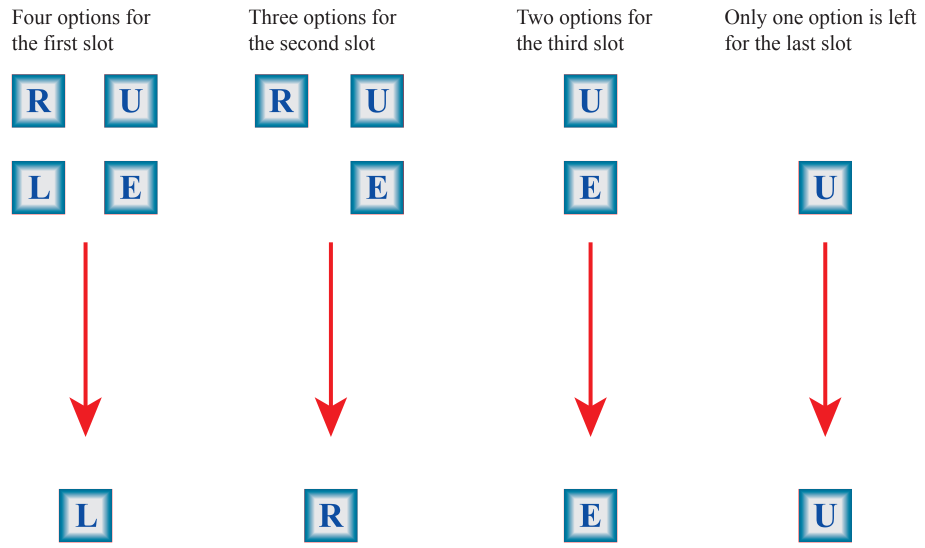
Jackets and ties are mutually exclusive so use $n(A \text{ OR } B) = n(A) + n(B)$

b Number of ways $= 4 + 7$
$= 11$

Permutations and combinations

To calculate how many ways the letters R, U, L, E can be arranged, you can consider the number of letter options available for the first position, second position and so on and use the AND rule:

- first position: 4 options
- second position: 3 options
- third position: 2 options
- fourth position: 1 option.



Total number of ways $= 4 \times 3 \times 2 \times 1 = 24$

Each of these 24 arrangements (RULE, RUEL, LUER, and so on) is called a **permutation**.

This method provides a useful result.

KEY POINT 1.3

The number of permutations of n items is $n!$

TOOLKIT: Problem Solving

How many ways are there of arranging five objects in a circle? Can you find a formula for the number of ways of arranging n objects in a circle?

This idea can be combined with the AND or the OR rule.

WORKED EXAMPLE 1.2

Jason wants to set up a new username consisting of the five letters J, A, S, O, N followed by the three digits 1, 2, 3.

Find the number of different ways he can do this.

There are 5! permutations of the letters AND 3! permutations of the numbers

$$\begin{aligned}\text{Number of ways} &= 5! \times 3! \\ &= 120 \times 6 \\ &= 720\end{aligned}$$

Instead of finding arrangements of a given number of items, you might be interested in finding the number of ways of choosing some items from a larger list, for example the number of ways of choosing four letters from the list R, U, L, E, S.

You might just want to know how many different groups of four letters can be chosen (so RULE, RUEL, LURE, and so on, would just count as one choice). A selection like this where the order does not matter is called a **combination**. There are only five combinations of four letters from this list: RULE, RULS, RUES, RLES, ULES.

The number of combinations can be calculated in general using the method you met when finding binomial coefficients.

KEY POINT 1.4

The number of ways of choosing r items from n when the order does not matter is

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Proof 1.1

Prove that $^nC_r = \dfrac{n!}{r!(n-r)!}$

One classic method of proof is finding the same thing in two different ways

Consider the number of ways of permuting n distinct objects. This is $n!$

However, it could also be done by

- choosing r objects from the n

AND

- putting these r objects into an order

AND

- ordering the remaining $n - r$ objects.

We are using 'choosing r objects out of n' as the defining feature of ^nC_r

This is done in the following number of ways:

We then apply the AND rule

$$^nC_r \times r! \times (n-r)!$$

Equating the two ways of permuting n objects

Therefore, $\quad n! = {}^nC_r \times r! \times (n-r)!$

Rearranging:

$$^nC_r = \frac{n!}{r!(n-r)!}$$

CONCEPT – GENERALIZATION

A very good way of understanding a new formula or proof is to try a specific example and see how it might **generalize**.

Consider the proof above and permuting the letters ABCD. Think about one pair of letters and arrange them in two ways, then arrange the remaining pair of letters in two ways. For each combination of the first pair of letters, there are therefore four ways of arranging them all. Given that there are six ways of picking two letters out of four, this makes the overall number of permutations 24.

Once you have understood the problem using a specific example, you can then generalize the idea using proof, as illustrated above.

See Chapter 13 of the Mathematics: analysis and approaches SL book for a reminder of how to work with the ${}^nC_r$ formula.

You might also want to count all the different orderings (permutations) of items picked from a larger list. With the example of choosing four letters from R, U, L, E, S, you have the five combinations (RULE, RULS, RUES, RLES, ULES) and each of these will have $4! = 24$ permutations, giving $5 \times 24 = 120$ ways of selecting four letters from RULES when the order matters.

The symbol for the number of ways of choosing r objects out of n distinct objects is ${}^nP_r$.

It can be calculated using:

$$ {}^nP_r = {}^nC_r \times r! = \frac{n!}{r!(n-r)!} \times r! = \frac{n!}{(n-r)!} $$

Tip

Remember that ${}^nP_r$ and ${}^nC_r$ are both most easily evaluated on a calculator.

KEY POINT 1.5

The number of ways of choosing r items from n when the order does matter is

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

WORKED EXAMPLE 1.3

A maths teacher needs to select a team of four students from a class of 19 to represent the school in a maths competition.

Find the number of different ways she can choose the team.

Picking Albert, Billy, Camille and Dani will result in the same team as Billy, Dani, Albert and Camille, so the order doesn't matter. Therefore, use ${}^nC_r$

Number of ways $= {}^{19}C_4$
$= 3876$

WORKED EXAMPLE 1.4

The board of directors of a company consists of 12 members. They need to appoint a Chair, a Chief Finance Officer and a Secretary.

Find the number of different ways this could be done.

Let us say that the first person chosen will be the Chair, the second will be Chief Finance Officer and the third will be the Secretary. Therefore, the order does matter, so use ${}^nP_r$

Number of ways $= {}^{12}P_3$
$= 1320$

You are the Researcher

Although counting is one of the first topics you meet when you start studying mathematics, it is also one of the hardest topics advanced mathematicians deal with. You might be interested in researching Hilbert's Hotel and how it helps to explain counting to infinity. Ramsey theory covers an area of mathematics that deals with counting on networks and the numbers used in this are some of the largest ever found to have a useful application.

The basic tools of ${}^nC_r$ and ${}^nP_r$ can be combined with the AND and OR rules to break down harder problems.

WORKED EXAMPLE 1.5

A class has ten girls and eight boys. A committee of six must have an equal number of boys and girls. In how many ways can this be done?

Break the problem down into smaller parts which are of the form required to use ${}^nC_r$ and ${}^nP_r$ ······ We need to choose 3 girls from 10 AND 3 boys from 8

Since order does not matter, we should use ${}^nC_r$ ······ This can be done in ${}^{10}C_3 \times {}^8C_3 = 120 \times 56$
$= 6720$ ways

Exercise 1A

In questions 1 to 3, use the method demonstrated in Worked Example 1.1 to find the number of ways of choosing

1 a one item from a list of eight and one item from a different list of five
  b one item from a list of seven and one item from a different list of four
2 a one item from a list of eight or one item from a different list of five
  b one item from a list of seven or one item from a different list of four
3 a one item from each of a list of eight, a list of five and a list of seven
  b one item from either a list of eight, a list of five or a list of seven.

In questions 4 to 6, use the method demonstrated in Worked Example 1.2 to find the number of permutations of

4 a nine items
  b six items
5 a four items followed by five different items
  b ten items followed by three different items
6 a either two items followed by five or three items followed by four
  b either three items followed by seven or four items followed by six.

In questions 7 to 9, use the method demonstrated in Worked Example 1.3 to find the number of ways of choosing

7 a five items from nine
  b three items from eight
8 a four items from seven, followed by two items from six other items
  b six items from eight, followed by four items from ten other items
9 a either three items from twelve or four items from eleven
  b either five items from eight or six items from ten.

In questions 10 to 12, use the method demonstrated in Worked Example 1.4 to find the number of ways of permuting

10 **a** five items from nine

b three items from eight

11 **a** four items from seven followed by two items from six other items

b six items from eight followed by four items from ten other items

12 **a** either three items from twelve or four items from eleven

b either five items from eight or six items from ten.

13 Amelie wants a pet cat and a pet dog. There are five breeds of cat and eleven breeds of dog at her local pet shop.

Find the number of possible ways she can choose a cat and dog.

14 There are seven men and four women who would like to play as a team of two in a bridge tournament.

Find the number of ways a pair can be chosen if one player has to be male and the other female.

15 A headteacher wants to choose a school council consisting of one student from each of Years 9, 10, 11, 12 and 13.

There are 95 students in Year 9, 92 in Year 10, 86 in Year 11, 115 in Year 12 and 121 in Year 13.

a Find the number of ways of choosing the school council.

The headteacher now decides that he only wants one student from either Year 9 or Year 10 and one each from Year 11, 12 and 13.

b Find the number of ways the school council can be chosen now.

16 A menu at a restaurant offers five starters, eight main courses and six desserts.

Find the number of different choices of meal you can make if you would like

a a starter, a main course and a desert

b a main course and either a starter or a desert

c any two different courses.

17 A mixed soccer team of six boys and five girls are having a team photo taken. The girls are arranged in the front row and the boys in the back row.

Find the number of possible arrangements.

18 **a** Find the number of seven-digit numbers that can be formed using the digits 1 to 7 exactly once each.

b How many of these are divisible by five?

19 David is planting a flower bed with six different types of rose and two different types of tulip.

They are all planted in a line.

a Find the number of possible arrangements.

b How many of these arrangements have the tulips at either end?

20 An exam paper consists of ten questions. Students can select any six questions to answer.

Find the number of different selections that can be made.

21 Ulrike is revising for seven subjects, but can only complete three in any one evening.

a Find the number of ways she could choose which subjects to revise in an evening.

b If she decides that one of her three subjects must be mathematics, find the number of ways she could choose the subjects to revise in an evening.

22 Find the number of four-digit numbers that can be made with the digits 1 to 9 if no digit can be repeated.

23 Find the number of ways the gold, silver and bronze medals could be awarded in a race consisting of eight athletes.

24 A teacher wants to award the Science Prize and the Humanities Prize to two different students in her class of 17.

Find the number of possible selections she could make.

25 Dr Walker has nine different shirts (three each of white, blue and green), six different pairs of trousers (two each of black, grey and blue) and four different waistcoats (one each of black, blue, beige and red).

He always wears a pair of trousers, a shirt and a waistcoat.

a Find the number of different outfits he can wear.

Dr Walker never wears blue and green together.

b Find the number of different outfits he can wear with this restriction.

26 a Find how many five-digit numbers can be formed using the digits 1 to 5 exactly once each.

b How many of these numbers are less than 40 000?

27 In a lottery, players select five numbers from the numbers 1 to 40 and then two further bonus numbers from the numbers 1 to 10.

Find the number of possible selections.

28 A hockey team consists of one goalkeeper, three defenders, five midfielders and two forwards.

The coach has three goalkeepers, six defenders, eight midfielders and four forwards in the squad.

Find the number of ways she can pick the team.

29 There are 15 places on a school trip to Paris, 12 on a trip to Rome and 10 on a trip to Athens.

There are 48 students in Year 12 who have applied to go on a trip, and they are all happy to go on any of the options available.

Find the number of ways the places can be allocated.

30 A committee of three boys and three girls needs to be chosen from 18 boys and 15 girls. Ahmed is the chairman, so he has to be on the committee.

a Find the number of ways the committee can be selected.

b If Baha or Connie but not both must be chosen from the girls, find the number of ways the committee can be selected now.

31 Eight athletes compete in a race. Find the number of possible ways the gold, silver and bronze medals can be awarded if Usain wins either gold or silver.

32 There are 16 girls and 14 boys in a class. Three girls and two boys are needed to play the lead roles in a play.

Find the number of ways these five roles can be cast.

33 Student ID codes must consist of either three different letters chosen from the letters A to Z followed by four different digits chosen from the numbers 1 to 9, or of four different letters followed by three different digits.

Find the number of possible ID codes available.

34 A class consists of eight girls and seven boys. A committee of five is chosen.

a How many possible committees can be chosen if there are no constraints?

b How many different committees are possible if

i it must contain Jamila

ii it must contain at least one girl.

c If the committee is chosen at random, what is the probability that it contains at least one girl?

35 Solve the equation ${}^{n}C_2 = 210$.

36 Solve the equation ${}^{n}P_2 = 132$.

37 Prove that ${}^{n}P_n = {}^{n}P_{n-1}$.

38 Find the number of ways that

a five presents can be put into two boxes

b three presents can be put into four boxes.

39 A class of 18 students are lining up in three rows of six for a class photo.

Find the number of different arrangements.

40 Ten points on a plane are drawn so that no three lie in a straight line. If the points are connected by straight lines and if vertices can only occur at the original points, find the number of different

a triangles that can be formed

b quadrilaterals that can be formed.

41 At a party, everyone shakes hands with everyone else. In total there are 465 handshakes.

Find the number of people at the party.

1B Problem solving

In addition to the techniques encountered in Section 1A, there are a few further techniques that are often useful in solving more-complex problems:

- **Count what you do not want and subtract that from the total to get what you do want.**

  For example, to find how many numbers between 1 and 100 are not divisible by five, count how many are divisible by five (20) and subtract from the total (100) to give 80 numbers that are not divisible by five.

- **Treat any items that must be together as a single item in the list.**

  For example, to find the number of permutations of ABCDEFG where A and B must be together, treat 'AB' as being on a single tile:

  C D AB E F G H

  But remember that the items that are together also need to be permuted.

  C D BA E F G H

WORKED EXAMPLE 1.6

Anushka, Beth, Caroline, Dina, Elizabeth, Freya and Georgie line up for a netball team photo.

Find the number of possible arrangements in which:

a Anushka and Beth are next to each other

b Anushka and Beth are not next to each other.

Treat A and B as one item in the list (X). There are 6! permutations of XCDEFG AND A and B can be arranged in 2! ways

a Number of arrangements with A and B together $= 6! \times 2!$

$= 1440$

This is given by the total number of permutations of ABCDEFG minus the number where A and B are together

b Number of arrangements with A and B apart

$= 7! - 1440$

$= 3600$

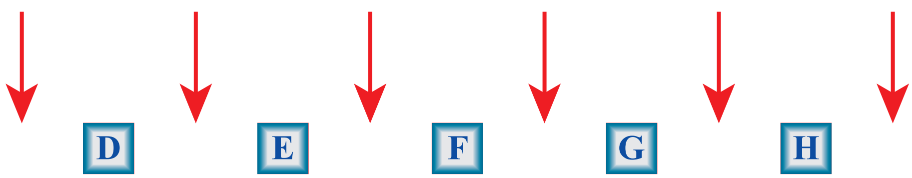
Tip

Notice that the number of permutations in which three (or more) items are kept apart can't be found by subtracting the number of permutations with the items together from the total. This would leave permutations with two of the items together still being counted.

The following technique is useful if more than two items need to be kept apart:

- **To separate three or more items, consider the gaps they can fit into.**

  For example, to separate A, B and C from the list of ABCDEFGH, consider the six gaps created by the other letters into which A, B and C could fit:



WORKED EXAMPLE 1.7

Find the number of possible arrangements of the single-row netball photograph line-up in which Anushka, Beth and Caroline are not next to each other.

We have five possible gaps for A, B and C to fill so they are not together:
_ D _ E _ F _ G _

Permute DEFG
AND
Permute ABC in 3 out of the 5 gaps

Number of arrangements with A, B, C apart

$= 4! \times {}^5P_3$

$= 1440$

You are the Researcher

Can you find out the number of ways to arrange n objects if some of them are identical?

TOK Links

Does having a symbol, such as ${}^5P_3$, to describe a standard calculation add to your mathematical knowledge? Does attaching a label to something help us to use an idea more effectively?

Exercise 1B

1 Find the number of ways ten different cheeses can be arranged on a shelf in a supermarket, if the brie must be next to the camembert.

2 Find the number of ways six different Standard Level IB textbooks and three different Higher Level IB textbooks can be arranged on a bookshelf if the three HL books have to be together.

3 Find the number of permutations of the letters A, B, C, D, E, F that do not start with A.

4 Find the number of seven-digit codes that do not end with '67' using each of the digits 1 to 7 once.

5 Alessia has seven different soft toys and five different toy cars she likes to play with.

Find the number of ways she can choose four of these to play with if at least one of them must be a toy car.

6 Find the number of permutations of the word 'COMPUTE' that do not have C, O and M in the first three letters.

7 A box contains 25 different chocolates: ten dark, eight milk and seven white.

Find the number of ways of choosing three chocolates such that

a they are all a different type

b they are not all dark.

8 Find the number of permutations of the letters D, I, P, L, O, M, A that do not begin with D or end with A.

9 Three letters are selected from the word 'COUNTED' and arranged in order.

Find the number of these arrangements that contain at least one vowel.

10 Eight students are to be chosen from 16 girls and 13 boys.

Find the number of ways this can be done if at least two girls must be chosen.

11 Seven different milk chocolate bars, five different white chocolate bars and four different dark chocolate bars are lined up on a shelf in a sweet shop.

Find the number of possible arrangements if all chocolate bars of the same type must be together.

12 Six students from class 12A, four students from 12B and three students from 12C all arrive to line up in the lunch queue.

Find the number of possible ways they can arrange themselves given that the students from 12C must be separated.

13 The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged at random.
Find the probability that no odd number is next to another odd number.

14 Mr and Mrs Semba and their four children line up for a family photo.
Find the number of ways they can do this if
- **a** Mr and Mrs Semba are at opposite ends of the line
- **b** Mr and Mrs Semba are together.

The family lines up at random.
- **c** Find the probability that Mr and Mrs Semba are not next to each other.

15 Five cards are dealt from a randomly shuffled standard deck of 52 playing cards. Find the probability of getting
- **a** all spades
- **b** all red cards
- **c** at least two black cards.

16 Six members of a family go to the cinema and all want to be seated on the same row next to each other. This row contains 20 seats.
Find the number of ways they can arrange themselves.

17 A six-a-side football team is to be selected from short-listed players from the top two sides in the league. Eight players are short-listed from Team A and seven from Team B.
Find the number of ways the side can be chosen if there must be at least two players from Team A and at least one from Team B.

18 In a word game, there are 26 tiles each printed with a different letter.
Find the number of ways of choosing seven tiles if at least two of them are vowels.

Checklist

- You should be able to find the number of ways of choosing an option from list A and an option from list B. The AND rule:
  - $n(A \text{ AND } B) = n(A) \times n(B)$
- You should be able to find the number of ways of choosing an option from list A or an option from list B. The OR rule:
  - If A and B are mutually exclusive:
    $n(A \text{ OR } B) = n(A) + n(B)$
- You should be able to find the number of permutations of n items.
  - The number of permutations of n items is $n!$
- You should be able to find the number of ways of choosing r items from a list of n items.
  - The number of combinations of r objects out of n is written as:
    $^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ when the order does not matter.
  - The number of permutations of r objects out of n is written as:
    $^{n}P_{r} = \dfrac{n!}{(n-r)!}$ when the order does matter.

Mixed Practice

1 Eight athletes take part in the 100 m final of the Olympic games.

Find the number of ways the three medals could be won.

2 A car registration number consists of four different letters followed by three different digits chosen from 1 to 9.

Find the number of possible registration numbers.

3 A team of six students is to be chosen from a class of 15 girls and 11 boys.

Find the number of possible selections if the team must contain the same number of boys and girls.

4 Rylan has written Christmas cards for five friends and sealed each of them in a separate envelope. However, he has forgotten which card is which.

If he now wrote the names on the envelopes at random, find the probability that all the cards and envelopes would match.

5 Three teachers and eight students want to join the lunch queue. Find the number of different ways this can be done if the three teachers must go first.

6 There are six places available on a school mathematics trip and 19 students who have applied to go on the trip.

a Find the number of possible selections that could be made.
b Find the probability that Jack and Jill are both chosen.

7 Find the number of three-digit numbers that contain no zeros.

8 A football manager has a squad consisting of three goalkeepers, seven defenders, eight midfielders and four forwards.

He wants to select a team of one goalkeeper, four defenders, four midfielders and two forwards. Find the number of possible selections.

9 Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

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10 Find the number of permutations of the word 'CHANGED' that start and end with a consonant.

11 A committee of five students is to be selected from a class of 28. The two youngest cannot both be on the committee.

Find the number of ways the committee can be selected.

12 A group of 19 students contains nine boys and ten girls. A team is said to be mixed if it contains at least one boy and at least one girl.

Find the number of ways a mixed team of seven can be chosen from the 19 students.

13 A class consists of six girls and nine boys. A committee of six is chosen.

a How many possible committees can be chosen if there are no constraints?
b How many different committees are possible if
  i it must not contain John
  ii it must contain at least two girls.
c If the committee is chosen at random, what is the probability that it contains at least two girls?

14 12 people need to travel to a hockey match in two vehicles: a car, which can carry four people and an SUV which can carry eight people.

Given that only two of the 12 can drive, find the number of ways they can be allocated to the two vehicles.

15 Henrik draws seven letter tiles from a bag: D, M, S, T, A, E, O.

a Find the number of arrangements of the letters.
b Find the number of arrangements with the three vowels together.
c Find the number of arrangements with the three vowels all separated.

16 Amit, Brian, Connor, Dan and Ed stand in a line.

Find the number of possible permutations in which

a Amit is at one end of the line
b Amit is not at either end
c Amit is at the left end of the line or Ed is at the right end, or both.

17 Players in a lottery choose six different numbers from 1 to 50 inclusive.

a Find the probability of matching all six numbers.

There is a prize for anyone matching three numbers or more.

b Find the probability of winning a cash prize.

18 There are n different letters in a bag. If 380 possible two-letter 'words' can be formed from these letters, find the value of n.

19 Three boys and three girls are to sit on a bench for a photograph.

a Find the number of ways this can be done if the three girls must sit together.
b Find the number of ways this can be done if the three girls must all sit apart.

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20 A set of positive integers {1, 2, 3, 4, 5, 6, 7, 8, 9} is used to form a pack of nine cards. Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

a Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7.
b Find the number of selections Grace could make if at least two of the four integers drawn are even.

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21 An SUV has eight seats: two at the front, a row of three in the middle and a row of three at the back. Seven people are travelling in the car, three of whom can drive.

Find the number of ways they can be seated.

22 In a doctor's waiting room, there are 10 seats in a row. Six people are waiting to be seen.

a Find the number of ways they can be seated.
b One of them has a bad cold and mustn't sit next to anyone else. Find the number of ways the six people can be seated now.

23 Twelve friends arrive at a quiz, but are told that the maximum number in a team is six. Find the number of ways they can split up into

a two teams of six
b three teams of four.

Answers

Chapter 1 Prior Knowledge

1 a 120 b 35

2 a $\frac{1}{12}$ b $\frac{7}{12}$

Exercise 1A

1 a 40 b 28

2 a 13 b 11

3 a 280 b 20

4 a 362 880 b 720

5 a 2880 b 2 177 280

6 a 384 b 47 520

7 a 126 b 56

8 a 525 b 5880

9 a 550 b 266

10 a 15 120 b 336

11 a 25 200 b 101 606 400

12 a 9240 b 157 920

13 55

14 28

15 a 1.05×10^{10} b 223 781 030

16 a 240 b 88 c 118

17 86 400

18 a 5040 b 720

19 a 40 320 b 1440

20 210

21 a 35 b 15

22 3024

23 336

24 272

25 a 216 b 180

26 a 120 b 72

27 29 610 360

28 20 160

29 1.37×10^{26}

30 a 61 880 b 21 216

31 84

32 611 520

33 228 009 600

34 a 3003 b i 1001 ii 2982 c 0.993

35 21

36 12

38 a 32 b 81

39 6.40×10^{15}

40 a 120 b 210

41 31

Exercise 1B

1 725 760

2 30 240

3 600

4 4920

5 460

6 4896

7 a 560 b 2180

8 3720

9 186

10 4 263 402

11 87 091 200

12 3 592 512 000

13 $\frac{1}{126}$

14 a 48 b 240 c $\frac{2}{3}$

15 a $\frac{33}{66\,640}$ b $\frac{253}{9996}$ c $\frac{2062}{2499}$

16 10 800

17 4802

18 270 200

Chapter 1 Mixed Practice

1 336

2 180 835 200

3 75 075

4 $\frac{1}{120}$

5 241 920

6 a 27 132 b $\frac{5}{57}$

7 729

8 44 100

9 210

10 2400

11 95 680

12 50 232

13 a 5005

b i 3003 ii 4165

c 0.832

14 240

15 a 5040 b 720

c 1440

16 a 48 b 72

c 42

17 a $\frac{1}{15\,890\,700}$ b $\frac{347}{19\,740}$

18 20

19 a 144 b 144

20 a 34 b 81

21 15 120

22 a 151 200

b 33 600

23 a 462

b 5775

Chapter 2 Prior Knowledge

1 $16-96x+216x^2-216x^3+81x^4$

2 $(5x-2)(x+3)$

3 $\frac{3x+5}{x^2+2x-3}$

4 $x=5,\ y=-2$

Exercise 2A

1 a $1-2x+3x^2+\dots,\ |x|<1$

b $1-3x+6x^2+\dots,\ |x|<1$

2 a $1+\frac{1}{3}x-\frac{1}{9}x^2+\dots,\ |x|<1$

b $1+\frac{1}{4}x-\frac{3}{32}x^2+\dots,\ |x|<1$

3 a $1+\frac{1}{4}x+\frac{1}{16}x^2+\dots,\ |x|<4$

b $1-2x+\frac{5}{2}x^2+\dots,\ |x|<2$

4 a $1-x+\frac{3}{2}x^2+\dots,\ |x|<\frac{1}{2}$

b $1+2x+5x^2+\dots,\ |x|<\frac{1}{3}$

5 a $\frac{1}{3}-\frac{1}{9}x+\frac{1}{27}x^2+\dots,\ |x|<3$

b $\frac{1}{25}-\frac{2}{125}x+\frac{3}{625}x^2+\dots,\ |x|<5$

6 a $2+\frac{1}{12}x-\frac{1}{288}x^2+\dots,\ |x|<8$

b $3+\frac{1}{6}x-\frac{1}{216}x^2+\dots,\ |x|<9$

7 a $\frac{1}{8}+\frac{9}{16}x+\frac{27}{16}x^2+\dots,\ |x|<\frac{2}{3}$

b $\frac{1}{3}-\frac{4}{9}x+\frac{16}{27}x^2+\dots,\ |x|<\frac{3}{4}$

8 a $128+7x+\frac{21}{256}x^2+\dots,\ |x|<32$

b $8-x+\frac{1}{48}x^2+\dots,\ |x|<12$

9 $1-x-\frac{1}{2}x^2-\frac{1}{2}x^3+\dots$

10 $1+\frac{3}{4}x+\frac{3}{8}x^2+\frac{5}{32}x^3+\dots$

11 $2-\frac{1}{12}x+\frac{1}{144}x^2+\dots$

12 $\frac{1}{2}+\frac{5}{4}x+\frac{25}{8}x^2+\dots$

13 a $3+\frac{1}{6}x-\frac{1}{216}x^2+\dots$

b $|x|<9$

c 3.162 04