Exercises 25.01 *[112 marks]*

1a. Find the roots of the equation $w^3 = 8\mathrm{i}$, $w \in \mathbb{C}$. Give your answers in Cartesian form. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$w^3 = 8\mathrm{i}$

writing $8\mathrm{i} = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to find cube roots of w using modulus-argument form.

cube roots $w = 2\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2}+2\pi k}{3}\right)\right)$ ***(M1)***

i.e. $w = \sqrt{3} + \mathrm{i},\ -\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

METHOD 2

$w^3 + (2\mathrm{i})^3 = 0$

$(w + 2\mathrm{i})\left(w^2 - 2w\mathrm{i} - 4\right) = 0$ ***M1***

$w = \frac{2\mathrm{i} \pm \sqrt{12}}{2}$ ***M1***

$w = \sqrt{3} + \mathrm{i},\ -\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

[4 marks]

1b. One of the roots w_1 satisfies the condition $\mathrm{Re}\,(w_1) = 0$. *[3 marks]*

Given that $w_1 = \frac{z}{z-\mathrm{i}}$, express z in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Q}$.

Markscheme

$w_1 = -2\mathrm{i}$

$\frac{z}{z-\mathrm{i}} = -2\mathrm{i}$ ***M1***

$z = -2\mathrm{i}\,(z - \mathrm{i})$

$z\,(1 + 2\mathrm{i}) = -2$

$z = \frac{-2}{1+2\mathrm{i}}$ ***A1***

$z = -\frac{2}{5} + \frac{4}{5}\mathrm{i}$ ***A1***

Note: Accept $a = -\frac{2}{5}$, $b = \frac{4}{5}$.

[3 marks]

2a. Express $-3 + \sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *[5 marks]*

Markscheme

attempt to find modulus ***(M1)***

$r = 2\sqrt{3}\left(= \sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3 + \sqrt{3}\mathrm{i} = \sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(= 2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

Let the roots of the equation $z^3 = -3 + \sqrt{3}\mathrm{i}$ be u, v and w.

2b. Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *[5 marks]*

Markscheme

attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}e^{\frac{5\pi i}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}e^{-\frac{7\pi i}{18}}$ ***A1***

$12^{\frac{1}{6}}e^{\frac{17\pi i}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

2c. Find the area of triangle UVW. *[4 marks]*

Markscheme

METHOD 1

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

Area $= 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$

= $\frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

METHOD 2

UV$^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

UV $= \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using Area = $\frac{1}{2}$ × UV × VW × sin α for example ***M1***

Area $= \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

= $\frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

2d. By considering the sum of the roots u, v and w, show that *[4 marks]*

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0.$

Markscheme

$u+v+w=0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{18}\right)+\cos\frac{5\pi}{18}+\mathrm{i}\sin\frac{5\pi}{18}+\cos\frac{17\pi}{18}+\mathrm{i}\sin\frac{17\pi}{18}\right)=0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right)+\cos\frac{5\pi}{18}+\cos\frac{17\pi}{18}\right)=0$

$\cos\left(-\frac{7\pi}{18}\right)=\cos\frac{17\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0$ ***AG***

[4 marks]

3a. Use the binomial theorem to expand $(\cos\theta+\mathrm{i}\sin\theta)^4$. Give your answer in the form $a+b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. *[3 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

uses the binomial theorem on $(\cos\theta+\mathrm{i}\sin\theta)^4$ **M1**

$={}_4C_0\cos^4\theta+{}_4C_1\cos^3\theta(\mathrm{i}\sin\theta)+{}_4C_2\cos^2\theta\left(\mathrm{i}^2\sin^2\theta\right)+{}_4C_3\cos\theta\left(\mathrm{i}^3\sin^3\theta\right)+{}_4C$
A1

$=\left(\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta\right)+\mathrm{i}\left(4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta\right)$ **A1**

[3 marks]

3b. Use de Moivre’s theorem and the result from part (a) to show that $\cot 4\theta=\frac{\cot^4\theta-6\cot^2\theta+1}{4\cot^3\theta-4\cot\theta}$. *[5 marks]*

Markscheme

(using de Moivre's theorem with $n = 4$ gives) $\cos 4\theta + \mathrm{i} \sin 4\theta$ **(A1)**

equates both the real and imaginary parts of $\cos 4\theta + \mathrm{i} \sin 4\theta$ and $\left(\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta\right) + \mathrm{i}\left(4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta\right)$ **M1**

$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$ and
$\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$

recognizes that $\cot 4\theta = \frac{\cos 4\theta}{\sin 4\theta}$ **(A1)**

substitutes for $\sin 4\theta$ and $\cos 4\theta$ into $\frac{\cos 4\theta}{\sin 4\theta}$ **M1**

$$\cot 4\theta = \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$$

divides the numerator and denominator by $\sin^4 \theta$ to obtain

$$\cot 4\theta = \frac{\frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{\sin^4\theta}}{\frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\sin^4\theta}}$$ **A1**

$$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$$ **AG**

[5 marks]

3c. Use the identity from part (b) to show that the quadratic equation $x^2 - 6x + 1 = 0$ has roots $\cot^2 \frac{\pi}{8}$ and $\cot^2 \frac{3\pi}{8}$. *[5 marks]*

Markscheme

setting $\cot 4\theta = 0$ and putting $x = \cot^2\theta$ in the numerator of $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$ gives $x^2 - 6x + 1 = 0$ **M1**

attempts to solve $\cot 4\theta = 0$ for θ **M1**

$4\theta = \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots \left(4\theta = \frac{1}{2}(2n+1)\pi,\ \text{n} = 0,\ 1,\ \ldots\right)$ **(A1)**

$\theta = \frac{\pi}{8},\ \frac{3\pi}{8}$ **A1**

Note: Do not award the final **A1** if solutions other than $\theta = \frac{\pi}{8},\ \frac{3\pi}{8}$ are listed.

finding the roots of $\cot 4\theta = 0 \left(\theta = \frac{\pi}{8},\ \frac{3\pi}{8}\right)$ corresponds to finding the roots of $x^2 - 6x + 1 = 0$ where $x = \cot^2\theta$ **R1**

so the equation $x^2 - 6x + 1 = 0$ as roots $\cot^2\frac{\pi}{8}$ and $\cot^2\frac{3\pi}{8}$ **AG**

[5 marks]

3d. Hence find the exact value of $\cot^2\frac{3\pi}{8}$. *[4 marks]*

Markscheme

attempts to solve $x^2 - 6x + 1 = 0$ for x **M1**

$x = 3 \pm 2\sqrt{2}$ **A1**

since $\cot^2\frac{\pi}{8} > \cot^2\frac{3\pi}{8}, \cot^2\frac{3\pi}{8}$ has the smaller value of the two roots **R1**

Note: Award **R1** for an alternative convincing valid reason.

so $\cot^2\frac{3\pi}{8} = 3 - 2\sqrt{2}$ **A1**

[4 marks]

3e. Deduce a quadratic equation with integer coefficients, having roots $\text{cosec}^2\ \frac{\pi}{8}$ and $\text{cosec}^2\ \frac{3\pi}{8}$. *[3 marks]*

Markscheme

let $y = \text{cosec}^2\ \theta$

uses $\cot^2\theta = \text{cosec}^2\ \theta - 1$ where $x = \cot^2\theta$ **(M1)**

$x^2 - 6x + 1 = 0 \Rightarrow (y-1)^2 - 6(y-1) + 1 = 0$ **M1**

$y^2 - 8y + 8 = 0$ **A1**

[3 marks]

Consider the equation $(z-1)^3 = \mathrm{i},\ z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

4a. Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation. *[2 marks]*

Markscheme

$\left(1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} - 1\right)^3$

$= \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$ ***A1***

$= \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$= \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$

$= \mathrm{i}$ ***AG***

Note: Candidates who solve the equation correctly can be awarded the above two marks. The working for part (i) may be seen in part (ii).

[2 marks]

4b. Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. *[4 marks]*

Markscheme

$(z-1)^3 = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi\mathrm{k}\right)}$ ***(M1)***

$z-1 = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+\frac{4\pi\mathrm{k}}{6}\right)}$ ***(M1)***

$(k=1) \Rightarrow \omega_2 = 1 + \mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ ***A1***

$(k=2) \Rightarrow \omega_3 = 1 + \mathrm{e}^{\mathrm{i}\frac{9\pi}{6}}$ ***A1***

[4 marks]

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

4c. Plot the points A, B and C on an Argand diagram. *[4 marks]*

Markscheme

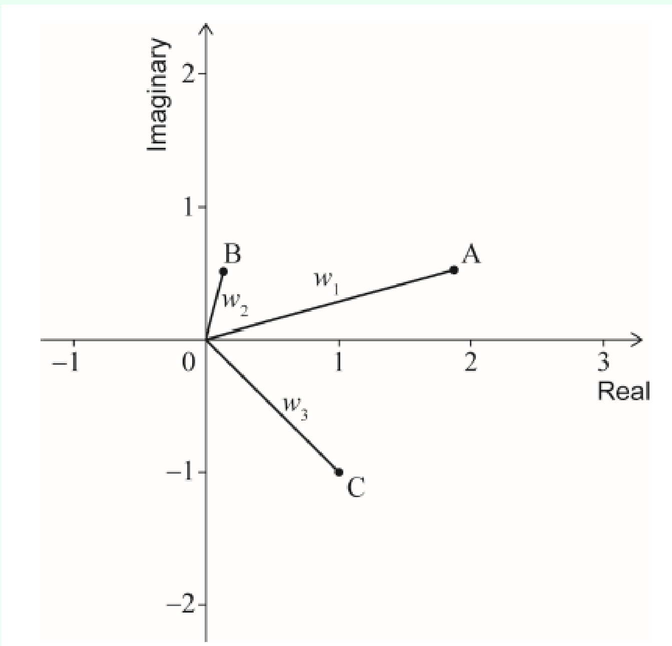
EITHER

attempt to express $\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$, $\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$, $\mathrm{e}^{\mathrm{i}\frac{9\pi}{6}}$ in Cartesian form and translate 1 unit in the positive direction of the real axis ***(M1)***

OR

attempt to express w_1, w_2 and w_3 in Cartesian form ***(M1)***

THEN



Note: To award ***A*** marks, it is not necessary to see A, B or C, the w_1, or the solid lines

A1A1A1

[4 marks]

4d. Find AC. *[3 marks]*

Markscheme

valid attempt to find $\omega_1 - \omega_3$ $(\text{or } \omega_3 - \omega_1)$ ***M1***

$\omega_1 - \omega_3 = \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right) - (1 - \text{i}) = \frac{\sqrt{3}}{2} + \frac{3}{2}\text{i}$ OR $\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6} + \text{i}\sin\frac{\pi}{2}$

valid attempt to find $\left|\frac{\sqrt{3}}{2} + \frac{3}{2}\text{i}\right|$ ***M1***

$= \sqrt{\frac{3}{4} + \frac{9}{4}}$

$\text{AC} = \sqrt{3}$ ***A1***

[3 marks]

Consider the equation $(z-1)^3 = \text{i}z^3,\ z \in \mathbb{C}$.

4e. By using de Moivre's theorem, show that $\alpha = \frac{1}{1-\text{e}^{\text{i}\frac{\pi}{6}}}$ is a root of this equation. *[3 marks]*

Markscheme

METHOD 1

$(z-1)^3 = \text{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \text{i}$ ***M1***

$\left(\frac{z-1}{z}\right)^3 = \text{e}^{\text{i}\frac{\pi}{2}}$ ***A1***

$\frac{\alpha-1}{\alpha} = \text{e}^{\text{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$\alpha - 1 = \alpha\text{e}^{\text{i}\frac{\pi}{6}}$

$\alpha - \alpha\text{e}^{\text{i}\frac{\pi}{6}} = 1$

$\alpha\left(1 - \text{e}^{\text{i}\frac{\pi}{6}}\right) = 1$

$\alpha = \frac{1}{1-\text{e}^{\text{i}\frac{\pi}{6}}}$ ***AG***

METHOD 2

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(1-\frac{1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$1-\frac{1}{z} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \frac{1}{\alpha}$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

METHOD 3

LHS$= (z-1)^3 = \left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}-1\right)^3$

$= \left(\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3$

$= \frac{\mathrm{i}}{\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3} \left(= \frac{\mathrm{i}}{\frac{5}{2}-\frac{3\sqrt{3}}{2}+\mathrm{i}\left(\frac{3\sqrt{3}}{2}-\frac{5}{2}\right)}\right)$ ***M1A1***

Note: Award ***M1*** for applying de Moivre's theorem (may be seen in modulus-argument form.)

RHS$= \mathrm{i}z^3 = \mathrm{i}\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3$

$= \frac{\mathrm{i}}{\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3}$ ***A1***

$(z-1)^3 = \mathrm{i}z^3$ ***AG***

METHOD 4

$(z-1)^3 = \mathrm{i}z^3$

$z^3-3z^2+3z-1 = \mathrm{i}z^3$

$(1-\mathrm{i})z^3-3z^2+3z-1 = 0$ ***(M1)***

$(1-\mathrm{i})\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3-3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^2+3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)-1$

$=(1-\mathrm{i})-3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)+3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2-\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$ ***(A1)***

$=(1-\mathrm{i})-3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)+3\left(1-2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)-\left(1-3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}-\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)$

A1

$=0$ ***AG***

Note: If the candidate does not interpret their conclusion, award ***(M1)(A1)A0*** as appropriate.

[3 marks]

4f. Determine the value of $\mathrm{Re}(\alpha)$. *[6 marks]*

Markscheme

METHOD 1

$\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)}$ ***M1***

$=\frac{2}{2-\sqrt{3}-\mathrm{i}}$ ***A1***

attempt to use conjugate to rationalise ***M1***

$=\frac{4-2\sqrt{3}+2\mathrm{i}}{\left(2-\sqrt{3}\right)^2+1}$ ***A1***

$=\frac{4-2\sqrt{3}+2\mathrm{i}}{8-4\sqrt{3}}$ ***A1***

$=\frac{1}{2}+\frac{1}{4-2\sqrt{3}}\mathrm{i}$

$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

METHOD 2

$\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)}$ ***M1*** attempt to use conjugate to rationalise **M1**

$$= \frac{1}{\left(1-\cos\frac{\pi}{6}\right)-\mathrm{i}\sin\frac{\pi}{6}} \times \frac{\left(1-\cos\frac{\pi}{6}\right)+\mathrm{i}\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)+\mathrm{i}\sin\frac{\pi}{6}}$$ **A1**

$$= \frac{\left(1-\cos\frac{\pi}{6}\right)+\mathrm{i}\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)^2+\sin^2\frac{\pi}{6}}$$ **A1**

$$= \frac{\left(1-\cos\frac{\pi}{6}\right)+\mathrm{i}\sin\frac{\pi}{6}}{1-2\cos\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}}$$

$$= \frac{\left(1-\cos\frac{\pi}{6}\right)+\mathrm{i}\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ **A1**

$$= \frac{1}{2} + \frac{\mathrm{i}\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ **A1**

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

METHOD 3

attempt to multiply through by $-\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}}{\mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}}$ **M1**

$$\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}} = -\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}}{\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}-\mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}}$$ **A1**

attempting to re-write in r-cis form **M1**

$$= -\frac{\cos\left(-\frac{\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{\pi}{12}\right)}{\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}-\left(\cos\left(-\frac{\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)}$$ **A1**

$$= -\frac{\cos\frac{\pi}{12}-\mathrm{i}\sin\frac{\pi}{12}}{2\mathrm{i}\sin\frac{\pi}{12}}$$ **A1**

$$= \frac{1}{2} - \frac{1}{2\mathrm{i}}\cot\frac{\pi}{12} \left(= \frac{1}{2} + \frac{1}{2}\mathrm{i}\cot\frac{\pi}{12}\right)$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ **A1**

METHOD 4

attempt to multiply through by $\frac{1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}}{1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}}$ **M1**

$$\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}} = \frac{1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}}{1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+1}$$ **A1**

attempting to re-write in r-cis form ***M1***

$= \frac{1-\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$ ***A1***

attempt to re-write in Cartesian form ***M1***

$= \frac{1-\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}}{2-\sqrt{3}} \left(= \frac{\frac{2-\sqrt{3}}{2}}{2-\sqrt{3}} + \mathrm{i}\frac{\frac{1}{2}}{2-\sqrt{3}}\right)$

$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final ***A*** mark to be awarded

[6 marks]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

5a. Find the modulus of zw. *[1 mark]*

Markscheme

$(|zw|=)16$ ***A1***

[1 mark]

5b. Find the argument of zw in terms of k. *[2 marks]*

Markscheme

attempt to find $\arg(z)+\arg(w)$ ***(M1)***

$\arg(zw) = \arg(z)+\arg(w)$

$= \frac{\pi}{5} - \frac{2k\pi}{5} \left(= \frac{(1-2k)\pi}{5}\right)$ ***A1***

[2 marks]

Suppose that $zw \in \mathbb{Z}$.

5c. Find the minimum value of k. *[3 marks]*

Markscheme

$zw \in \mathbb{Z} \Rightarrow \arg(zw)$ is a multiple of π ***(M1)***

$\Rightarrow 1 - 2k$ is a multiple of 5 ***(M1)***

$k = 3$ ***A1***

[3 marks]

5d. For the value of k found in part (i), find the value of zw. *[1 mark]*

Markscheme

$zw = 16(\cos(-\pi)+\mathrm{i}\sin(-\pi))$

-16 ***A1***

[1 mark]

6a. Find the roots of $z^{24} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where $r, \theta \in \mathbb{R}^+$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$(r(\cos\theta + \mathrm{i}\sin\theta))^{24} = 1(\cos 0 + \mathrm{i}\sin 0)$

use of De Moivre's theorem ***(M1)***

$r^{24} = 1 \Rightarrow r = 1$ ***(A1)***

$24\theta = 2\pi n \Rightarrow \theta = \frac{\pi n}{12}, (n \in \mathbb{Z})$ ***(A1)***

$0 < \arg(z) < \frac{\pi}{2} \Rightarrow n =$ 1, 2, 3, 4, 5

$z = \mathrm{e}^{\frac{\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{2\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{3\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{4\pi\mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{5\pi\mathrm{i}}{12}}$ ***A2***

Note: Award ***A1*** if additional roots are given or if three correct roots are given with no incorrect (or additional) roots.

[5 marks]

Let *S* be the sum of the roots found in part (a).

6b. Show that Re *S* = Im *S*. *[4 marks]*

Markscheme

Re *S* = $\cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

Im *S* = $\sin\frac{\pi}{12} + \sin\frac{2\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{4\pi}{12} + \sin\frac{5\pi}{12}$ ***A1***

Note: Award ***A1*** for both parts correct.

but $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12},\ \sin\frac{4\pi}{12} = \cos\frac{2\pi}{12},\ \sin\frac{3\pi}{12} = \cos\frac{3\pi}{12},\ \sin\frac{2\pi}{12} = \cos\frac{4\pi}{12}$ and $\sin\frac{\pi}{12} = \cos\frac{5\pi}{12}$ ***M1A1***

⇒ Re *S* = Im *S* ***AG***

Note: Accept a geometrical method.

[4 marks]

6c. By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$, find the value of cos $\frac{\pi}{12}$ in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$, where a, b and c are integers to be determined. *[3 marks]*

Markscheme

$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6}$ ***M1A1***

$=\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}$

$=\frac{\sqrt{6}+\sqrt{2}}{4}$ ***A1***

[3 marks]

6d. Hence, or otherwise, show that $S=\frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)(1+\mathrm{i})$. *[4 marks]*

Markscheme

$\cos\frac{5\pi}{12}=\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\cos\frac{\pi}{6}\cos\frac{\pi}{4}-\sin\frac{\pi}{6}\sin\frac{\pi}{4}$ ***(M1)***

Note: Allow alternative methods *eg* $\cos\frac{5\pi}{12}=\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$.

$=\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}-\frac{1}{2}\frac{\sqrt{2}}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)***

Re S = $\cos\frac{\pi}{12}+\cos\frac{2\pi}{12}+\cos\frac{3\pi}{12}+\cos\frac{4\pi}{12}+\cos\frac{5\pi}{12}$

Re S = $\frac{\sqrt{2}+\sqrt{6}}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}+\frac{1}{2}+\frac{\sqrt{6}-\sqrt{2}}{4}$ ***A1***

$=\frac{1}{2}\left(\sqrt{6}+1+\sqrt{2}+\sqrt{3}\right)$ ***A1***

$=\frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)$

S = Re(S)(1 + i) since Re S = Im S, ***R1***

S = $\frac{1}{2}\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)(1+\mathrm{i})$ ***AG***

[4 marks]

7a. Solve $2\sin(x+60°) = \cos(x+30°),\ 0° \leqslant x \leqslant 180°$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2\sin(x+60°) = \cos(x+30°)$

$2(\sin x\cos 60° + \cos x\sin 60°) = \cos x\cos 30° - \sin x\sin 30°$ ***(M1)(A1)***

$2\sin x \times \frac{1}{2} + 2\cos x \times \frac{\sqrt{3}}{2} = \cos x \times \frac{\sqrt{3}}{2} - \sin x \times \frac{1}{2}$ ***A1***

$\Rightarrow \frac{3}{2}\sin x = -\frac{\sqrt{3}}{2}\cos x$

$\Rightarrow \tan x = -\frac{1}{\sqrt{3}}$ ***M1***

$\Rightarrow x = 150°$ ***A1***

[5 marks]

7b. Show that $\sin 105° + \cos 105° = \frac{1}{\sqrt{2}}$. *[3 marks]*

Markscheme

EITHER

choosing two appropriate angles, for example 60° and 45° ***M1***

$\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$ and

$\cos 105^\circ = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$ ***(A1)***

$\sin 105^\circ + \cos 105^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$ ***A1***

$= \frac{1}{\sqrt{2}}$ ***AG***

OR

attempt to square the expression ***M1***

$(\sin 105^\circ + \cos 105^\circ)^2 = \sin^2 105^\circ + 2\sin 105^\circ \cos 105^\circ + \cos^2 105^\circ$

$(\sin 105^\circ + \cos 105^\circ)^2 = 1 + \sin 210^\circ$ ***A1***

$= \frac{1}{2}$ ***A1***

$\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta, z \in \mathbb{C}, 0 \leqslant \theta \leqslant \pi$.

7c. Find the modulus and argument of z in terms of θ. Express each answer in its simplest form. *[9 marks]*

Markscheme

EITHER

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1 - \cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1 - \cos 2\theta)}$ ***A1***

$= \sqrt{2(2\sin^2\theta)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\frac{\sin 2\theta}{1 - \cos 2\theta}$ ***M1***

$= \frac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)$ ***A1***

$= \theta - \frac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta(\sin\theta - \mathrm{i}\cos\theta)$ ***(A1)***

$= -2\mathrm{i}\sin\theta(\cos\theta + \mathrm{i}\sin\theta)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \frac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \frac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \frac{\pi}{2}$ ***A1***

[9 marks]

7d. Hence find the cube roots of z in modulus-argument form. *[5 marks]*

Markscheme

attempt to apply De Moivre's theorem ***M1***

$$(1-\cos 2\theta-\mathrm{i}\sin 2\theta)^{\frac{1}{3}}=2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\theta-\frac{\pi}{2}+2n\pi}{3}\right)\right]$$

A1A1A1

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n=-1,\ 0,\ 1$. Equivalent forms are acceptable. ***A1***

[5 marks]