Differential equations *[155 marks]*

Consider the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right), x > 0$

1a. Use the substitution $y = vx$ to show that $\int \frac{\mathrm{d}v}{f(v)-v} = \ln x + C$ where C is an arbitrary constant. *[3 marks]*

The curve $y = f(x)$ for $x > 0$ has a gradient function given by

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+3xy+2x^2}{x^2}$.

The curve passes through the point $(1,\ -1)$.

1b. By using the result from part (a) or otherwise, solve the differential equation and hence show that the curve has equation $y = x(\tan(\ln x) - 1)$. *[9 marks]*

1c. The curve has a point of inflexion at $(x_1,\ y_1)$ where $\mathrm{e}^{-\frac{\pi}{2}} < x_1 < \mathrm{e}^{\frac{\pi}{2}}$. Determine the coordinates of this point of inflexion. *[6 marks]*

1d. Use the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+3xy+2x^2}{x^2}$ to show that the points of zero gradient on the curve lie on two straight lines of the form $y = mx$ where the values of m are to be determined. *[4 marks]*

Consider the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

2a. Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 1.5$. *[4 marks]*

2b. Use the substitution $y = vx$ to show that $x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - v - 2$. *[3 marks]*

2c. By solving the differential equation, show that $y = \frac{8x+x^4}{4-x^3}$. *[10 marks]*

2d. Find the actual value of y when $x = 1.5$. *[1 mark]*

2e. Using the graph of $y = \frac{8x+x^4}{4-x^3}$, suggest a reason why the approximation given by Euler's method in part (a) is not a good estimate to the actual value of y at $x = 1.5$. *[1 mark]*

3. Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x},\ x > 0$, given that $y = 4$ at $x = \frac{1}{2}$. *[7 marks]*

Give your answer in the form $y = f(x)$.

The function f has a derivative given by $f'(x) = \frac{1}{x(k-x)}, x \in \mathbb{R}, x \neq o, x \neq k$ where k is a positive constant.

4a. The expression for $f'(x)$ can be written in the form $\frac{a}{x} + \frac{b}{k-x}$, where $a, b \in \mathbb{R}$. Find a and b in terms of k. *[3 marks]*

4b. Hence, find an expression for $f(x)$. *[3 marks]*

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P(k-P)}{5k}$, where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

4c. By solving the differential equation, show that $P = \frac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}}+1200}$. *[8 marks]*

At $t = 10$ the population of the colony has doubled in size from its initial value.

4d. Find the value of k, giving your answer correct to four significant figures. *[3 marks]*

4e. Find the value of t when the rate of change of the population is at its maximum. *[3 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-x}{y+x}$, where $x, y > 0$.

It is given that $y = 2$ when $x = 1$.

5a. Solve the differential equation, giving your answer in the form $f(x,\ y) = 0$. *[9 marks]*

5b. The graph of y against x has a local maximum between $x = 2$ and $x = 3$. Determine the coordinates of this local maximum. *[4 marks]*

5c. Show that there are no points of inflexion on the graph of y against x. *[4 marks]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^2+y^2-xy}{x^2}$, with $y = 2$ when $x = 1$.

6a. Use Euler's method, with step length $h = 0.1$, to find an approximate value of y when $x = 1.4$. *[5 marks]*

6b. Express $m^2 - 2m + 4$ in the form $(m-a)^2 + b$, where $a, b \in \mathbb{Z}$. *[1 mark]*

6c. Solve the differential equation, for $x > 0$, giving your answer in the form $y = f(x)$. *[10 marks]*

6d. Sketch the graph of $y = f(x)$ for $1 \leqslant x \leqslant 1.4$. *[1 mark]*

6e. With reference to the curvature of your sketch in part (c)(iii), and without further calculation, explain whether you conjecture $f(1.4)$ will be less than, equal to, or greater than your answer in part (a). *[2 marks]*

Consider the differential equation $2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2$, where $x > 0$.

7a. Solve the differential equation and show that a general solution is $x^2 + y^2 = cx$ where c is a positive constant. *[11 marks]*

7b. Prove that there are two horizontal tangents to the general solution curve and state their equations, in terms of c. *[5 marks]*

Consider the differential equation $\frac{dy}{dx} = 1 + \frac{y}{x}$, where $x \neq 0$.

8a. Given that $y(1) = 1$, use Euler's method with step length h = 0.25 to find an approximation for $y(2)$. Give your answer to two significant figures. *[4 marks]*

8b. Solve the equation $\frac{dy}{dx} = 1 + \frac{y}{x}$ for $y(1) = 1$. *[6 marks]*

8c. Find the percentage error when $y(2)$ is approximated by the final rounded value found in part (a). Give your answer to two significant figures. *[3 marks]*

Consider the differential equation $x\frac{dy}{dx} - y = x^p + 1$ where $x \in \mathbb{R}$, $x \neq 0$ and p is a positive integer, $p > 1$.

9a. Solve the differential equation given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$. *[8 marks]*

9b. Show that the x-coordinate(s) of the points on the curve $y = f(x)$ where $\frac{dy}{dx} = 0$ satisfy the equation $x^{p-1} = \frac{1}{p}$. *[2 marks]*

9c. Deduce the set of values for p such that there are two points on the curve $y = f(x)$ where $\frac{dy}{dx} = 0$. Give a reason for your answer. *[2 marks]*

Consider the differential equation $\frac{dy}{dx} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

10a. Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. *[4 marks]*

10b. Solve the differential equation giving your answer in the form $y = f(x)$. *[6 marks]*