Differential equations *[155 marks]*

Consider the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right), x > 0$

1a. Use the substitution $y = vx$ to show that $\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + C$ where C is an arbitrary constant. *[3 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ **M1**

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v)$ **A1**

$\int \frac{\mathrm{d}v}{f(v) - v} = \int \frac{\mathrm{d}x}{x}$ **A1**

integrating the RHS, $\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + C$ **AG**

[3 marks]

The curve $y = f(x)$ for $x > 0$ has a gradient function given by

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + 3xy + 2x^2}{x^2}$.

The curve passes through the point $(1, -1)$.

1b. By using the result from part (a) or otherwise, solve the differential equation and hence show that the curve has equation $y = x(\tan(\ln x) - 1)$. *[9 marks]*

Markscheme

EITHER

attempts to find $f(v)$ **M1**

$f(v) = v^2 + 3v + 2$ **(A1)**

substitutes their $f(v)$ into $\int \frac{\mathrm{d}v}{f(v) - v}$ **M1**

$\int \frac{\mathrm{d}v}{f(v) - v} = \int \frac{\mathrm{d}v}{v^2 + 2v + 2}$

attempts to complete the square **(M1)**

$\int \frac{\mathrm{d}v}{(v+1)^2 + 1}$ **A1**

$\arctan(v + 1) (= \ln x + C)$ **A1**

OR

attempts to find $f(v)$ **M1**

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 + 3v + 2$ **A1**

$\int \frac{\mathrm{d}v}{v^2 + 2v + 2} = \frac{\mathrm{d}x}{x}$ **M1**

attempts to complete the square **(M1)**

$\int \frac{\mathrm{d}v}{(v+1)^2 + 1} \left(= \int \frac{\mathrm{d}x}{x}\right)$ **A1**

$\arctan(v + 1) (= \ln x + C)$ **A1**

THEN

when $x = 1$, $v = -1$ (or $y = -1$) and so $C = 0$ **M1**

substitutes for v into their expression **M1**

$\arctan\left(\frac{y}{x} + 1\right) = \ln x$

$\frac{y}{x} + 1 = \tan(\ln x)$ **A1**

so $y = x(\tan(\ln x) - 1)$ **AG**

[9 marks]

1c. The curve has a point of inflexion at $(x_1,\ y_1)$ where $\mathrm{e}^{-\frac{\pi}{2}} < x_1 < \mathrm{e}^{\frac{\pi}{2}}$. Determine the coordinates of this point of inflexion. *[6 marks]*

Markscheme

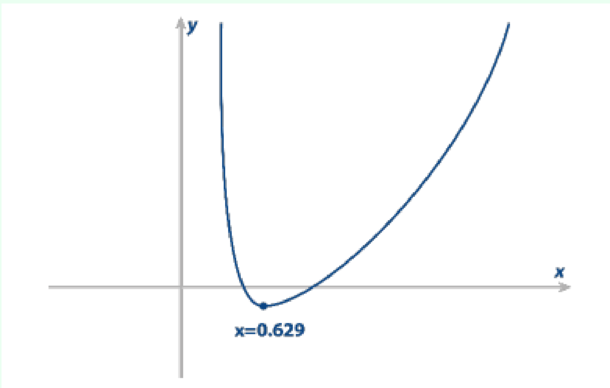
METHOD 1

EITHER

a correct graph of $y = f'(x)$ (for approximately $e^{-\frac{\pi}{2}} < x < e^{\frac{\pi}{2}}$) with a local minimum point below the x-axis **A2**

Note: Award **M1A1** for $\frac{dy}{dx} = \tan(\ln x) + \sec^2(\ln x) - 1$.

attempts to find the x-coordinate of the local minimum point on the graph of $y = f'(x)$ **(M1)**



OR

a correct graph of $y = f''(x)$ (for approximately $e^{-\frac{\pi}{2}} < x < e^{\frac{\pi}{2}}$) showing the location of the x-intercept **A2**

Note: Award **M1A1** for $\frac{d^2y}{dx^2} = \frac{\sec^2(\ln x)}{x} + \frac{2\sec^2(\ln x)\tan(\ln x)}{x}$.

attempts to find the x-intercept **(M1)**

THEN

$x = 0.629\left(= e^{-\arctan\frac{1}{2}}\right)$ **A1**

attempts to find $f(0.629)\left(f\left(e^{-\arctan\frac{1}{2}}\right)\right)$ **(M1)**

the coordinates are $(0.629, -0.943)\left(e^{-\arctan\frac{1}{2}}, -\frac{3}{2}e^{-\arctan\frac{1}{2}}\right)$ **A1**

METHOD 2

attempts implicit differentiation on $\frac{dy}{dx}$ to find $\frac{d^2y}{dx^2}$ **M1**

$\frac{d^2y}{dx^2} = \frac{(2y+3x)\left(x\frac{dy}{dx}-y\right)}{x^3}$ (or equivalent)

$\frac{d^2y}{dx^2} = 0 \Rightarrow y = -\frac{3x}{2}$ $\left(\frac{dy}{dx} \neq \frac{y}{x}\right)$ **A1**

attempts to solve $-\frac{3x}{2} = x(\tan(\ln x)-1)$ for x where $e^{-\frac{\pi}{2}} < x < e^{\frac{\pi}{2}}$ **M1**

$x = 0.629\left(= e^{-\arctan\frac{1}{2}}\right)$ **A1**

attempts to find $f(0.629)\left(f\left(= e^{-\arctan\frac{1}{2}}\right)\right)$ **(M1)**

the coordinates are $(0.629,\ -0.943)\left(e^{-\arctan\frac{1}{2}},\ -\frac{3}{2}e^{-\arctan\frac{1}{2}}\right)$ **A1**

[6 marks]

1d. Use the differential equation $\frac{dy}{dx} = \frac{y^2+3xy+2x^2}{x^2}$ to show that the points of zero gradient on the curve lie on two straight lines of the form $y = mx$ where the values of m are to be determined. *[4 marks]*

Markscheme

$\frac{dy}{dx} = 0 \Rightarrow y^2 + 3xy + 2x^2 = 0$ **M1**

attempts to solve $y^2 + 3xy + 2x^2 = 0$ for y **M1**

$(y+2x)(y+x) = 0$ or $y = \frac{-3x \pm \sqrt{(3x)^2 - 4(2x^2)}}{2}\left(= \frac{-3x \pm x}{2}, (x > 0)\right)$ **A1**

$y = -2x$ and $y = -x\ (m = -2,\ -1)$ **A1**

Note: Award **M1** for stating $\frac{dy}{dx} = 0$, **M1** for substituting $y = mx$ into $\frac{dy}{dx}(=0)$, **A1** for $(m+2)(m+1) = 0$ and **A1** for $m = -2,\ -1 \Rightarrow y = -2x$ and $y = -x$.

[4 marks]

Consider the differential equation $x^2\frac{dy}{dx} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

2a. Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 1.5$. *[4 marks]*

Markscheme

attempt to use Euler's method ***(M1)***

$x_{n+1} = x_n + 0.1$; $y_{n+1} = y_n + 0.1 \times \frac{dy}{dx}$, where $\frac{dy}{dx} = \frac{y^2 - 2x^2}{x^2}$

correct intermediate y-values ***(A1)(A1)***

$3.7,\ 4.63140\ldots,\ 5.92098,\ 7.79542\ldots$

Note: ***A1*** for any two correct y-values seen

$y = 10.6958\ldots$

$y = 10.7$ ***A1***

Note: For the final ***A1***, the value 10.7 must be the last value in a table or a list, or be given as a final answer, not just embedded in a table which has further lines.

[4 marks]

2b. Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = v^2 - v - 2$. *[3 marks]*

Markscheme

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ***(A1)***

replacing y with vx and $\frac{dy}{dx}$ with $v + x\frac{dv}{dx}$ ***M1***

$x^2\frac{dy}{dx} = y^2 - 2x^2 \Rightarrow x^2\left(v + x\frac{dv}{dx}\right) = v^2x^2 - 2x^2$ ***A1***

$v + x\frac{dv}{dx} = v^2 - 2$ (since $x > 0$)

$x\frac{dv}{dx} = v^2 - v - 2$ ***AG***

[3 marks]

2c. By solving the differential equation, show that $y = \frac{8x+x^4}{4-x^3}$. *[10 marks]*

Markscheme

attempt to separate variables v and x ***(M1)***

$\int \frac{\mathrm{d}v}{v^2-v-2} = \int \frac{\mathrm{d}x}{x}$

$\int \frac{\mathrm{d}v}{(v-2)(v+1)} = \int \frac{\mathrm{d}x}{x}$ ***(A1)***

attempt to express in partial fraction form ***M1***

$\frac{1}{(v-2)(v+1)} \equiv \frac{A}{v-2} + \frac{B}{v+1}$

$\frac{1}{(v-2)(v+1)} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)$ ***A1***

$\frac{1}{3}\int\left(\frac{1}{v-2} - \frac{1}{v+1}\right)\mathrm{d}v = \int \frac{\mathrm{d}x}{x}$

$\frac{1}{3}(\ln|v-2| - \ln|v+1|) = \ln|x|(+c)$ ***A1***

Note: Condone absence of modulus signs throughout.

EITHER

attempt to find c using $x = 1,\ y = 3,\ v = 3$ ***M1***

$c = \frac{1}{3}\ln\frac{1}{4}$

$\frac{1}{3}(\ln|v-2| - \ln|v+1|) = \ln|x| + \frac{1}{3}\ln\frac{1}{4}$

expressing both sides as a single logarithm ***(M1)***

$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(\frac{|x|^3}{4}\right)$

OR

expressing both sides as a single logarithm ***(M1)***

$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(A|x|^3\right)$

attempt to find A using $x = 1,\ y = 3,\ v = 3$ ***M1***

$A = \frac{1}{4}$

THEN

$\left|\frac{v-2}{v+1}\right| = \frac{1}{4}x^3$ (since $x > 0$)

substitute $v = \frac{y}{x}$ (seen anywhere) ***M1***

$\frac{\frac{y}{x}-2}{\frac{y}{x}+1} = \frac{1}{4}x^3$ (since $y > 2x$)

$\left(\Rightarrow \frac{y-2x}{y+x} = \frac{1}{4}x^3\right)$

attempt to make y the subject ***M1***

$y - \frac{x^3 y}{4} = 2x + \frac{x^4}{4}$ ***A1***

$y = \frac{8x+x^4}{4-x^3}$ ***AG***

[10 marks]

2d. Find the actual value of y when $x = 1.5$. *[1 mark]*

Markscheme

actual value at $y(1.5) = 27.3$ ***A1***

[1 mark]

2e. Using the graph of $y = \frac{8x+x^4}{4-x^3}$, suggest a reason why the approximation given by Euler's method in part (a) is not a good estimate to the actual value of y at $x = 1.5$. *[1 mark]*

Markscheme

gradient changes rapidly (during the interval considered) OR

the curve has a vertical asymptote at $x = \sqrt[3]{4}$ $(= 1.5874\ldots)$ ***R1***

[1 mark]

3. Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$. *[7 marks]*

Give your answer in the form $y = f(x)$.

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2y}{x} = \frac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$\left(\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = \mathrm{e}^{2\ln x}\right) = x^2$ ***(A1)***

$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \ln 2x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y\right) = \ln 2x$

$x^2 y = \int \ln 2x \,\mathrm{d}x$

attempt to use integration by parts ***(M1)***

$x^2 y = x \ln 2x - x (+c)$ ***A1***

$y = \frac{\ln 2x}{x} - \frac{1}{x} + \frac{c}{x^2}$

substituting $x = \frac{1}{2},\ y = 4$ into an integrated equation involving c ***M1***

$4 = 0 - 2 + 4c$

$\Rightarrow c = \frac{3}{2}$

$y = \frac{\ln 2x}{x} - \frac{1}{x} + \frac{3}{2x^2}$ ***A1***

[7 marks]

The function f has a derivative given by $f'(x) = \frac{1}{x(k-x)}, x \in \mathbb{R}, x \neq o, x \neq k$ where k is a positive constant.

4a. The expression for $f'(x)$ can be written in the form $\frac{a}{x} + \frac{b}{k-x}$, where $a, b \in \mathbb{R}$. Find a and b in terms of k. *[3 marks]*

Markscheme

$\frac{1}{x(k-x)} \equiv \frac{a}{x} + \frac{b}{k-x}$

$a(k-x)+bx = 1$ ***(A1)***

attempt to compare coefficients OR substitute $x = k$ and $x = 0$ and solve ***(M1)***

$a = \frac{1}{k}$ and $b = \frac{1}{k}$ ***A1***

$f'\left(x\right) = \frac{1}{kx} + \frac{1}{k(k-x)}$

[3 marks]

4b. Hence, find an expression for $f(x)$. *[3 marks]*

Markscheme

attempt to integrate their $\frac{a}{x} + \frac{b}{k-x}$ ***(M1)***

$f(x)\frac{1}{k}\int\left(\frac{1}{x} + \frac{1}{k-x}\right)\mathrm{d}\,x$

$= \frac{1}{k}(\ln|x| - \ln|k-x|)(+c)\left(= \frac{1}{k}\ln\left|\frac{x}{k-x}\right|(+c)\right)$ ***A1A1***

Note: Award ***A1*** for each correct term. Award ***A1A0*** for a correct answer without modulus signs. Condone the absence of $+c$.

[3 marks]

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P(k-P)}{5k}$, where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

4c. By solving the differential equation, show that $P = \frac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}}+1200}$. *[8 marks]*

Markscheme

attempt to separate variables and integrate both sides ***M1***

$5k\int \frac{1}{P(k-P)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$

$5(\ln P - \ln(k-P)) = t + c$ ***A1***

Note: There are variations on this which should be accepted, such as $\frac{1}{k}(\ln P - \ln(k-P)) = \frac{1}{5k}t + c$. Subsequent marks for these variations should be awarded as appropriate.

EITHER

attempt to substitute $t = 0, P = 1200$ into an equation involving c ***M1***

$c = 5(\ln 1200 - \ln(k-1200))\left(= 5\ln\left(\frac{1200}{k-1200}\right)\right)$ ***A1***

$5(\ln P - \ln(k-P)) = t + 5(\ln 1200 - \ln(k-1200))$ ***A1***

$\ln\left(\frac{P(k-1200)}{1200(k-P)}\right) = \frac{t}{5}$

$\frac{P(k-1200)}{1200(k-P)} = \mathrm{e}^{\frac{t}{5}}$ ***A1***

OR

$\ln\left(\frac{P}{k-P}\right) = \frac{t+c}{5}$

$\frac{P}{k-P} = A\mathrm{e}^{\frac{t}{5}}$ ***A1***

attempt to substitute $t = 0, P = 1200$ ***M1***

$\frac{1200}{k-1200} = A$ ***A1***

$\frac{P}{k-P} = \frac{1200\mathrm{e}^{\frac{t}{5}}}{k-1200}$ ***A1***

THEN

attempt to rearrange and isolate P ***M1***

$Pk - 1200P = 1200k\mathrm{e}^{\frac{t}{5}} - 1200P\mathrm{e}^{\frac{t}{5}}$ OR $Pk\mathrm{e}^{-\frac{t}{5}} - 1200P\mathrm{e}^{-\frac{t}{5}} = 1200k - 1200P$ OR $\frac{k}{P} - 1 = \frac{k-1200}{1200\mathrm{e}^{\frac{t}{5}}}$

$P\left(k - 1200 + 1200\mathrm{e}^{\frac{t}{5}}\right) = 1200k\mathrm{e}^{\frac{t}{5}}$ OR $P\left(k\mathrm{e}^{-\frac{t}{5}} - 1200\mathrm{e}^{-\frac{t}{5}} + 1200\right) = 1200k$ ***A1***

$P = \frac{1200k}{(k-1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$ ***AG***

[8 marks]

At $t = 10$ the population of the colony has doubled in size from its initial value.

4d. Find the value of k, giving your answer correct to four significant figures. *[3 marks]*

Markscheme

attempt to substitute $t = 10, P = 2400$ ***(M1)***

$2400 = \frac{1200k}{(k-1200)\mathrm{e}^{-2}+1200}$ ***(A1)***

$k = 2845.34\ldots$

$k = 2845$ ***A1***

Note: Award ***(M1)(A1)A0*** for any other value of k which rounds to 2850

[3 marks]

4e. Find the value of t when the rate of change of the population is at its maximum. *[3 marks]*

Markscheme

attempt to find the maximum of the first derivative graph OR zero of the second derivative graph OR that $P = \frac{k}{2}(= 1422.67\ldots)$ ***(M1)***

$t = 1.57814\ldots$

$= 1.58$ (days) ***A2***

Note: Accept any value which rounds to 1.6.

[3 marks]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-x}{y+x}$, where $x, y > 0$.

It is given that $y = 2$ when $x = 1$.

5a. Solve the differential equation, giving your answer in the form $f(x,\ y) = 0$. *[9 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

puts $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx-x}{vx+x}\ \left(= \frac{v-1}{v+1}\right)$ ***A1***

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single rational fraction in v

$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{v+1}$ ***M1***

attempts to separate variables ***M1***

$\int \frac{v+1}{v^2+1}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$

$\frac{1}{2}\ln\left(v^2+1\right) + \arctan v = -\ln x (+C)$ ***A1A1***

substitutes $y = 2,\ x = 1$ and attempts to find the value of C ***M1***

$C = \frac{1}{2}\ln 5 + \arctan 2$ ***A1***

the solution is

$\frac{1}{2}\ln\left(\frac{y^2}{x^2} + 1\right) + \arctan\left(\frac{y}{x}\right) + \ln x - \frac{1}{2}\ln 5 - \arctan 2 = 0$ ***A1***

[9 marks]

5b. The graph of y against x has a local maximum between $x = 2$ and $x = 3$ *[4 marks]* . Determine the coordinates of this local maximum.

Markscheme

at a maximum, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

attempts to substitute $y = x$ into their solution ***M1***

$\frac{1}{2}\ln 2 + \arctan 1 + \ln x = \frac{1}{2}\ln 5 + \arctan 2$

attempts to solve for x, y ***(M1)***

$(2.18,\ 2.18)$ $\left(\frac{\sqrt{10}}{2}\mathrm{e}^{\arctan 2 - \frac{\pi}{4}}, \frac{\sqrt{10}}{2}\mathrm{e}^{\arctan 2 - \frac{\pi}{4}}\right)$ ***A1***

Note: Accept all answers that round to the correct $2\ \mathrm{sf}$ answer.
Accept $x = 2.18,\ y = 2.18$.

[4 marks]

5c. Show that there are no points of inflexion on the graph of y against x. *[4 marks]*

Markscheme

METHOD 1

attempts (quotient rule) implicit differentiation ***M1***

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}-1\right)(y+x)-(y-x)\left(\frac{dy}{dx}+1\right)}{(y+x)^2}$$

correctly substitutes $\frac{dy}{dx} = \frac{y-x}{y+x}$ into $\frac{d^2y}{dx^2}$

$$= \frac{\left(\frac{y-x}{y+x}-1\right)(y+x)-(y-x)\left(\frac{y-x}{y+x}+1\right)}{(y+x)^2}$$ ***A1***

$$= -\frac{2\left(x^2+y^2\right)}{(x+y)^3}$$ ***A1***

this expression can never be zero therefore no points of inflexion ***R1***

METHOD 2

attempts implicit differentiation on $(y+x)\frac{dy}{dx} = y - x$ ***M1***

$$\left(\frac{dy}{dx}+1\right)\frac{dy}{dx} + (y+x)\frac{d^2y}{dx^2} = \frac{dy}{dx} - 1$$ ***A1***

$$(y+x)\frac{d^2y}{dx^2} = \frac{dy}{dx} - 1 - \left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}$$

$$= -1 - \left(\frac{dy}{dx}\right)^2$$ ***A1***

$-1 - \left(\frac{dy}{dx}\right)^2 < 0$ and $x + y > 0,\ \frac{d^2y}{dx^2} \neq 0$ therefore no points of inflexion ***R1***

Note: Accept putting $\frac{d^2y}{dx^2} = 0$ and obtaining contradiction.

[4 marks]

Consider the differential equation $\frac{dy}{dx} = \frac{4x^2+y^2-xy}{x^2}$, with $y = 2$ when $x = 1$.

6a. Use Euler's method, with step length $h = 0.1$, to find an approximate value of y when $x = 1.4$. *[5 marks]*

Markscheme

x	y	$\frac{dy}{dx}$
1	2	6
1.1	2.6	7.22
1.2	3.32	8.89652
1.3	4.21	11.26
1.4	5.34	

(M1)(A1)(A1)(A1)A1

$y(1.4) \approx 5.34$

Note: Award ***A1*** for each correct y value.
For the intermediate y values, accept answers that are accurate to 2 significant figures.
The final y value must be accurate to 3 significant figures or better.

[5 marks]

6b. Express $m^2 - 2m + 4$ in the form $(m-a)^2 + b$, where a,$b \in \mathbb{Z}$. *[1 mark]*

Markscheme

$m^2 - 2m + 4 = (m-1)^2 + 3 \quad (a = 1,\ b = 3)$ ***A1***

[1 mark]

6c. Solve the differential equation, for $x > 0$, giving your answer in the form $y = f(x)$. *[10 marks]*

Markscheme

recognition of homogeneous equation,
let $y = vx$ ***M1***

the equation can be written as

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 4 + v^2 - v$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - 2v + 4$

$\int \frac{1}{v^2-2v+4}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***M1***

Note: Award ***M1*** for attempt to separate the variables.

$\int \frac{1}{(v-1)^2+3}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ from part (c)(i) ***M1***

$\frac{1}{\sqrt{3}}\arctan\left(\frac{v-1}{\sqrt{3}}\right) = \ln x \ \ (+c)$ ***A1A1***

$x = 1, y = 2 \Rightarrow v = 2$

$\frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) = \ln 1 + c$ ***M1***

Note: Award ***M1*** for using initial conditions to find c.

$\Rightarrow c = \frac{\pi}{6\sqrt{3}} \ (= 0.302)$ ***A1***

$\arctan\left(\frac{v-1}{\sqrt{3}}\right) = \sqrt{3}\ln x + \frac{\pi}{6}$

substituting $v = \frac{y}{x}$ ***M1***

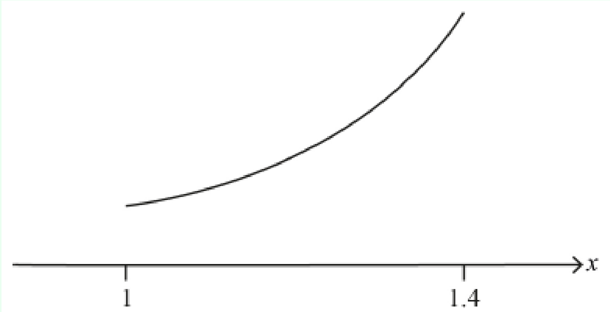
Note: This ***M1*** may be awarded earlier.

$y = x\left(\sqrt{3}\tan\left(\sqrt{3}\ln x + \frac{\pi}{6}\right) + 1\right)$ ***A1***

[10 marks]

6d. Sketch the graph of $y = f(x)$ for $1 \leqslant x \leqslant 1.4$. *[1 mark]*

Markscheme



curve drawn over correct domain ***A1***

[1 mark]

6e. With reference to the curvature of your sketch in part (c)(iii), and without further calculation, explain whether you conjecture $f\left(1.4\right)$ will be less than, equal to, or greater than your answer in part (a). *[2 marks]*

Markscheme

the sketch shows that f is concave up ***A1***

Note: Accept f' is increasing.

this means the tangent drawn using Euler's method will give an underestimate of the real value, so $f\left(1.4\right)$ > estimate in part (a) ***R1***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

Consider the differential equation $2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2$, where $x > 0$.

7a. Solve the differential equation and show that a general solution is $x^2 + y^2 = cx$ where c is a positive constant. *[11 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - x^2}{2xy}$

let $y = vx$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2 - x^2}{2vx^2}$ ***(M1)***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2 - 1}{2v} \quad \left(= \frac{v}{2} - \frac{1}{2v}\right)$ ***(A1)***

Note: Or equivalent attempt at simplification.

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-v^2 - 1}{2v} \quad \left(= -\frac{v}{2} - \frac{1}{2v}\right)$ ***A1***

$\frac{2v}{1+v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{x}$ ***(M1)***

$\int \frac{2v}{1+v^2}\mathrm{d}v = \int -\frac{1}{x}\mathrm{d}x$ ***(A1)***

$\ln\left(1 + v^2\right) = -\ln x + \ln c$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS and a constant.

$\ln\left(1 + \left(\frac{y}{x}\right)^2\right) = -\ln x + \ln c$ ***M1***

Note: Award ***M1*** for substituting $v = \frac{y}{x}$. May be seen at a later stage.

$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}$ ***A1***

Note: Award ***A1*** for any correct equivalent equation without logarithms.

$x^2 + y^2 = cx$ ***AG***

[11 marks]

7b. Prove that there are two horizontal tangents to the general solution curve and state their equations, in terms of c. *[5 marks]*

Markscheme

METHOD 1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - x^2}{2xy}$

(for horizontal tangents) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$\left(\Rightarrow y^2 = x^2\right) \Rightarrow y = \pm x$

EITHER

using $x^2 + y^2 = cx \Rightarrow 2x^2 = cx$ ***M1***

$2x^2 - cx = 0 \Rightarrow x = \frac{c}{2}$ ***A1***

Note: Award ***M1A1*** for $2y^2 = \pm cy$.

OR

using implicit differentiation of $x^2 + y^2 = cx$

$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = c$ ***M1***

Note: Accept differentiation of $y = \sqrt{cx - x^2}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = \frac{c}{2}$ ***A1***

THEN

tangents at $y = \frac{c}{2},\ y = -\frac{c}{2}$ ***A1A1***

hence there are two tangents ***AG***

METHOD 2

$x^2 + y^2 = cx$

$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$ ***M1A1***

this is a circle radius $\frac{c}{2}$ centre $\left(\frac{c}{2},\ 0\right)$ ***A1***

hence there are two tangents ***AG***

tangents at $y = \frac{c}{2},\ y = -\frac{c}{2}$ ***A1A1***

[5 marks]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$, where $x \neq 0$.

8a. Given that $y(1) = 1$, use Euler's method with step length h = 0.25 to find an approximation for $y(2)$. Give your answer to two significant figures. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to apply Euler's method ***(M1)***

$x_{n+1} = x_n + 0.25;\ y_{n+1} = y_n + 0.25 \times \left(1 + \frac{y_n}{x_n}\right)$

x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
1.00	1.00000	2.00000
1.25	1.50000	2.20000
1.50	2.05000	2.36667
1.75	2.64167	2.50952
2.00	3.26905	

(A1)(A1)

Note: Award ***A1*** for correct x values, **A1** for first three correct y values.

y = 3.3 ***A1***

[4 marks]

8b. Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$ for $y(1) = 1$. *[6 marks]*

Markscheme

METHOD 1

$I\left(x\right) = \mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}$ ***(M1)***

$= \mathrm{e}^{-\ln x}$

$= \frac{1}{x}$ ***(A1)***

$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = \frac{1}{x}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \frac{1}{x}$

$\frac{y}{x} = \ln\left|x\right| + C$ ***A1***

$y\left(1\right) = 1 \Rightarrow C = 1$ ***M1***

$y = x\ln\left|x\right| + x$ ***A1***

METHOD 2

$v = \frac{y}{x}$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}y$ ***(A1)***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v$ ***M1***

$\int 1\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$

$v = \ln\left|x\right| + C$

$\frac{y}{x} = \ln\left|x\right| + C$ ***A1***

$y\left(1\right) = 1 \Rightarrow C = 1$ ***M1***

$y = x\ln\left|x\right| + x$ ***A1***

[6 marks]

8c. Find the percentage error when $y\left(2\right)$ is approximated by the final rounded value found in part (a). Give your answer to two significant figures. *[3 marks]*

Markscheme

$y(2) = 2\ln 2 + 2 = 3.38629\ldots$

percentage error $= \frac{3.38629\ldots - 3.3}{3.38629\ldots} \times 100\%$ ***(M1)(A1)***

= 2.5% ***A1***

[3 marks]

Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^p + 1$ where $x \in \mathbb{R}$, $x \neq 0$ and p is a positive integer, $p > 1$.

9a. Solve the differential equation given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$. *[8 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} = x^{p-1} + \frac{1}{x}$ ***(M1)***

integrating factor $= \mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}$ ***M1***

$= \mathrm{e}^{-\ln x}$ ***(A1)***

= $\frac{1}{x}$ ***A1***

$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = x^{p-2} + \frac{1}{x^2}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = x^{p-2} + \frac{1}{x^2}$

$\frac{y}{x} = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of *C*.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1$, $y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of *C*.

$y = \frac{1}{p-1}(x^p - x) - 1$ ***A1***

[8 marks]

METHOD 2

put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1(A1)***

substituting, ***M1***

$x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - vx = x^p + 1$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-1} + \frac{1}{x}$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-2} + \frac{1}{x^2}$

$v = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of *C*.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1$, $y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of *C*.

$y = \frac{1}{p-1}(x^p - x) - 1$ ***A1***

[8 marks]

9b. Show that the x-coordinate(s) of the points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ satisfy the equation $x^{p-1} = \frac{1}{p}$. *[2 marks]*

Markscheme

METHOD 1

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p-1}\left(px^{p-1} - 1\right)$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow px^{p-1} - 1 = 0$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

METHOD 2

substitute $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and their y into the differential equation and solve for x

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow -\left(\frac{x^p - x}{p-1}\right) + 1 = x^p + 1$ ***M1***

$x^p - x = x^p - px^p$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

[2 marks]

9c. Deduce the set of values for p such that there are two points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Give a reason for your answer. *[2 marks]*

Markscheme

there are two solutions for x when p is odd (and $p > 1$ ***A1***

if $p-1$ is even there are two solutions (to $x^{p-1} = \frac{1}{p}$)

and if $p-1$ is odd there is only one solution (to $x^{p-1} = \frac{1}{p}$) ***R1***

Note: Only award the ***R1*** if both cases are considered.

[4 marks]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

10a. Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

integrating factor $= \mathrm{e}^{\int \frac{x}{x^2+1}\mathrm{d}x}$ ***(M1)***

$\int \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\ln(x^2+1)$ ***(M1)***

Note: Award ***M1*** for use of $u = x^2 + 1$ for example or $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x)$.

integrating factor $= \mathrm{e}^{\frac{1}{2}\ln(x^2+1)}$ ***A1***

$= \mathrm{e}^{\ln\left(\sqrt{x^2+1}\right)}$ ***A1***

Note: Award ***A1*** for $\mathrm{e}^{\ln\sqrt{u}}$ where $u = x^2 + 1$.

$= \sqrt{x^2+1}$ ***AG***

METHOD 2

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{x^2+1}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{x^2+1} + \frac{x}{\sqrt{x^2+1}}y$ ***M1A1***

$\sqrt{x^2+1}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y\right)$ ***M1A1***

Note: Award ***M1*** for attempting to express in the form $\sqrt{x^2+1} \times (\text{LHS of de})$.

so $\sqrt{x^2+1}$ is an integrating factor for this differential equation ***AG***

[4 marks]

10b. Solve the differential equation giving your answer in the form $y = f(x)$. *[6 marks]*

Markscheme

$\sqrt{x^2+1}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{\sqrt{x^2+1}}y=x\sqrt{x^2+1}$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{x^2+1}\right)=x\sqrt{x^2+1}$

$y\sqrt{x^2+1}=\int x\sqrt{x^2+1}\mathrm{d}x\ \left(y=\frac{1}{\sqrt{x^2+1}}\int x\sqrt{x^2+1}\mathrm{d}x\right)$ ***A1***

$=\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$ ***(M1)A1***

Note: Award ***M1*** for using an appropriate substitution.

Note: Condone the absence of C.

substituting $x=0, y=1\Rightarrow C=\frac{2}{3}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y=\frac{1}{3}(x^2+1)+\frac{2}{3\sqrt{x^2+1}}\ \left(y=\frac{(x^2+1)^{\frac{3}{2}}+2}{3\sqrt{x^2+1}}\right)$ ***A1***

[6 marks]