Exercises 25.01 *[112 marks]*

1a. Find the roots of the equation $w^3 = 8\mathrm{i}$, $w \in \mathbb{C}$. Give your answers in Cartesian form. *[4 marks]*

1b. One of the roots w_1 satisfies the condition $\mathrm{Re}\,(w_1) = 0$. *[3 marks]*

Given that $w_1 = \frac{z}{z-\mathrm{i}}$, express z in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

2a. Express $-3 + \sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *[5 marks]*

Let the roots of the equation $z^3 = -3 + \sqrt{3}\mathrm{i}$ be u, v and w.

2b. Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *[5 marks]*

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

2c. Find the area of triangle UVW. *[4 marks]*

2d. By considering the sum of the roots u, v and w, show that *[4 marks]*

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$.

3a. Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^4$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. *[3 marks]*

3b. Use de Moivre’s theorem and the result from part (a) to show that $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. *[5 marks]*

3c. Use the identity from part (b) to show that the quadratic equation $x^2 - 6x + 1 = 0$ has roots $\cot^2\frac{\pi}{8}$ and $\cot^2\frac{3\pi}{8}$. *[5 marks]*

3d. Hence find the exact value of $\cot^2 \frac{3\pi}{8}$. *[4 marks]*

3e. Deduce a quadratic equation with integer coefficients, having roots $\text{cosec}^2\ \frac{\pi}{8}$ and $\text{cosec}^2\ \frac{3\pi}{8}$. *[3 marks]*

Consider the equation $(z-1)^3 = \mathrm{i},\ z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

4a. Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation. *[2 marks]*

4b. Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. *[4 marks]*

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

4c. Plot the points A, B and C on an Argand diagram. *[4 marks]*

4d. Find AC. *[3 marks]*

Consider the equation $(z-1)^3 = \mathrm{i}z^3,\ z \in \mathbb{C}$.

4e. By using de Moivre's theorem, show that $\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. *[3 marks]*

4f. Determine the value of $\mathrm{Re}(\alpha)$. *[6 marks]*

Consider the complex numbers $z = 2\left(\cos \frac{\pi}{5} + \mathrm{i} \sin \frac{\pi}{5}\right)$ and $w = 8\left(\cos \frac{2k\pi}{5} - \mathrm{i} \sin \frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

5a. Find the modulus of zw. *[1 mark]*

5b. Find the argument of zw in terms of k. *[2 marks]*

Suppose that $zw \in \mathbb{Z}$.

5c. Find the minimum value of k. *[3 marks]*

5d. For the value of k found in part (i), find the value of zw. *[1 mark]*

6a. Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where r, $\theta \in \mathbb{R}^{+}$. *[5 marks]*

Let *S* be the sum of the roots found in part (a).

6b. Show that Re *S* = Im *S*. *[4 marks]*

6c. By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of cos $\frac{\pi}{12}$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are integers to be determined. *[3 marks]*

6d. Hence, or otherwise, show that *S* = $\frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)(1+\mathrm{i})$. *[4 marks]*

7a. Solve $2\sin(x + 60^\circ) = \cos(x + 30^\circ),\ 0^\circ \leqslant x \leqslant 180^\circ$. *[5 marks]*

7b. Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$. *[3 marks]*

Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta, z \in \mathbb{C}, 0 \leqslant \theta \leqslant \pi$.

7c. Find the modulus and argument of z in terms of θ. Express each answer in its simplest form. *[9 marks]*

7d. Hence find the cube roots of z in modulus-argument form. *[5 marks]*