7 Functions

ESSENTIAL UNDERSTANDINGS

- Creating different representations of functions to model relationships between variables, visually and symbolically, as graphs, equations and tables represents different ways to communicate mathematical ideas.

In this chapter you will learn…

- about rational functions of the form $f(x) = \frac{ax+b}{cx^2+dx+e}$ and $f(x) = \frac{ax^2+bx+c}{dx+e}$
- how to solve cubic inequalities
- how to solve other inequalities graphically
- how to sketch graphs of the functions $y = |f(x)|$ and $y = f(|x|)$
- how to solve modulus equations and inequalities
- how to sketch graphs of the form $y = \frac{1}{f(x)}$
- how to sketch graphs of the form $y = f(ax + b)$
- how to sketch graphs of the form $y = [f(x)]^2$
- about even and odd functions
- about restricting the domain so that the inverse function exists
- about self-inverse functions.

CONCEPTS

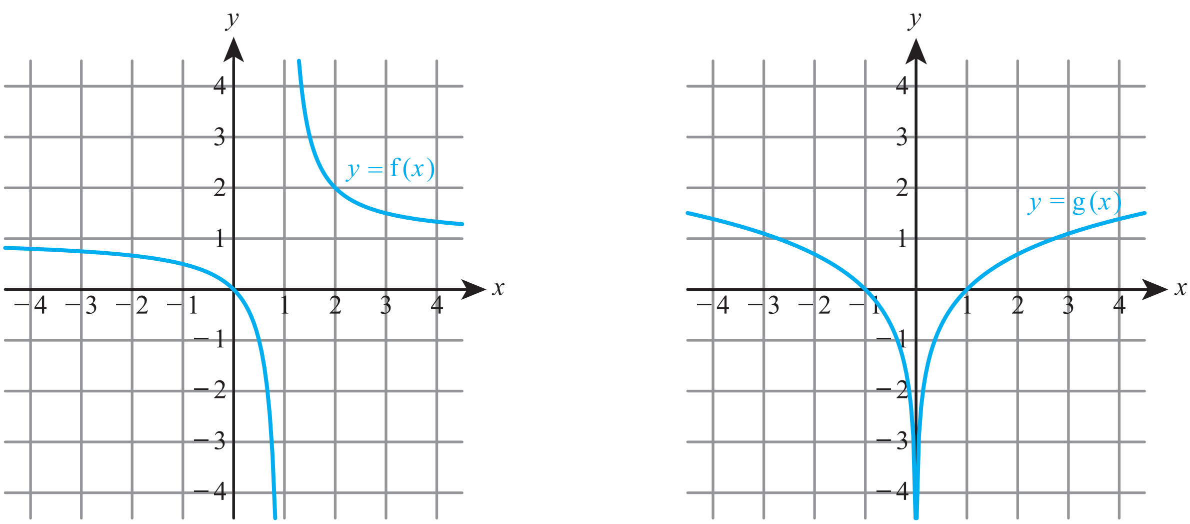
The following concepts will be addressed in this chapter:

- Different **representations** of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph.

LEARNER PROFILE – Risk-takers

Do you learn more from getting a problem right or wrong?

Figure 7.1 Graphs with different symmetries



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Sketch the graph of $y = \frac{2x-1}{x+3}$, labelling all asymptotes and axis intercepts.

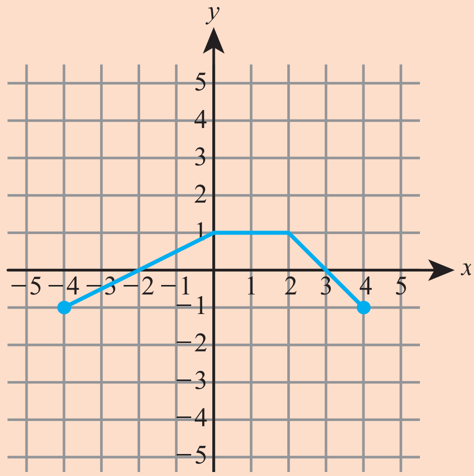
2 Solve the inequality $x^2 + 2x - 8 < 0$.

3 Solve, to 3 significant figures, with x in radians, the equation $\ln x = \sin 2x$.

4 The graph of $y = \mathrm{f}(x)$ is shown.

Sketch the graph of:

a $y = 2\mathrm{f}(x) + 3$ b $y = -\mathrm{f}(2x)$.



5 The function f is given by $\mathrm{f}(x) = \frac{2x-1}{x+3}$, $x \neq -3$.

Find f^{-1} and give its domain.

Starter Activity

Look at the pictures in Figure 7.1. In small groups, discuss any similarities you can identify between these functions.

Now look at this problem:

Use technology to investigate transformations of $\mathrm{f}(x) = x^3 - 4x$.

a Draw the graph of $y = \mathrm{f}(x)$.

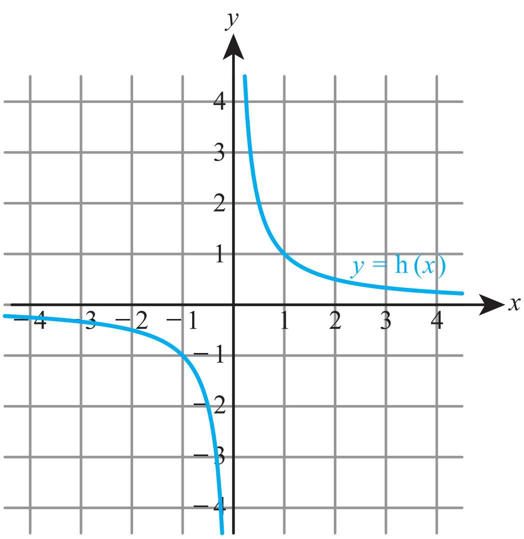
b Draw each of the following, and describe their relationship to $y = \mathrm{f}(x)$.

i $y = \mathrm{f}(2x + 3)$ ii $y = \mathrm{f}\left(\frac{1}{2}x - 3\right)$

c Draw $y = \mathrm{f}(-x)$ and $y = -\mathrm{f}(x)$. What do you notice?

You are already familiar with translations, stretches and reflections in the coordinate axes of graphs, but there are many other useful transformations that can be applied. Relating the algebraic representation of a function to possible symmetries of its graph also leads to different categorizations of functions.

It is also important to be able to apply sequences of transformations to graphs, including where there is more than one x-transformation.

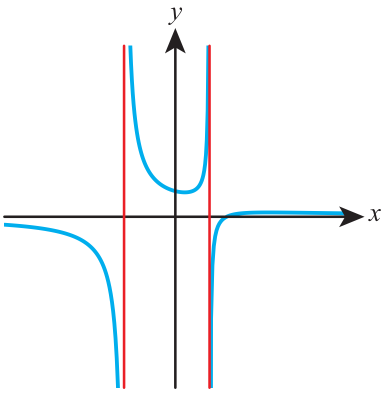


7A Rational functions of the form $f(x) = \dfrac{ax + b}{cx^2 + dx + e}$ and $f(x) = \dfrac{ax^2 + bx + c}{dx + e}$

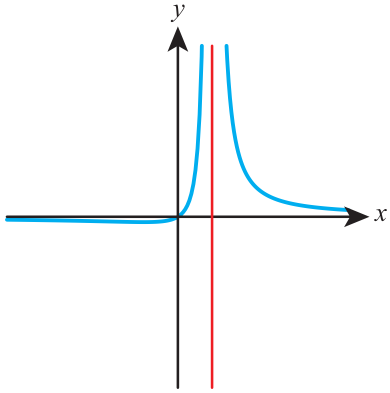
> You saw how to sketch the graph of the hyperbola $y = \dfrac{ax+b}{cx+d}$ in Section 16B of Mathematics: analysis and approaches SL.

Whereas the hyperbola $y = \dfrac{ax+b}{cx+d}$ always has one vertical asymptote, the graph of the rational function $y = \dfrac{ax+b}{cx^2+dx+e}$ could have two vertical asymptotes, one vertical asymptote or no vertical asymptotes, depending on the number of real solutions to the quadratic in the denominator.

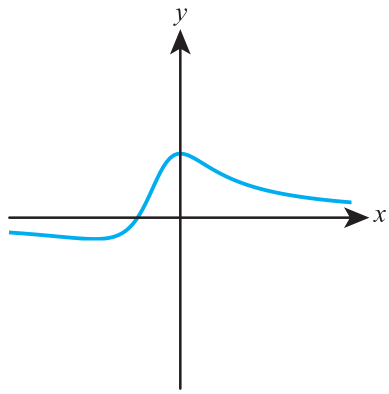
You also know that the horizontal asymptote will now always be $y = 0$, since the x^2 term in the denominator dominates and causes $y \to 0$ as $x \to \pm\infty$.



■ 2 vertical asymptotes



■ 1 vertical asymptote



■ 0 vertical asymptotes

KEY POINT 7.1

If $y = \dfrac{ax+b}{cx^2+dx+e}$ then

- the y-intercept is $\left(\dfrac{b}{e}, 0\right)$
- the x-intercept is $\left(0, -\dfrac{b}{a}\right)$
- the horizontal asymptote is at $y = 0$
- any vertical asymptotes occur at solutions of $cx^2 + dx + e = 0$.

WORKED EXAMPLE 7.1

Sketch the graph of $y = \dfrac{x-3}{2x^2+x-3}$, labelling any axis intercepts and asymptotes.

x-intercepts occur when $y = 0$

x-intercepts:

$x - 3 = 0$

$x = 3$

So, $(3, 0)$

y-intercepts occur when $x = 0$

y-intercepts:

$y = \dfrac{0-3}{2 \times 0^2 + 0 - 3} = \dfrac{-3}{-3} = 1$

So, $(0, 1)$

Vertical asymptotes occur when the denominator is zero

Vertical asymptotes:

$$2x^2+x-3=0$$
$$(2x+3)(x-1)=0$$
$$x=-\frac{3}{2} \text{ or } x=1$$

Because the degree of the denominator is larger then the degree of the numerator, the function tends to zero as $x \to \pm\infty$

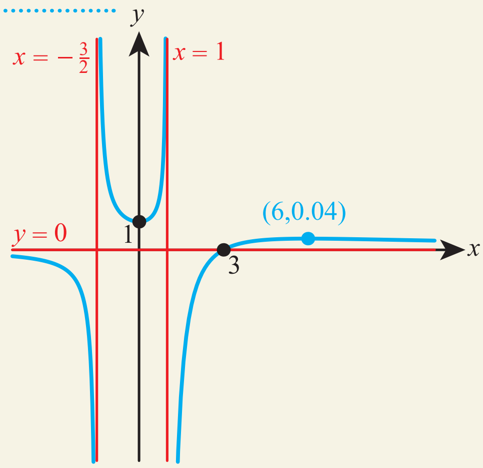
As $x \to \pm\infty$, $y \to 0$

So, horizontal asymptote at $y = 0$

Sketch the graph using all these features

To the right of the $x = 1$ asymptote: you can check that $y < 0$ when $x = 2$, so the graph starts below the x-axis, crosses it at $x = 3$ and then approaches it again for large x. There is therefore a maximum point with $x > 3$

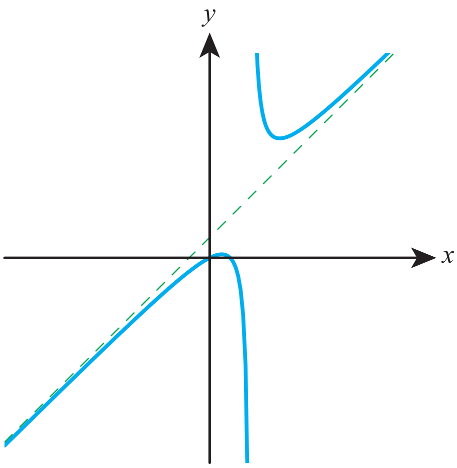
To the left of the $x=-\frac{3}{2}$ asymptote: you can check that $y < 0$ when $x = -2$, so the graph is below the x-axis



You can find the coordinates of the turning point using calculus or check using your calculator.

If $y=\dfrac{ax^2+bx+c}{dx+e}$, there will always be one vertical asymptote, and now when x becomes very large (either positive or negative) the x^2 term in the numerator dominates, so y tends to ∞ (or $-\infty$ if $\frac{a}{d}<0$) rather than zero, i.e. there is no horizontal asymptote.

However, the graph does tend to a non-horizontal asymptote as x becomes large. This is called an **oblique asymptote**.



You used polynomial division or comparing coefficients in Section 4C and Section 6B.

To find the equation of the oblique asymptote, we need to use polynomial division (or comparing coefficients) to express the rational function in an appropriate form.

KEY POINT 7.2

If $y = \frac{ax^2 + bx + c}{dx + e}$, then

- the y-intercept is $\left(0, \frac{c}{e}\right)$
- any x-intercepts occur at solutions of $ax^2 + bx + c = 0$
- the vertical asymptote is at $x = -\frac{e}{d}$
- there will be an oblique asymptote of the form $y = px + q$.

WORKED EXAMPLE 7.2

Sketch the graph of $y = \frac{x^2 - x - 2}{x - 3}$, labelling any axis intercepts and asymptotes.

x-intercepts occur when $y = 0$

x-intercepts:

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, -1$$

So, (2, 0) and (–1, 0)

y-intercepts occur when $x = 0$

y-intercepts:

$$y = \frac{0^2 - 0 - 2}{0 - 3} = \frac{2}{3}$$

So, $\left(0, \frac{2}{3}\right)$

Vertical asymptotes occur when the denominator is zero

Vertical asymptote:

$$x - 3 = 0$$
$$x = 3$$

Because the degree of the numerator is larger than the degree of the denominator, we need to do polynomial division (or comparing coefficients)

As $x \to \pm\infty$:

$$x^2 - x - 2 = (x - 3)(x + 2) + 4$$

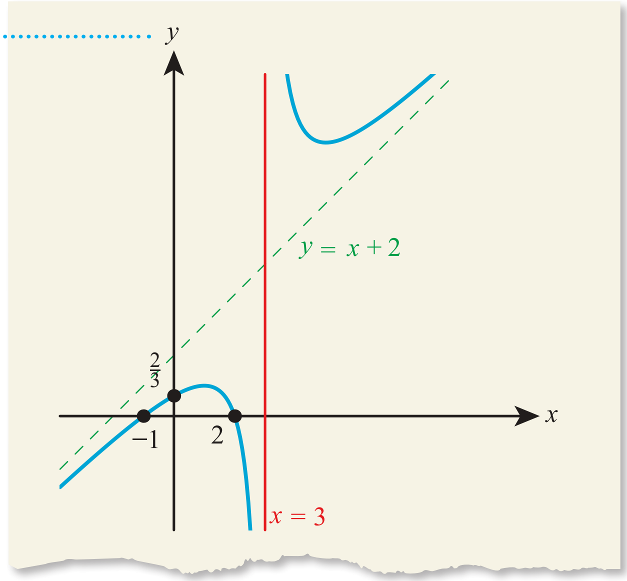
So,

$$\frac{x^2 - x - 2}{x - 3} = x + 2 + \frac{4}{x - 3}$$

We can now see that when $x \to \pm\infty$, $x + 2 + \frac{4}{x - 3} \to x + 2 + 0 = x + 2$

Therefore, oblique asymptote: $y = x + 2$.

Sketch the graph using all these features



Exercise 7A

For questions 1 to 4, use the method demonstrated in Worked Example 7.1 to sketch the graph. In each case label any vertical asymptotes and axis intercepts.

1 a $y=\frac{3x-2}{x^2+4x}$ b $y=\frac{5-2x}{x^2+x-2}$

2 a $y=\frac{1-x}{x^2-2x-3}$ b $y=\frac{x-3}{x^2-6x+8}$

3 a $y=\frac{4x}{x^2+2x+5}$ b $y=\frac{-3x}{x^2-x+2}$

4 a $y=\frac{x+3}{x^2+4x+4}$ b $y=\frac{x-2}{x^2-2x+1}$

For questions 5 to 8, use the method demonstrated in Worked Example 7.2 to sketch the graph. In each case label any vertical and oblique asymptotes and axis intercepts.

5 a $y=x-\frac{4}{x+3}$ b $y=2x-\frac{6}{x-2}$

6 a $y=3-x+\frac{6}{x+2}$ b $y=2x+1-\frac{9}{x-1}$

7 a $y=\frac{x^2+4x-5}{x-2}$ b $y=\frac{x^2-4}{x+3}$

8 a $y=\frac{x^2+4x}{x+1}$ b $y=\frac{x^2-3x-10}{x-4}$

9 For the graph of $y=\frac{4x+5}{4x^2-9}$

a find the equations of the vertical asymptotes

b sketch the graph, labelling the coordinates of any axis intercepts.

10 For the graph of $y=\frac{5x-10}{3x^2+2x-8}$

a find the equations of the vertical asymptotes

b sketch the graph, labelling the coordinates of any axis intercepts.

11 One of the asymptotes of the graph of $y=\frac{x+1}{2x^2+kx-12}$ is $x=-4$.

a Find the value of k.

b Find the equation of the other vertical asymptote.

c Sketch the graph, labelling any axis intercepts.

12 a On the same set of axes, sketch the graph of $y = \dfrac{3x}{x^2 - 2x + 1}$ and the graph of $y = x + 2$, labelling any vertical asymptotes and axis intercepts.

b Hence state the number of solutions of the equation $\dfrac{3x}{x^2 - 2x + 1} = x + 2$.

13 a On the same set of axes, sketch the graph of $y = \dfrac{x - 1}{2x^2 + 5x - 3}$ and the graph of $y = 2x - 1$, labelling any vertical asymptotes and axis intercepts.

b Hence state the number of solutions of the equation $\dfrac{x - 1}{2x^2 + 5x - 3} = 2x - 1$.

14 A curve has equation $y = \dfrac{2x - 3}{x^2 + 4}$.

a i If the line $y = k$ intersects the curve, show that $4k^2 + 3k - 1 \leqslant 0$.

ii Hence find the coordinates of the turning points of the curve.

b Sketch the curve.

15 a i Find the set of values of k for which the equation $ks^2 - (k + 1)x - 2k - 2 = 0$ has real roots.

ii Hence determine the range of the function $\mathrm{f}: x \mapsto \dfrac{x + 2}{x^2 - x - 2}$.

b State the equations of the vertical asymptotes of $y = \mathrm{f}(x)$ and the coordinates of any axis intercepts.

c Sketch the graph of $y = \mathrm{f}(x)$.

16 The curve C has equation $y = \dfrac{x - a}{(x - b)(x - c)}$.

Sketch C when:

a $a < b < c$

b $b < a < c$.

17 Let $\mathrm{f}(x) = \dfrac{x^2 - 6x + 10}{x - 3}$.

a Show that $y = \mathrm{f}(x)$ has an oblique asymptote at $y = Ax + B$, where A and B are constants to be found.

b Find the turning points of $\mathrm{f}(x)$.

c State the coordinates of any axis intercepts and the equation of the vertical asymptote of $y = \mathrm{f}(x)$.

d Sketch the graph of $y = \mathrm{f}(x)$.

18 a Show that the function $\mathrm{f}(x) = \dfrac{2x^2 - x - 3}{2x - 5}$ can be written as $\mathrm{f}(x) = Ax + B + \dfrac{C}{2x - 5}$, where A, B and C are constants to be found.

b Write down the equation of the oblique asymptote of $y = \mathrm{f}(x)$.

c By finding a condition on k for there to be real solutions to the equation $\mathrm{f}(x) = k$, find the range of f.

d State the axis intercepts and equation of the vertical asymptote of $y = \mathrm{f}(x)$.

e Sketch the graph of $y = \mathrm{f}(x)$.

19 The function f is defined by $\mathrm{f}(x) = \dfrac{x + c}{x^2 - 3x - c}$.

The range of f is $\mathrm{f}(x) \in \mathbb{R}$.

Find the possible values of c.

20 The function f is given by $\mathrm{f}(x) = \dfrac{x^2 + 2ax + a^2 - 1}{x + a}$, $a > 0$.

a Find the equation of the oblique asymptote of $y = \mathrm{f}(x)$.

b Show that f has no stationary points.

c Sketch the graph of $y = \mathrm{f}(x)$, labeling all asymptotes and axis intercepts.

7B Solutions of $g(x) \geqslant f(x)$, both analytically and graphically

In Chapter 15 of Mathematics: analysis and approaches SL you solved quadratic inequalities by sketching the graph and identifying the relevant region. This same method can be used with cubic inequalities.

WORKED EXAMPLE 7.3

Solve the inequality $x^3 > x^2 + 6x$.

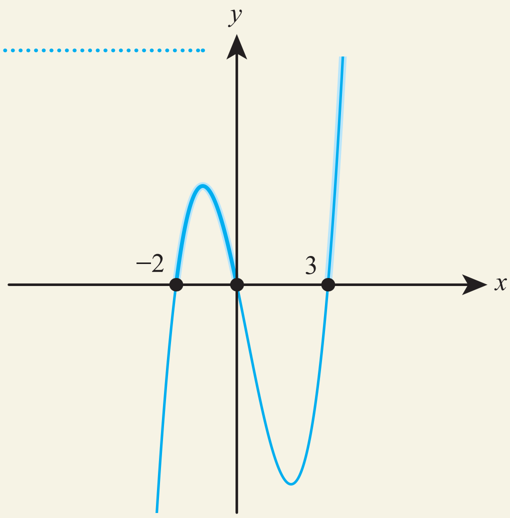
Rearrange in the form $f(x) > 0$

$$x^3 > x^2 + 6x$$
$$x^3 - x^2 - 6x > 0$$

Factorize in order to find roots of $f(x) = 0$

$$x(x^2 - x - 6) > 0$$
$$x(x + 2)(x - 3) > 0$$

Now sketch the graph of $y = f(x)$ and identify regions where the y-value is greater than zero



Describe the highlighted sections in terms of x

So, $-2 < x < 0$ or $x > 3$.

This method can be applied to inequalities involving other functions whose graphs you can sketch, such as rational functions.

WORKED EXAMPLE 7.4

Solve the inequality $\frac{7}{x^2 + x - 2} > 4$.

Sketch the graph of $y = \frac{7}{x^2 + x - 2}$ and add the line $y = 4$

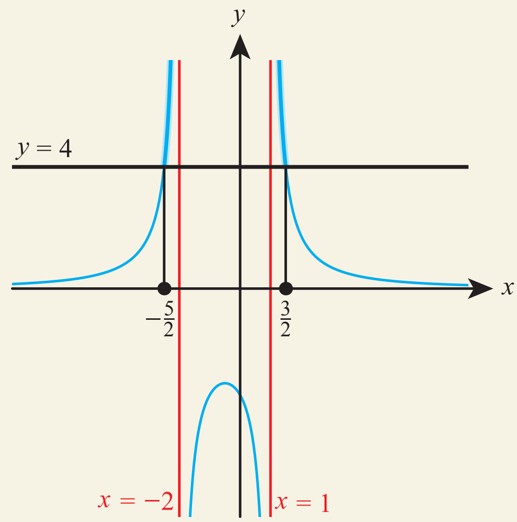
No x-intercept

y-intercept: $\left(-\frac{7}{2}, 0\right)$

Vertical asymptotes:

$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$
$$x = 1, -2$$

Horizontal asymptote: $y = 0$



Find the points of intersection

When $y = 4$

$$\frac{7}{x^2 + x - 2} = 4$$
$$4x^2 + 4x - 8 = 7$$
$$4x^2 + 4x - 15 = 0$$
$$(2x - 3)(2x + 5) = 0$$
$$x = \frac{3}{2}, -\frac{5}{2}$$

Describe the part of the graph above the line $y = 4$ in terms of x

Note that x can not be equal to −2 or 1 due to the asymptotes there

So, $-\frac{5}{2} \leqslant x < -2$ or $1 < x \leqslant \frac{3}{2}$.

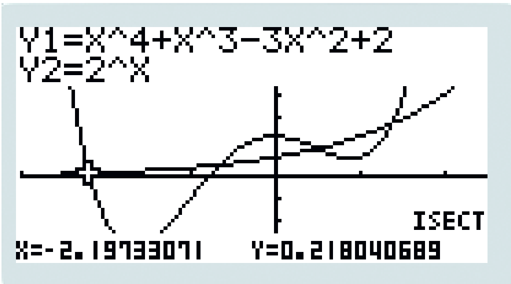
You can use your GDC to solve more complicated inequalities graphically.

WORKED EXAMPLE 7.5

Solve the inequality $x^4 + x^3 - 3x^2 + 2 < 2^x$.

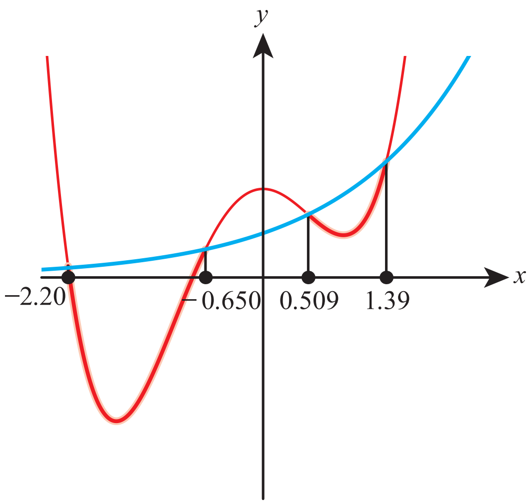
Use the GDC to find the intersection points of $y = x^4 + x^3 - 3x^2 + 2$ and $y = 2^x$

The expressions are equal when $x = -2.20, -0.650, 0.509, 1.39$



Now highlight the regions where $y = x^4 + x^3 - 3x^2 + 2$ is below $y = 2^x$

So, $-2.20 < x < -0.650$ or $0.509 < x < 1.39$.



Exercise 7B

For questions 1 to 5, use the method demonstrated in Worked Example 7.3 to solve the inequality.

1 a $x^3 + 2x > 3x^2$

b $x^3 > 4x^2 + 5x$

2 a $x^3 < 6x^2 - 8x$

b $x^3 + 18x < 9x^2$

3 a $(x+3)(x-2)(x-5) \leqslant 0$

b $(x-1)(x-3)(x-4) \leqslant 0$

4 a $(4-x)(x-3)(x+2) \geqslant 0$

b $(2-x)(x-1)(x-8) \geqslant 0$

5 a $(x+1)(x-2)^2 > 0$

b $(x+3)^2(x-4) > 0$

For questions 6 to 10, use the method demonstrated in Worked Example 7.5 to solve the inequality.

6 a $x^3 + 3x^2 - 2 \leqslant 2^{-x}$

b $x^3 + 8x^2 + 20x + 16 \leqslant 3^{-x}$

7 a $2e^{-x} \geqslant x^2 - 3$

b $e^{-x+1} \geqslant x^2 - 1$

8 a $4\ln x > x - 2$

b $\ln(x-1) > 2x - 5$

9 a $x^4 + 2x^3 + 1 < 3x^2 + 4x$

b $3x^2 + 10x - 9 > x^4 + 4x^3$

10 a $\cos 4x \leqslant e^{x+1} - 2$

b $\sin 3x \geqslant e^{\frac{x}{3}} - 2$

11 Solve the inequality $2x^3 + x^2 > 6x$.

12 a Show that $(x-2)$ is a factor of $2x^3 + x^2 - 7x - 6$

b Hence solve the inequality $3x^3 + 2x^2 \leqslant x^3 + x^2 + 7x + 6$.

13 a Show that $(x+3)$ is a factor of $2x^3 + 11x^2 + 12x - 9$.

b Hence solve the inequality $11x^2 - 4 > 5 - 12x - 2x^3$.

14 Given that $a < b < c$, solve the inequality $(x-a)(x-b)(x-c) > 0$.

15 Given that $a < b$, solve the inequality $(x-a)(x-b)^2 < 0$.

16 Find the set of values of x for which $x^4 - 4x^2 + 3x + 1 \leqslant 0$.

17 Solve the inequality $2x^5 - 6x^4 + 8x^2 - 1 \geqslant 0$.

18 Find the set of values of x for which $3\ln(x^2 + 1) < x + 2$.

19 The solution of the inequality $x^3 + bx^2 + cx + d < 2$ is $x < 3$, $x \neq 1$. Find the values of the integers a, b and c.

20 The solution of the inequality $ax^3 + bx^2 + cx + d > 3$ is $x < -4$ or $-1 < x < \frac{3}{2}$. Find the values of the integers a, b, c and d where $|a|$ is as small as possible.

21 Solve the inequality $\frac{3x}{x^2 + x - 6} \geqslant 2^x$.

22 Solve the inequality $\frac{x-2}{3x^2 + 2x - 8} \leqslant \ln(x+4)$.

23 Find the set of values of x for which the function $f(x) = 2 + 8x^3 - x^4$ is decreasing.

24 Find the set of values of x for which the function $f(x) = x^4 - 4x^3 - 2x^2 + 12x - 5$ is increasing.

25 a On the same axes draw the graphs of $y = \frac{3x+5}{x-2}$ and $y = 4$.

b Hence solve the inequality $\frac{3x+5}{x-2} \geqslant 4$.

26 a On the same axes draw the graphs of $y = \frac{2x-7}{x+1}$ and $y = x - 3$.

b Hence solve the inequality $\frac{2x-7}{x+1} < x - 3$.

27 A curve has equation $y = \frac{2x-a}{x+b}$, where $a, b > 0$.

a Sketch the curve, labelling all asymptotes and the coordinates of all axis intercepts.

b Hence solve the inequality $\frac{2x-a}{x+b} > 3$.

28 The solution of the inequality $\frac{16x+1}{px+1} > x + 4$ is $x < q$ or $r < x < 3$. Find p, q and r.

7C The graphs of the functions $y = |f(x)|$ and $y = f(|x|)$

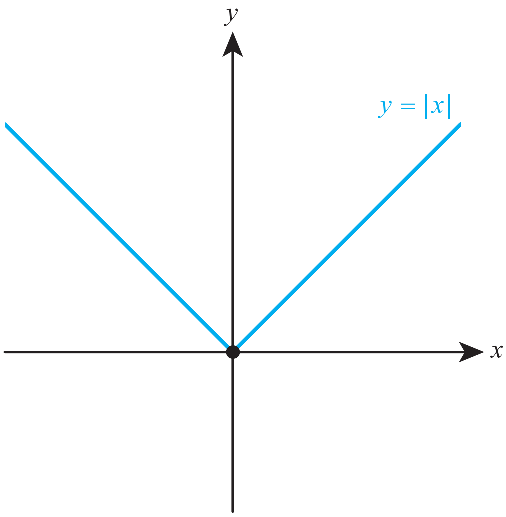
The modulus function leaves positive numbers unaffected but reverses the sign of negative numbers.

KEY POINT 7.3

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$

This means that the graph is of $y = |x|$ is given by $y = x$ when $x \geqslant 0$ and $y = -x$ when $y < 0$.

The domain of $|x|$ is all real numbers, whilst the range is all positive numbers and zero.



This idea can be applied to other functions involving the modulus function. The graph of $y = |f(x)|$ will be identical to that of $y = f(x)$ when $f(x) \geqslant 0$ but will be $y = -f(x)$ whenever $f(x) < 0$.

Since $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis this means that any part of $y = f(x)$ below the x-axis is just reflected in the x-axis.

KEY POINT 7.4

To sketch the graph of $y = |f(x)|$, start with the graph of $y = f(x)$ and reflect in the x-axis any parts that are below the x-axis.

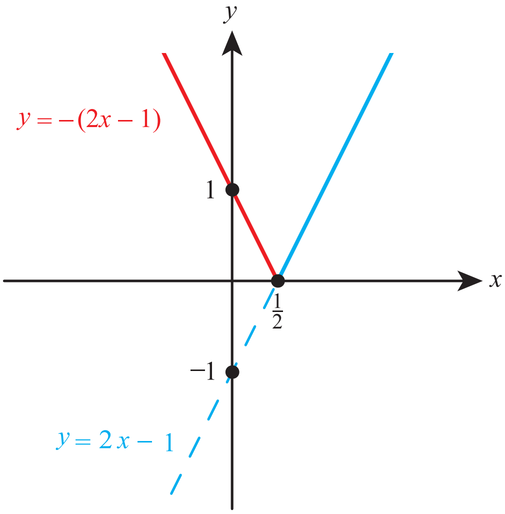
WORKED EXAMPLE 7.6

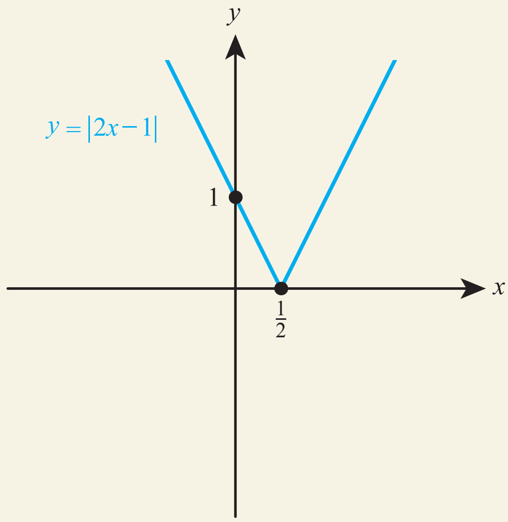
Sketch the graph of

a $y = |2x - 1|$

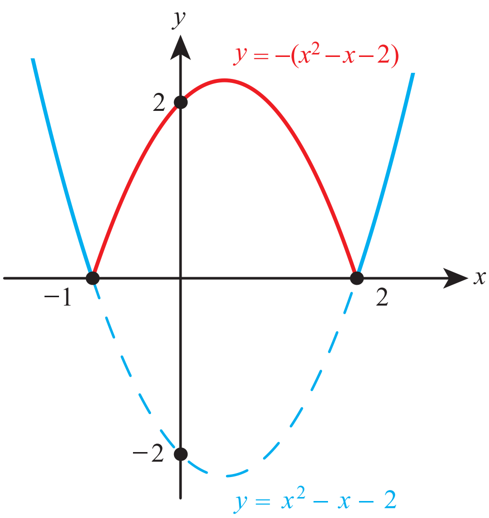
b $y = |x^2 - x - 2|$.

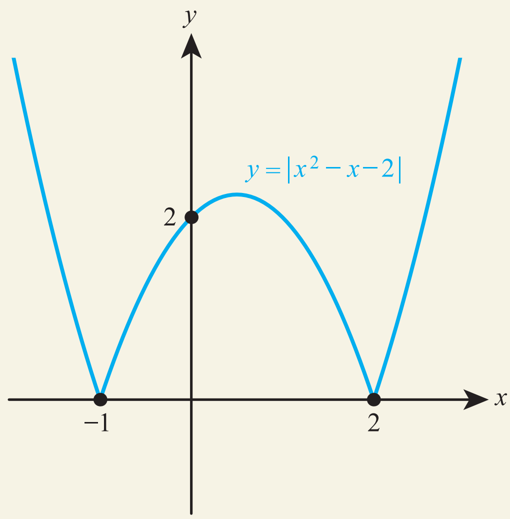
Sketch $y = 2x - 1$ and reflect the part below the x-axis to be above it **a**





Sketch $y = x^2 - x - 2$ and reflect the part below the x-axis to be above it **b**





To sketch the graph of $y = f(|x|)$, note that $f(|-x|) = f(|x|)$. Therefore $y = f(|x|)$ is symmetric in the y-axis.

KEY POINT 7.5

To sketch the graph of $y = f(|x|)$, start with the graph of $y = f(x)$ for $x \geqslant 0$ and reflect that in the y-axis.

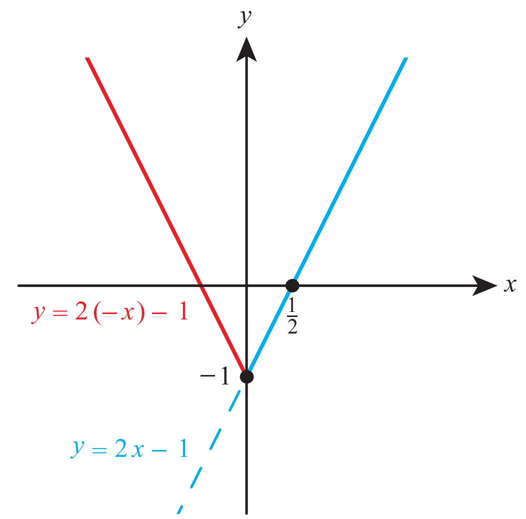
WORKED EXAMPLE 7.7

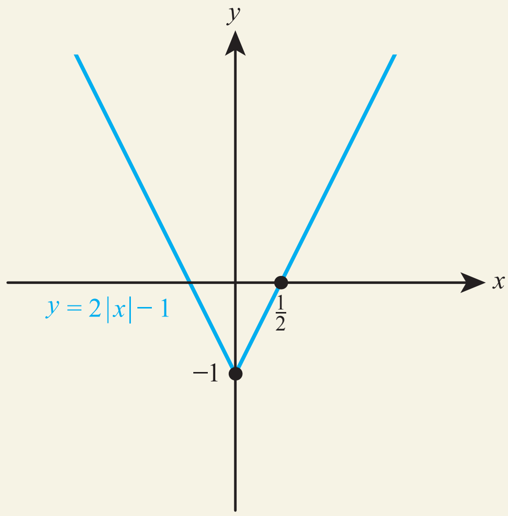
Sketch the graph of

a $y = 2|x| - 1$

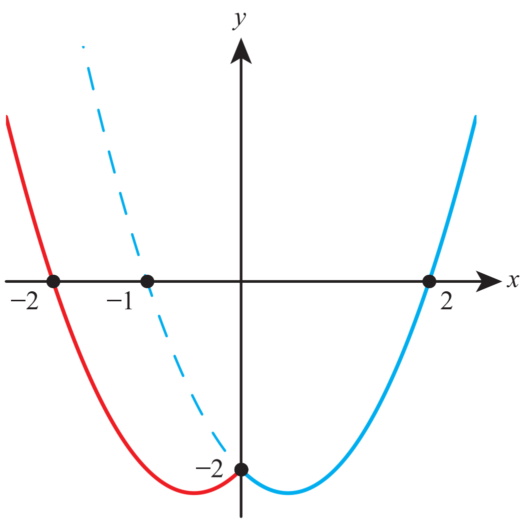
b $y = |x|^2 - |x| - 2$.

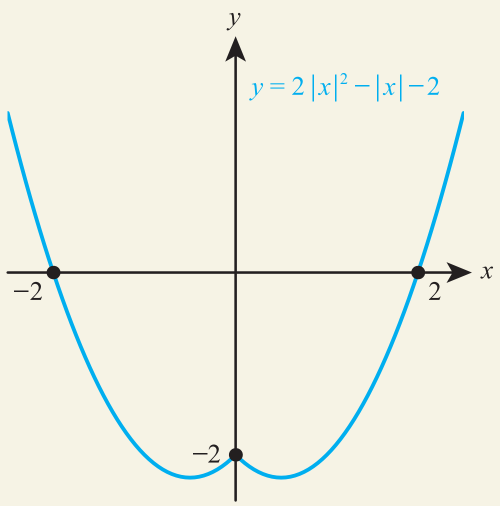
Sketch $y = 2x - 1$ for $x \geqslant 0$ and reflect it in the y-axis ····· **a**





Sketch $y = x^2 - x - 2$ for $x \geqslant 0$ and reflect it in the y-axis ····· **b**





Solutions of modulus equations and inequalities

It is useful to sketch the relevant graphs before solving equations and inequalities involving the modulus function.

The graphs enable you to decide whether any intersections are on the reflected or the original part of the graph. If on the original part you can rewrite the equation without the modulus sign in; if on the reflected part you need to replace the modulus sign by a minus sign.

WORKED EXAMPLE 7.8

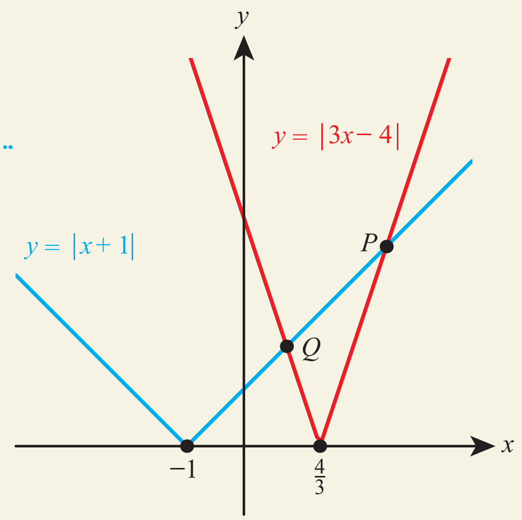
Solve the equation $|3x - 4| = |x + 1|$.

Sketch the graphs of $y = |3x - 4|$ and $y = |x + 1|$

There are two intersections:

Q is on the original part of the red graph and the original part of the blue graph

P is on the reflected part of the red graph and the original part of the blue graph



For the intersection of two original parts, just remove the modulus signs

$$3x - 4 = x + 1$$
$$2x = 5$$
$$x = \frac{5}{2}$$

For a reflected part, multiply the equation by -1

$$-(3x - 4) = x + 1$$
$$-3x + 4 = x + 1$$
$$4x = 3$$
$$x = \frac{3}{4}$$

So, solutions are $x = \frac{3}{4}$, $x = \frac{5}{2}$.

The next Worked Example illustrates how useful the graphs are in identifying the solutions of the modulus equation (and therefore the solution of the inequality).

WORKED EXAMPLE 7.9

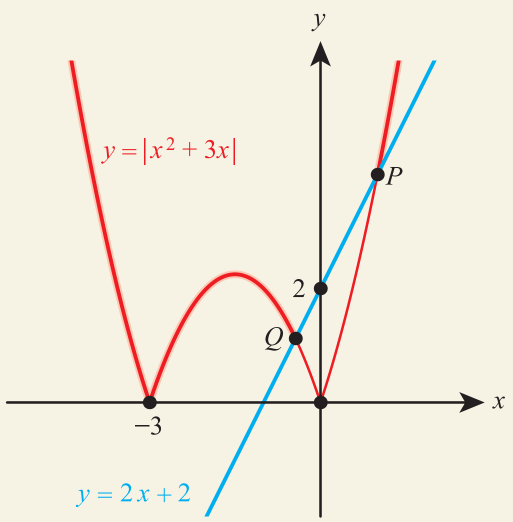
Solve the inequality $|x^2 + 3x| > 2x + 2$.

Sketch the graphs of $y = |x^2 + 3x|$ and $y = 2x + 2$ and highlight the required region

There are two intersections:

P is on the original part of the red graph

Q is on the reflected part of the red graph



For the original part, just remove the modulus sign from the equation

$$x^2 + 3x = 2x + 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, 1$$

We can see from the graph that $x = -2$ is not an intersection so disregard it

So, $x = 1$

For the reflected part, multiply by −1

$$-(x^2 + 3x) = 2x + 2$$
$$x^2 + 5x + 2 = 0$$
$$x = \frac{-5 \pm \sqrt{25 - 4 \times 1 \times 2}}{2}$$
$$= \frac{-5 \pm \sqrt{17}}{2}$$

Both solutions are negative but we can see from the graph that the intersection is the least negative of the two

So, $x = \dfrac{-5 + \sqrt{17}}{2}$

Refer back to the graph to describe the required region

So, $x < \dfrac{-5 + \sqrt{17}}{2}$ or $x > 1$.

Tip

You can also use your GDC to solve modulus equations and inequalities.

Be the Examiner 7.1

Solve the equation $|x-2| = 2x-3$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$x-2=2x-3$ $x=1$ $-(x-2)=2x-3$ $-x+2=2x-3$ $3x=5$ $x=\frac{5}{3}$ So, $x=1, \frac{5}{3}$	$x-2=2x-3$ $x=1$ So, $x=1$	$-(x-2)=2x-3$ $-x+2=2x-3$ $3x=5$ $x=\frac{5}{3}$ So, $x=\frac{5}{3}$

Exercise 7C

For questions 1 to 6, use the method demonstrated in Worked Example 7.6 to sketch the graph of $y = |f(x)|$, marking on axis intercepts.

1 a $y=|x+4|$
b $y=|x-1|$

2 a $y=|3x-2|$
b $y=|2x+5|$

3 a $y=|x^2-x-12|$
b $y=|x^2-4x+3|$

4 a $y=|x^3-4x|$
b $y=|x^3-6x^2+8x|$

5 a $y=|\sin x|, -2\pi \leqslant x \leqslant 2\pi$
b $y=|\tan x|, -\pi \leqslant x \leqslant \pi$

6 a $y=|\ln x|$
b $y=|\ln(x+2)|$

For questions 7 to 12, use the method demonstrated in Worked Example 7.7 to sketch the graph of $y = f(|x|)$, marking on axis intercepts.

7 a $y=|x|+4$
b $y=|x|-1$

8 a $y=3|x|-2$
b $y=2|x|+5$

9 a $y=|x|^2+|x|-12$
b $y=|x|^2-4|x|+3$

10 a $y=|x|^3-4|x|$
b $y=|x|^3-6|x|^2+8|x|$

11 a $y=\sin|x|, -2\pi \leqslant x \leqslant 2\pi$
b $y=\tan|x|, -\pi \leqslant x \leqslant \pi$

12 a $y=\ln|x|$
b $y=\ln|x+2|$

For questions 13 to 18, use the method demonstrated in Worked Example 7.8 to solve the modulus equation.

13 a $|2x-1|=5$
b $|3x+2|=8$

14 a $|3x-5|=|x+2|$
b $|4x+1|=|x-3|$

15 a $|5+2x|=3-4x$
b $|3x-4|=8-x$

16 a $|x^2+x-6|=6$
b $|x^2-5x+3|=3$

17 a $|x^2-3x-10|=x+2$
b $|x^2+5x+6|=x+3$

18 a $2|\cos x|=1, -\pi \leqslant x \leqslant \pi$
b $\sqrt{2}|\sin x|=1, -\pi \leqslant x \leqslant \pi$

For questions 19 to 24, use the method demonstrated in Worked Example 7.9 to solve the modulus inequality.

19 a $|2x-1|>5$
b $|3x+2|<8$

20 a $|3x-5| \leqslant |x+2|$
b $|4x+1| \geqslant |x-3|$

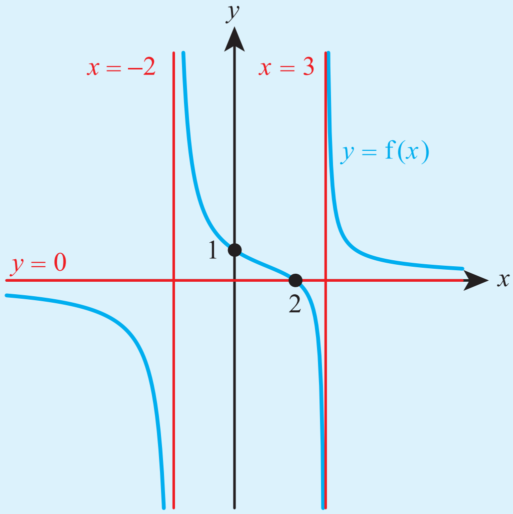
21 a $|5+2x|>3-4x$
b $|3x-4|<8-x$

22 a $|x^2+x-6| \geqslant 6$
b $|x^2-5x+3| \leqslant 3$

23 a $|x^2-3x-10|<x+2$
b $|x^2+5x+6|>x+3$

24 a $|\cos x|<\frac{1}{2}, -\pi \leqslant x \leqslant \pi$
b $|\sin x|>\frac{1}{\sqrt{2}}, -\pi \leqslant x \leqslant \pi$

25 The graph of $y = f(x)$ is shown below.

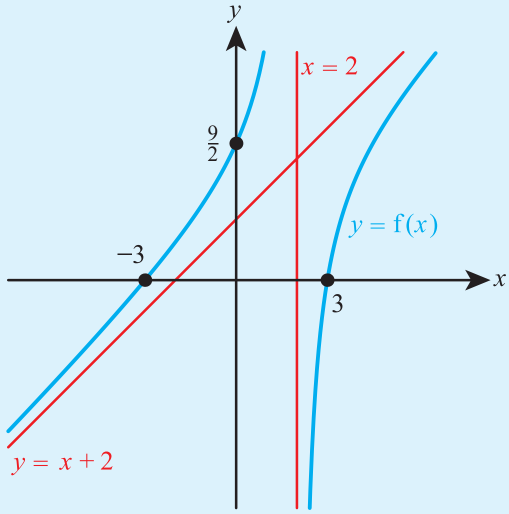


On separate axes, sketch the graph of

a $y = |f(x)|$

b $y = f(|x|)$.

26 The graph of $y = f(x)$ is shown below.



On separate axes, sketch the graph of

a $y = |f(x)|$

b $y = f(|x|)$.

27 Sketch the graph of $y = 2 - 3|x+1|$, labelling the axis intercepts.

28 Given that $a < 0 < b$, sketch the graph of $y = |(x-a)(x-b)|$.

29 Given that $a < 0 < b < c$, sketch the graph of $y = |(x-a)(x-b)(x-c)|$.

30 Solve the inequality $|xe^x| \leqslant 2 - x^2$.

31 Find the set of values of x for which $\left|\frac{3x-2}{x+4}\right| < 11 - 2x$.

32 Solve the inequality $|4x \arccos x| > 1$.

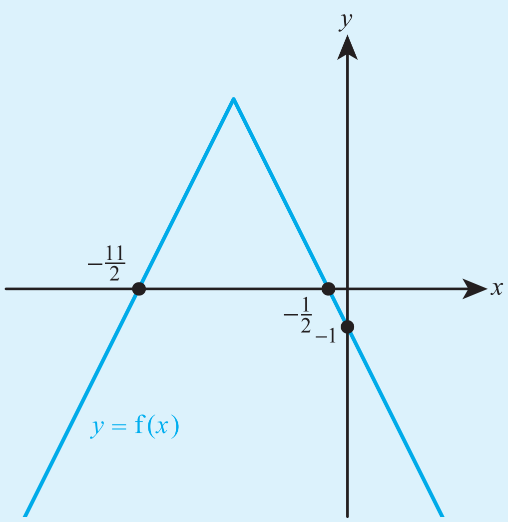
33 **a** On the same axes, sketch the graphs of $y = |2x-5|$ and $y = 3|x| + 1$.

b Hence solve the inequality $|2x-5| < 3|x| + 1$.

34 **a** Sketch the graph of $y = 3e^{|x|}$.

b Hence find the set of values of x for which $3e^{|x|} > 5$.

35 The graph of $y = a|x+b| + c$ is shown below.



Find the values of a, b and c.

36 Solve the inequality $|2x+3| > 3x+7$.

37 Find the values of x for which $|x^2-3x-5| = 3-x$.

38 Solve the inequality $|x^2-5x+4| > 2$.

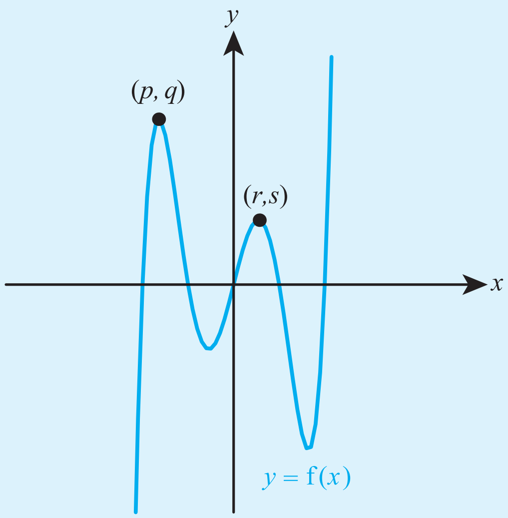
39 By sketching appropriate graphs or otherwise, solve the equation $|x+1|+|x-1| = x+4$.

40 By sketching appropriate graphs or otherwise, solve the equation $|3x-1| = x+|2-x|$.

41 **a** Sketch the graph of $y = x|x|$.

b Solve the equation $x|x| = kx$ where $k > 0$.

42 The graph of $y = f(x)$ is shown below.



Sketch the graph of $y = f(x) + |f(x)|$.

43 Solve the equation $|x+a^2| = |x-2a^2|$, giving your answer in terms of a.

44 Find the condition on the constant k such that the equation $|x^2+4x-7| = k$ has four solutions.

45 Find the condition on the constant k such that the equation $|x^3-12x+4| = k$ has four solutions.

7D The graphs of the functions $y = \frac{1}{f(x)}$, $y = f(ax + b)$ and $y = [f(x)]^2$

The graph of $y = \frac{1}{f(x)}$

Given the graph of $y = f(x)$ we can draw the graph of $y = \frac{1}{f(x)}$ by considering a few key features.

KEY POINT 7.6

To sketch the graph of $y = \frac{1}{f(x)}$ consider the following key features:

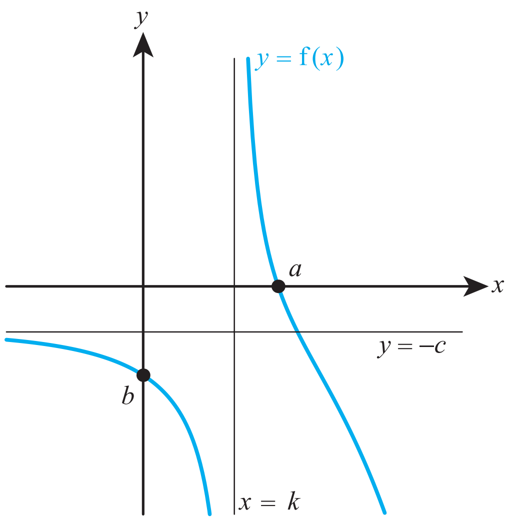
Feature of $y = f(x)$	**Feature of $y = \frac{1}{f(x)}$**
x-intercept at $(a, 0)$	$x = a$ is a vertical asymptote
y-intercept at $(0, b)$, $b \neq 0$	y-intercept at $\left(0, \frac{1}{b}\right)$
$x = a$ is a vertical asymptote	x-intercept at $(a, 0)$
$y = a$ is a horizontal asymptote, $a \neq 0$	$y = \frac{1}{a}$ is a horizontal asymptote
$y = 0$ is a horizontal asymptote	$y \to \pm\infty$
$y \to \pm\infty$	$y = 0$ is a horizontal asymptote
(a, b) is a turning point, $b \neq 0$)	$\left(a, \frac{1}{b}\right)$ is the opposite turning point

Tip

If you are unsure about which side of an asymptote the graph lies on, check a few points.

WORKED EXAMPLE 7.10

The diagram shows the graph of $y = f(x)$.



Sketch the graph of $y = \frac{1}{f(x)}$.

$y = f(x)$ has an x-intercept at $(a, 0)$ so $y = \frac{1}{f(x)}$ has a vertical asymptote at $x = a$

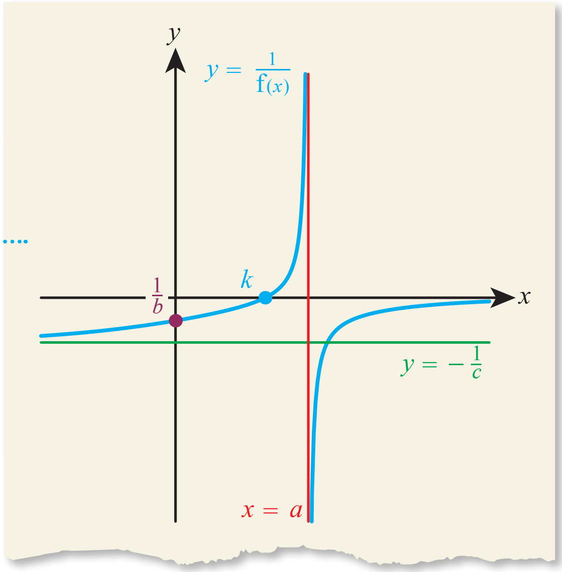
$y = f(x)$ has a y-intercept at $(0, b)$ so $y = \frac{1}{f(x)}$ has a y-intercept at $\left(0, \frac{1}{b}\right)$

$y = f(x)$ has a vertical asymptote at $x = k$ so $y = \frac{1}{f(x)}$ has an x-intercept at $(k, 0)$

$y = f(x)$ has a horizontal asymptote at $y = -c$ so $y = \frac{1}{f(x)}$ has a horizontal asymptote at $y = -\frac{1}{c}$

As $x \to \infty$, $f(x) \to -\infty$ so $y = \frac{1}{f(x)}$ has a horizontal asymptote at $y = 0$

$y = f(x)$ has no turning points so $y = \frac{1}{f(x)}$ has no turning points



The graph of $y = f(ax + b)$

In Section 16A of Mathematics: analysis and approaches SL you saw how to apply two vertical transformations, or one vertical and one horizontal transformation.

We now need to be able to apply two horizontal transformations.

Tip

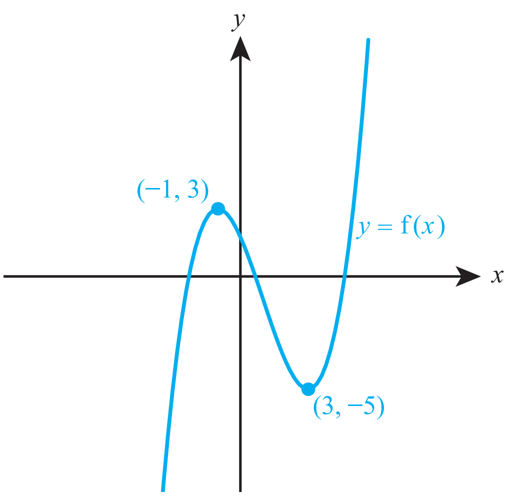
Notice that the transformations are in the 'wrong' order: the addition is done before the multiplication.

KEY POINT 7.7

When two horizontal transformations are applied, the order matters: $y = f(ax + b)$ is a horizontal translation by $-b$ followed by a horizontal stretch with scale factor $\frac{1}{a}$.

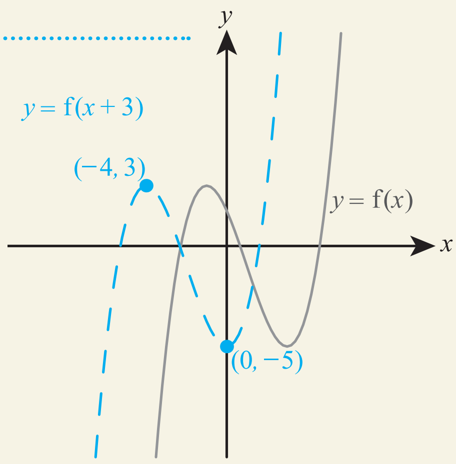
WORKED EXAMPLE 7.11

Below is the graph of $y = f(x)$. Sketch the graph of $y = f(2x + 3)$.

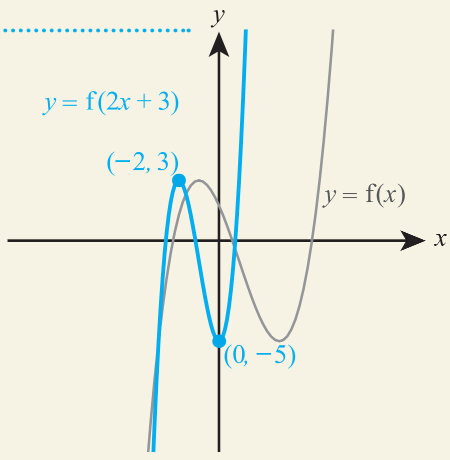


There are two horizontal transformations so the order is important.

First translate horizontally by −3...



... and then stretch horizontally with scale factor $\frac{1}{2}$



WORKED EXAMPLE 7.12

Describe a sequence of two horizontal transformations that maps the graph of $y = 9x^2$ to the graph of $y = x^2 - 6x + 9$.

Let $f(x) = 9x^2$

Express the second equation in function notation, related to the first

Then, $y = x^2 - 6x + 9$

$= (x-3)^2$

$= 9\left[\frac{1}{9}(x-3)^2\right]$

The factor of $\frac{1}{9}$ goes inside the squared bracket as a factor of $\frac{1}{3}$

$= 9\left(\frac{1}{3}x - 1\right)^2$

$= f\left(\frac{1}{3}x - 1\right)$

State the transformation, making sure the translation comes before the stretch

Horizontal translation by 1 followed by horizontal stretch with scale factor 3.

Be the Examiner 7.2

The graph of $y = f(x)$ is stretched horizontally with scale factor 2 and translated horizontally by 3. Find the equation of the transformed graph.

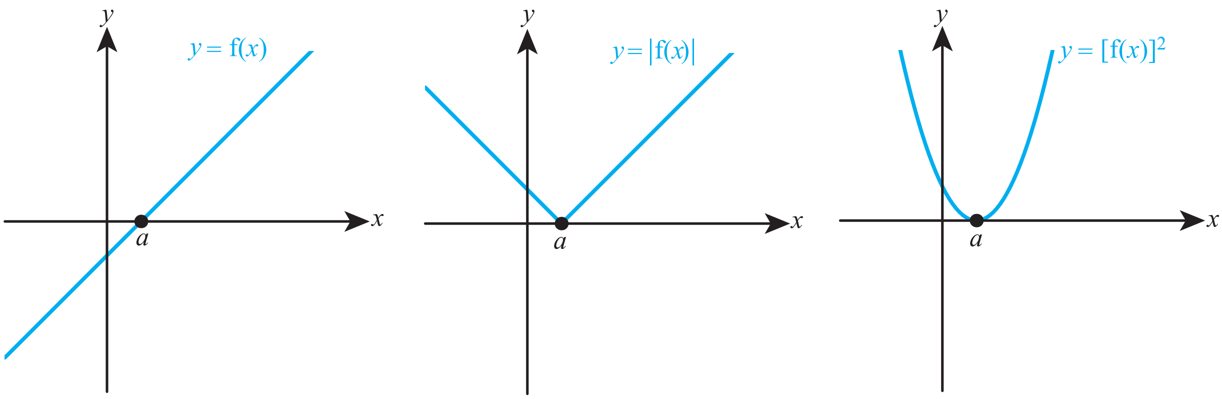
Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$y = f\left(\frac{1}{2}x - 3\right)$	$y = f\left(\frac{x-3}{2}\right)$	$y = f(2x - 6)$

The graph of $y = [f(x)]^2$

Sketching the graph of $y = [f(x)]^2$ is rather like sketching the graph of $y = |f(x)|$ in the sense that any parts of the graph of $y = f(x)$ that are below the x-axis will now be above the x-axis.

The difference is that all the y-values will change magnitude since they are being squared, except for any points where $f(x) = \pm 1$. This has the effect of smoothing the function where it touches the x-axis so that such points become turning points.



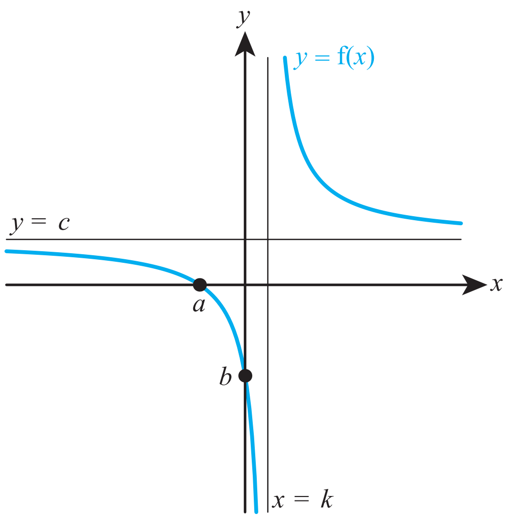
KEY POINT 7.8

To sketch the graph of $y = [f(x)]^2$ consider the following key features:

Feature of $y = f(x)$	Feature of $y = [f(x)]^2$
$y < 0$	$y > 0$
x-intercept at $(a, 0)$	Local minimum at $(a, 0)$
y-intercept at $(0, b)$	y-intercept at $(0, b^2)$
$x = a$ is a vertical asymptote	$x = a$ is a vertical asymptote
$y = a$ is a horizontal asymptote	$y = a^2$ is a horizontal asymptote
$y \to \pm\infty$	$y \to \infty$

WORKED EXAMPLE 7.13

Below is the graph of $y = f(x)$. Sketch the graph of $y = [f(x)]^2$.



All negative values on $y = f(x)$ now become positive on $y = [f(x)]^2$

$y = f(x)$ has an x-intercept at $(a, 0)$ so $y = [f(x)]^2$ has a local minimum at $(a, 0)$

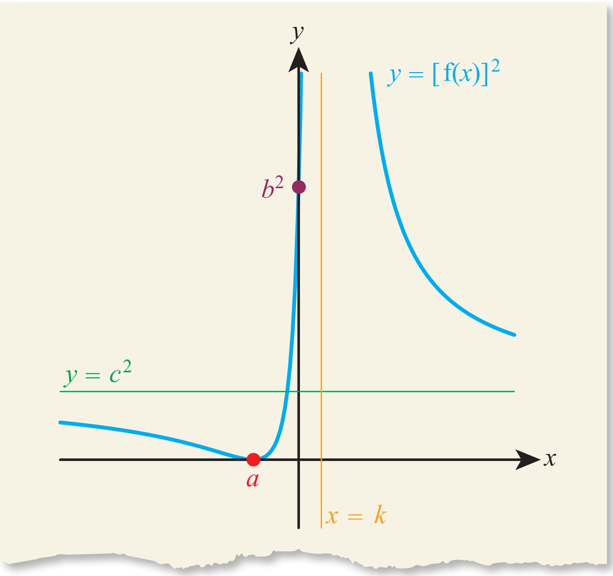
$y = f(x)$ has a y-intercept at $(0, b)$ so $y = [f(x)]^2$ has a y-intercept at $(0, b^2)$

The vertical asymptote is unaffected at $x = k$

$y = f(x)$ has a horizontal asymptote at $y = c$ so $y = [f(x)]^2$ has a horizontal asymptote at $y = c^2$

Where $f(x) > 1$, the graph of $y = [f(x)]^2$ will be above the graph of $y = f(x)$

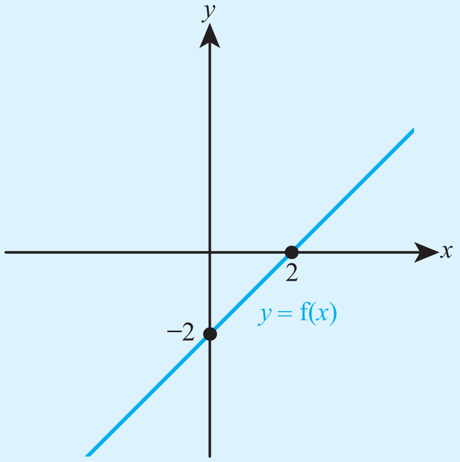
Where $0 < f(x) < 1$, the graph of $y = [f(x)]^2$ will be below the graph of $y = f(x)$



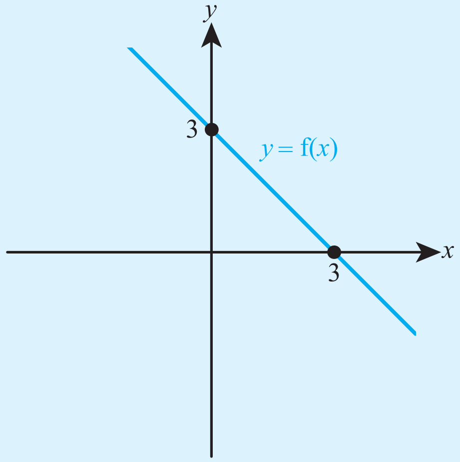
Exercise 7D

For questions 1 to 5, use the method demonstrated in Worked Example 7.10 to sketch the graph of $y = \frac{1}{f(x)}$. Label any axis intercepts, turning points and asymptotes.

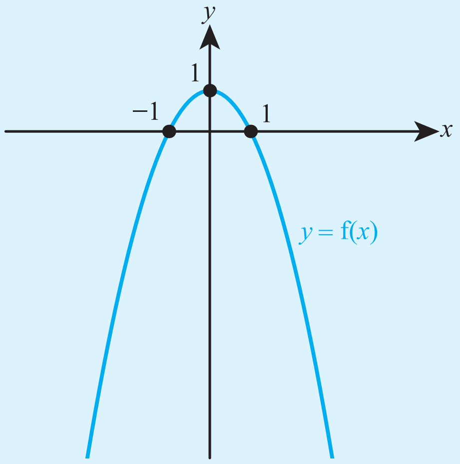
1 a



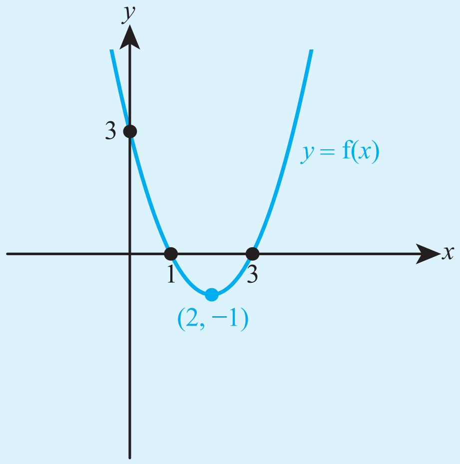
b



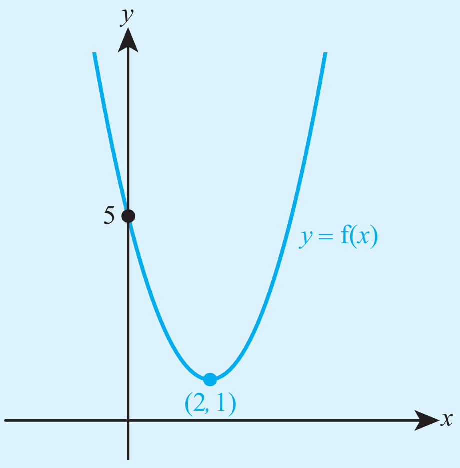
2 a



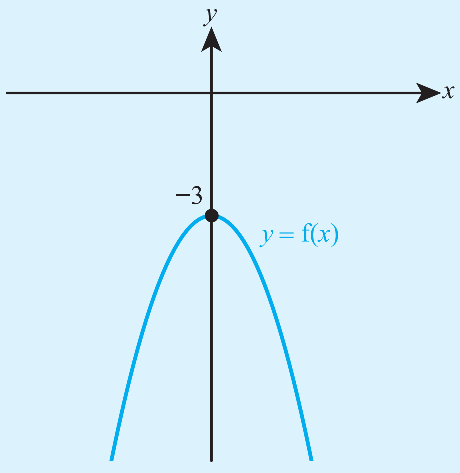
b



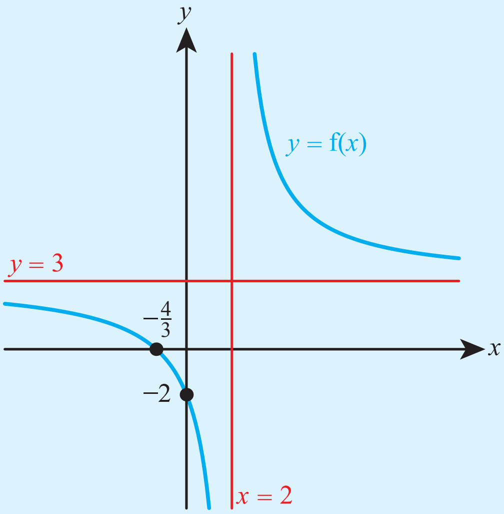
3 a



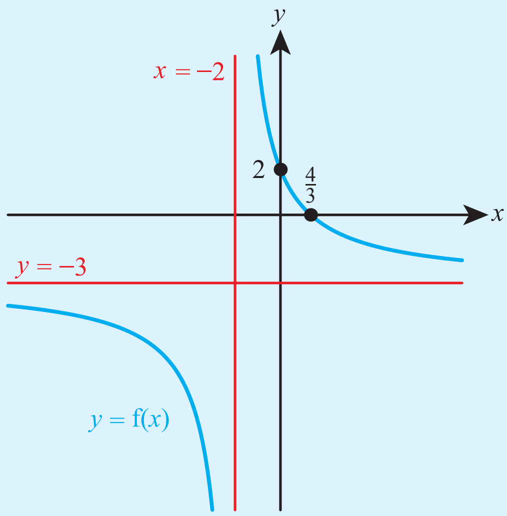
b



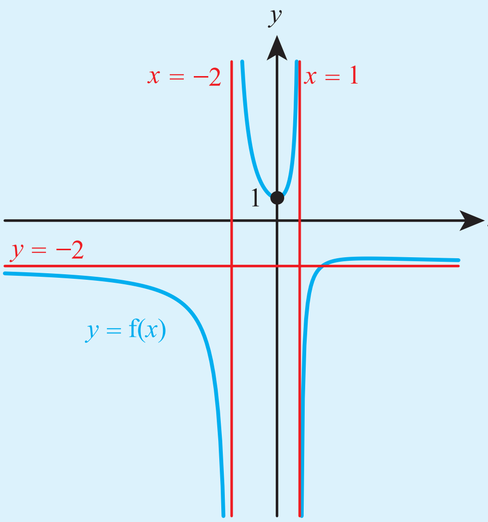
4 a



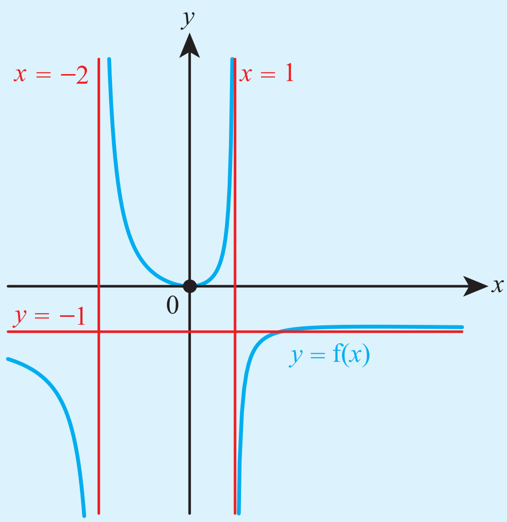
b



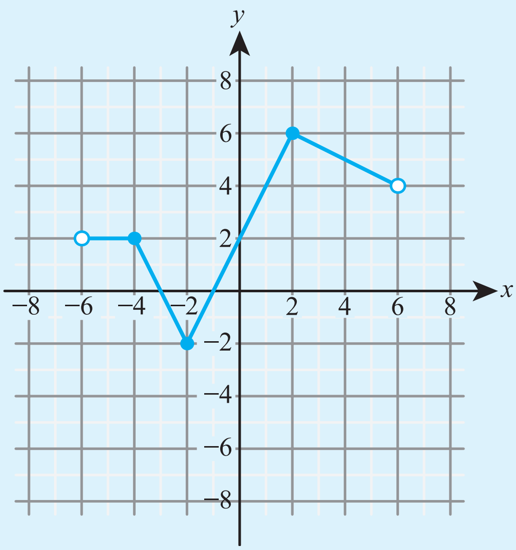
5 a



b



For questions 6 to 8, use the method demonstrated in Worked Example 7.11 to sketch the required graph.
The graph of $y = f(x)$ is given below.



6 **a** $y = f(3x - 2)$

b $y = f(4x + 1)$

7 **a** $y = f\left(\frac{1}{2}x + 3\right)$

b $y = f\left(\frac{1}{3}x - 1\right)$

8 **a** $y = f\left(\frac{x+1}{4}\right)$

b $y = f\left(\frac{x-4}{3}\right)$

For questions 9 to 11, use the method demonstrated in Worked Example 7.12 to describe a sequence of two horizontal transformations that maps the graph of $y = x^2$ to the given graph.

9 **a** $\frac{1}{16}x^2 - x + 4$

b $\frac{1}{9}x^2 + \frac{2}{3}x + 1$

10 **a** $9x^2 - 6x + 1$

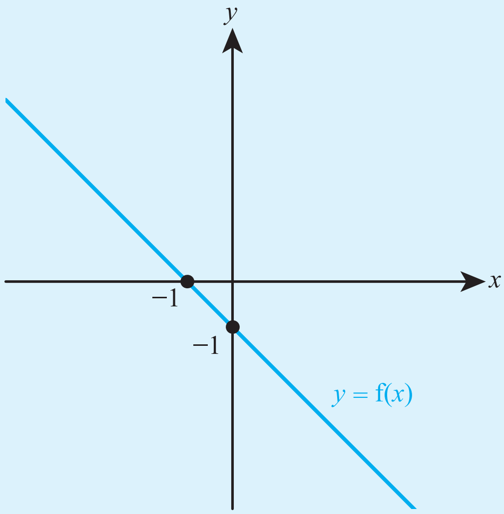
b $4x^2 + 12x + 9$

11 **a** $x^2 + 6x + 9$

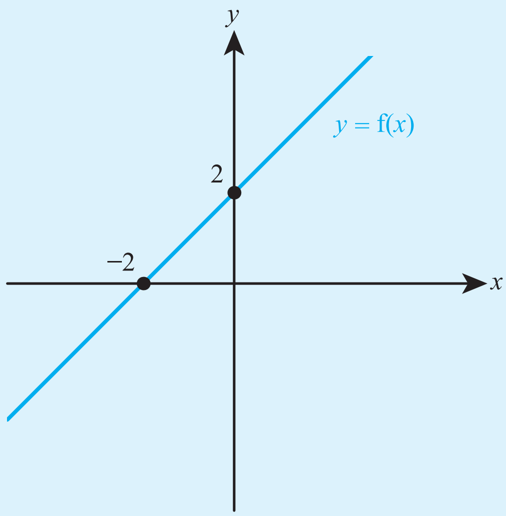
b $x^2 - 4x + 4$

For questions 12 to 16, use the method demonstrated in Worked Example 7.10 to sketch the graph of $y = [f(x)]^2$. Label any axis intercepts, turning points and asymptotes.

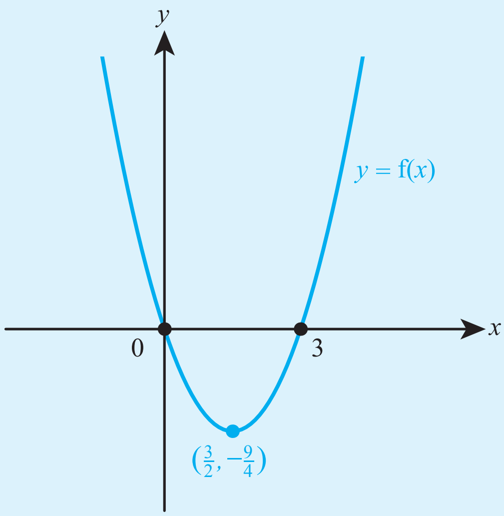
12 **a**



b



13 **a**



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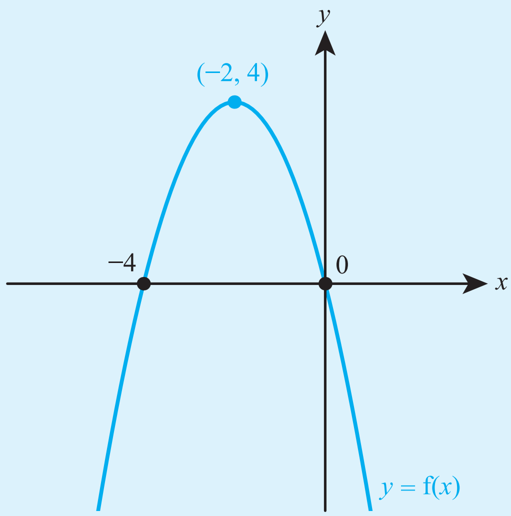
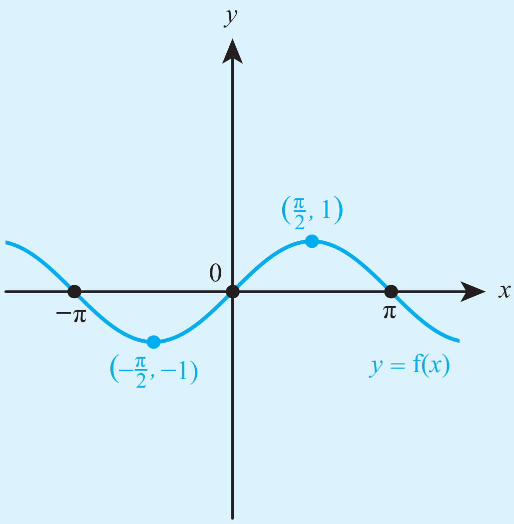


14 a



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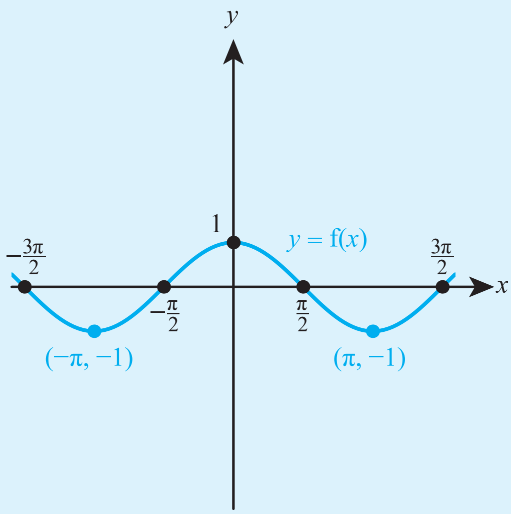

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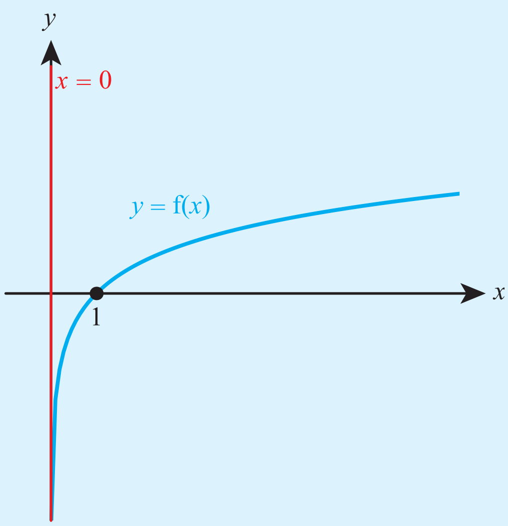

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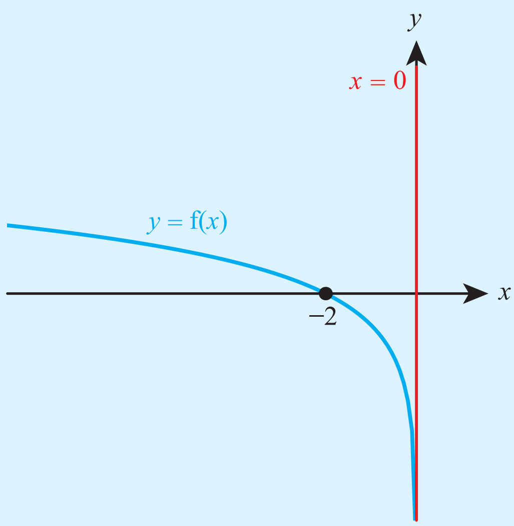
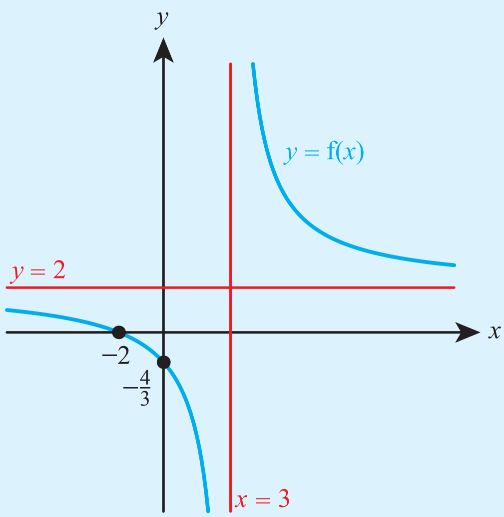


16 a



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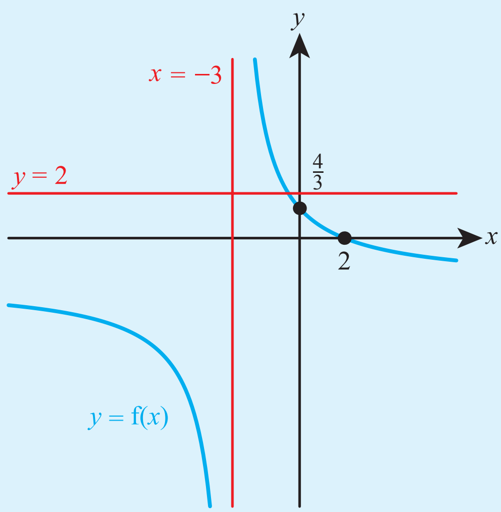
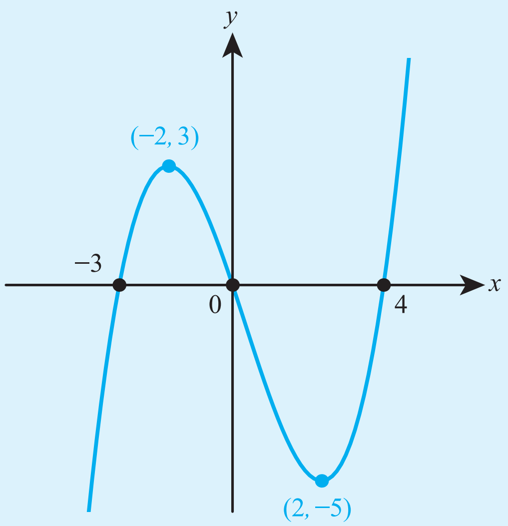


17 The graph of $y = \mathrm{f}(x)$ is shown.

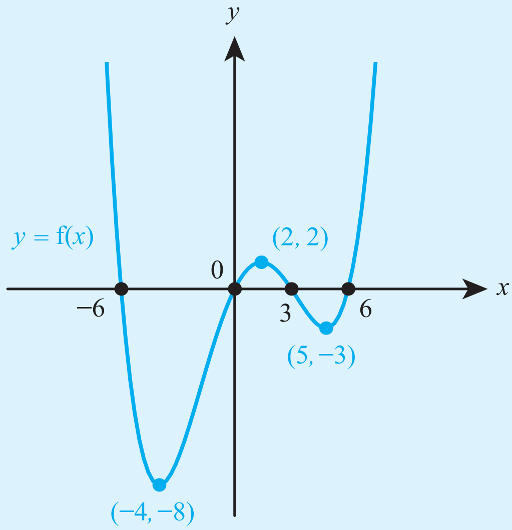


Labelling any axis intercepts, turning points and asymptotes, on separate axes, sketch the graph of:

a $y = \frac{1}{\mathrm{f}(x)}$

b $y = [\mathrm{f}(x)]^2$

18 The graph of $y = \mathrm{f}(x)$ is shown.

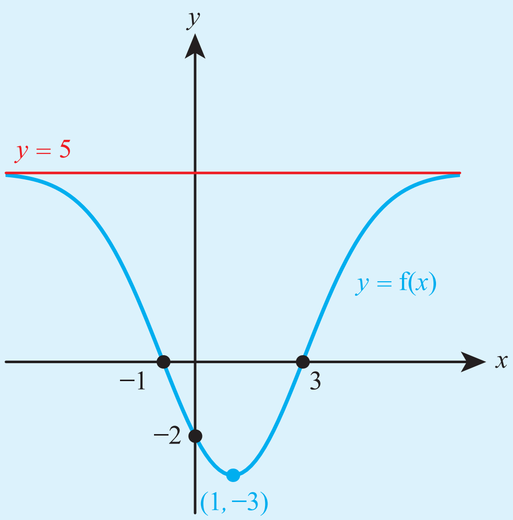


Labelling any axis intercepts, turning points and asymptotes, on separate axes, sketch the graph of:

a $y = \frac{1}{\mathrm{f}(x)}$

b $y = [\mathrm{f}(x)]^2$

19 The graph of $y = \mathrm{f}(x)$ is shown.



Labelling any axis intercepts, turning points and asymptotes, on separate axes, sketch the graph of:

a $y = \frac{1}{\mathrm{f}(x)}$

b $y = [\mathrm{f}(x)]^2$

20 The point P with coordinates $(3, -4)$ lies on the graph $y = \mathrm{f}(x)$. Find the coordinates of the transformed point P' on the graph of:

a $y = \frac{1}{\mathrm{f}(x)}$ **b** $y = [\mathrm{f}(x)]^2$ **c** $y = \mathrm{f}\left(\frac{1}{2}x + 2\right)$

21 State a sequence of two transformations that maps the graph of $y = \cos x$ onto the graph of $y = \cos\left(3x - \frac{\pi}{4}\right)$.

22 State a sequence of two transformations that maps the graph of $y = \ln x$ onto the graph of $y = \ln\left(\frac{2}{5}x + 3\right)$.

23 The graph of $y = 3x^2 + 4x$ is stretched horizontally with scale factor $\frac{1}{2}$ and then translated in the positive x-direction by 1 unit.

Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

24 The following sequence of transformations is applied to the graph of $y = f(x)$:

- Horizontal translation by 4 units in the positive x direction
- Horizontal stretch by scale factor $\frac{1}{2}$
- Vertical translation by 3 units in the positive y direction
- Vertical stretch by scale factor 2

Find the equation of the resulting graph.

25 The following sequence of transformations is applied to the graph of $y = f(x)$:

- Vertical stretch by scale factor $\frac{1}{3}$
- Vertical translation by 4 units in the negative y direction
- Horizontal stretch by scale factor 2
- Horizontal translation by 1 unit in the negative x direction

Find the equation of the resulting graph.

26 Transformation A is reflection in the y-axis.

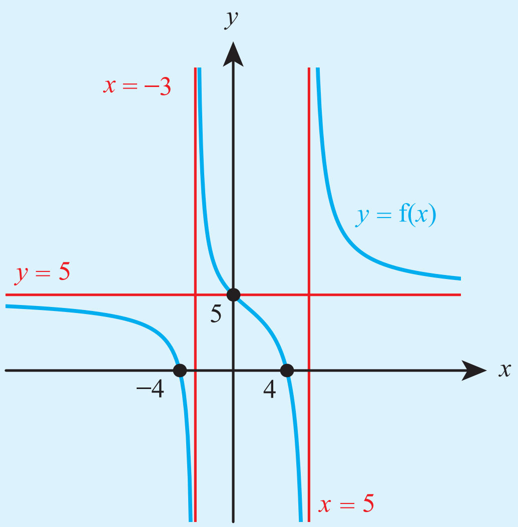
Transformation B is translation right by 5 units.

Find the equation of the resulting graph if $y = f(x)$ is transformed by:

a A and then B

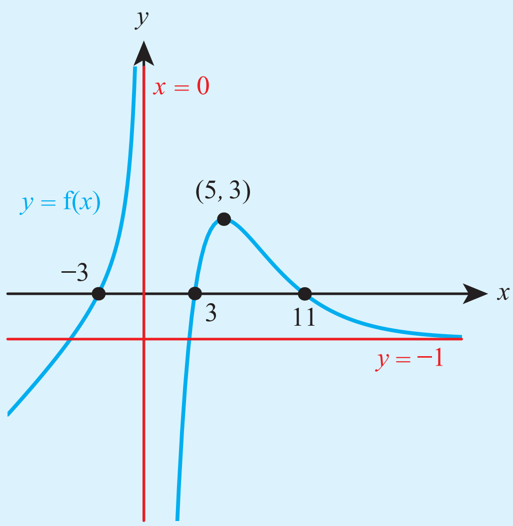
b B and then A

27 The graph of $y = \frac{1}{f(x)}$ is shown.



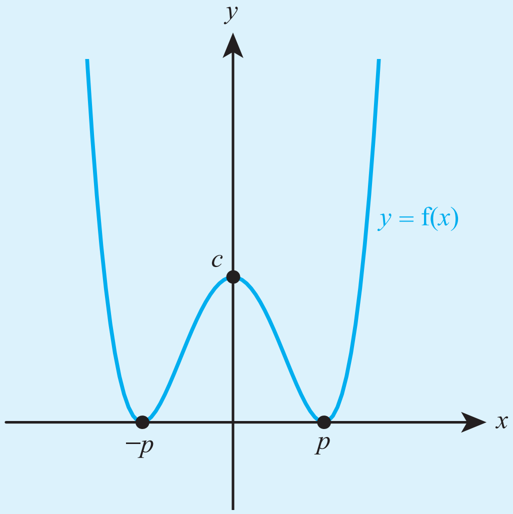
Sketch the graph of $y = f(x)$, labelling all axis intercepts and asymptotes.

28 The graph of $y = f(x)$ is shown.



Sketch the graph of $y = [f(x)]^2$, labelling all axis intercepts, turning points and asymptotes.

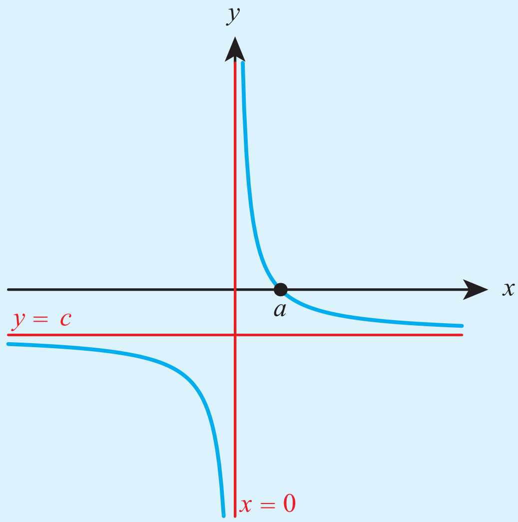
29 The graph of $y = \mathrm{f}(x)$ is shown.



Sketch the graph of $y = \dfrac{1}{\mathrm{f}(ax - b)}$, where $a, b > 0$.

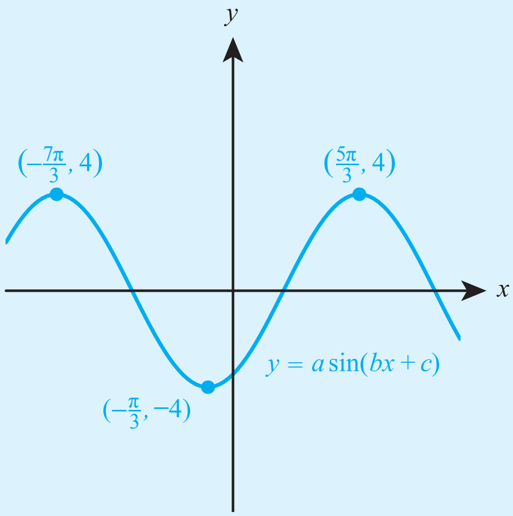
Label any axis intercepts, turning points and asymptotes.

30 The graph of $y = \mathrm{f}(x)$ is shown.



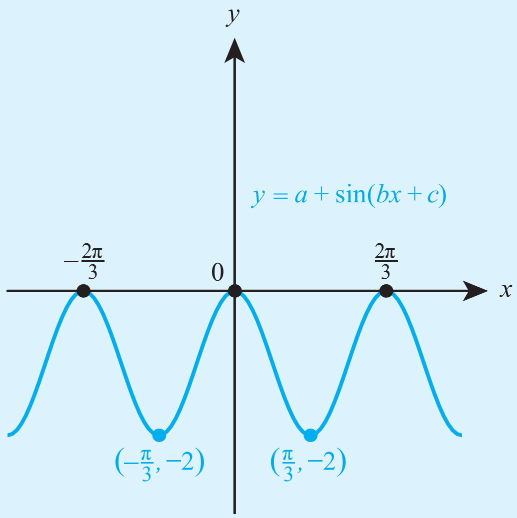
Sketch the graph of $y = \mathrm{f}(ax + b)^2$, where $a, b > 0$. Label any axis intercepts, turning points and asymptotes.

31 The graph of $y = a\sin(bx + c)$ is shown.



Find the values of the constants a, b and c.

32 The graph of $y = a + \sin(bx + c)$ is shown.



Find the values of the constants a, b and c.

33 The graph of $y = \mathrm{f}(x)$ is translated right by 2 and then stretched horizontally with a scale factor $\frac{1}{5}$ to give the graph $y = \mathrm{g}(x)$.

Find a different sequence of two transformations that maps $y = \mathrm{f}(x)$ onto $y = \mathrm{g}(x)$.

34 The graph of $y = ax^2 + bx + c$ is transformed by the following sequence:

- Translation by 2 units in the positive x direction
- Horizontal stretch with scale factor 3

The resulting graph is $y = x^2 + cx + 14$. Find the values of a, b and c.

35 The graph of $y = ax^3 + bx + c$ is transformed by the following sequence:

- Horizontal stretch with scale factor $\frac{1}{2}$
- Translation by 1 unit in the negative x direction

The resulting graph is $y = 2x^3 + 6x^2 - bx - 2$. Find the values of a, b and c.

36 State a sequence of two transformations that map the graph of $y = f(2x + 1)$ onto the graph of $y = f(3x)$.

37 The graph of $y = \tan\left(3x - \frac{\pi}{2}\right)$ is translated by $\frac{\pi}{6}$ in the negative x direction and then reflected in the y-axis. Find the equation of the transformed graph.

38 The graph of $y = 8^x$ is stretched vertically with scale factor 5.
The resulting graph is the same as that found when the graph of $y = 2^x$ is translated right by c units and then stretched horizontally with scale factor $\frac{1}{3}$. Find the value of c.

7E Properties of functions

Odd and even functions

Among the many features of trigonometric functions that you met in Mathematics: analysis and approaches SL Chapter 18 were the following:

- $\cos(-x) = \cos x$
- $\sin(-x) = -\sin x$
- $\tan(-x) = -\tan x$.

We say that sin and tan are **odd functions**, while cos is an **even function**.

Tip

Note that a function can be neither odd nor even if it does not satisfy either of the conditions in Key Point 7.9.

KEY POINT 7.9

A function is
- odd if $f(-x) = -f(x)$ for all x in the domain of f
- even if $f(-x) = f(x)$ for all x in the domain of f.

WORKED EXAMPLE 7.14

Determine algebraically whether $f(x) = x \sin x$ is an odd function, an even function or neither.

Find an expression for $f(-x)$ ...... $f(-x) = (-x)\sin(-x)$

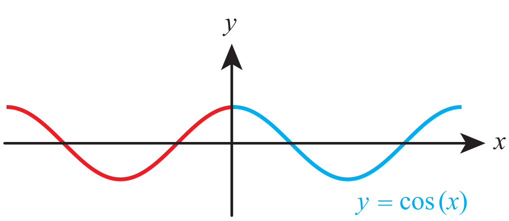
Use $\sin(-x) = -\sin x$ ...... $= (-x)(-\sin x)$

$= x \sin x$

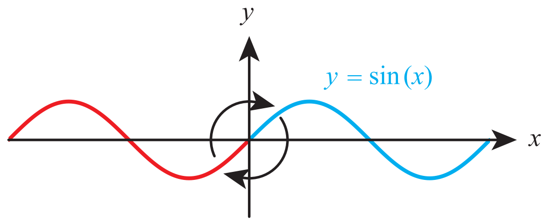
$f(-x) = f(x)$ so the function is even ...... $= f(x)$

So, f is an even function.

When you met the properties above for sin, cos and tan you related them to the graphs of these functions.



The cos graph is symmetric with respect to the y-axis, i.e. its graph remains unchanged after reflection in the y-axis.



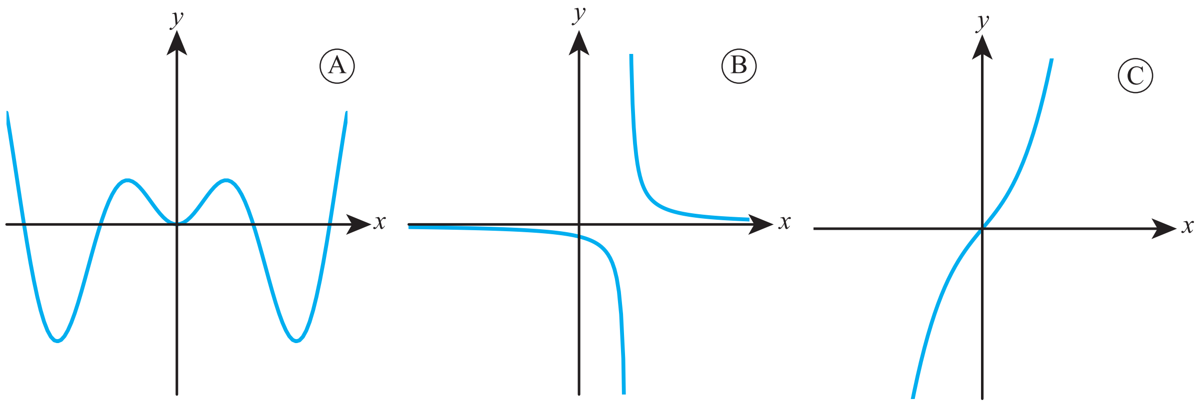
The sin graph is symmetric with respect to the origin, that is, its graph remains unchanged after rotation of 180° about the origin.

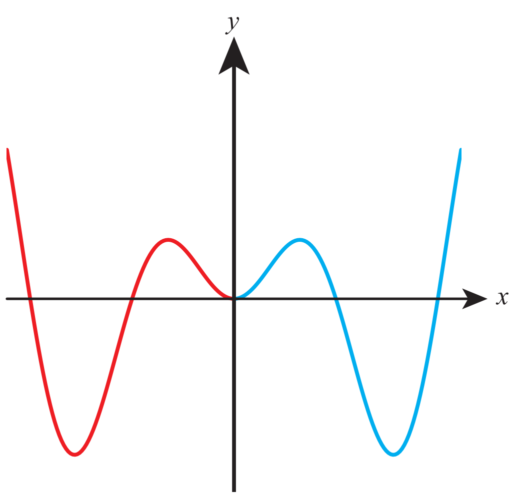
KEY POINT 7.10

The graph of
- an odd function is symmetric with respect to the origin
- an even function is symmetric with respect to the *y*-axis.

WORKED EXAMPLE 7.15

From their graphs, identify whether the functions are odd, even or neither.

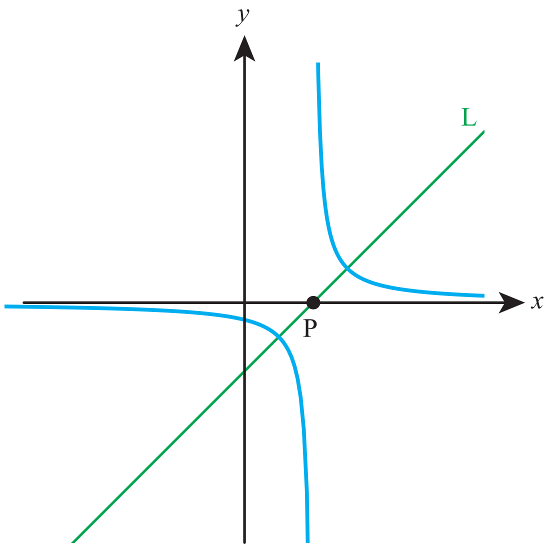


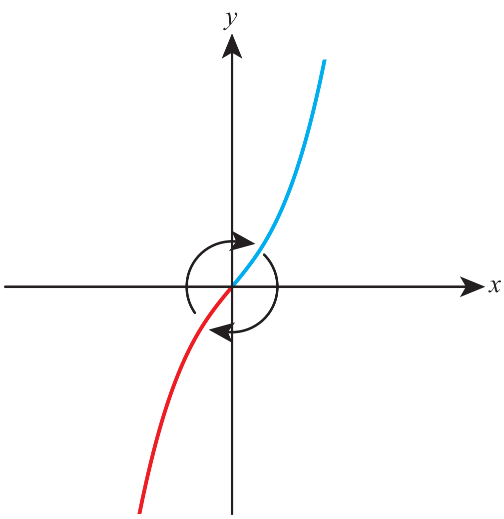


A is symmetric in the *y*-axis so is an even function.

The graph is symmetric in the line ℓ and about the point P but it does not have the symmetry required to be odd or even

B is not symmetric in the origin and not symmetric in the y-axis so it is neither odd nor even.





C is symmetric in the origin so is an odd function.

Tip

Since a function with maximum or minimum points cannot be one-to-one, when restricting the domain you want to start by looking for turning points.

Finding the inverse function $f^{-1}(x)$, including domain restriction

You know from Mathematics: analysis and approaches SL Chapter 14 that for an inverse function, f^{-1}, to exist, the original function, f, must be one-to-one.

We can make a function one-to-one by restricting its domain.

WORKED EXAMPLE 7.16

The function f is defined by $f(x) = x^2 + 6x + 4$, $x \geqslant k$.

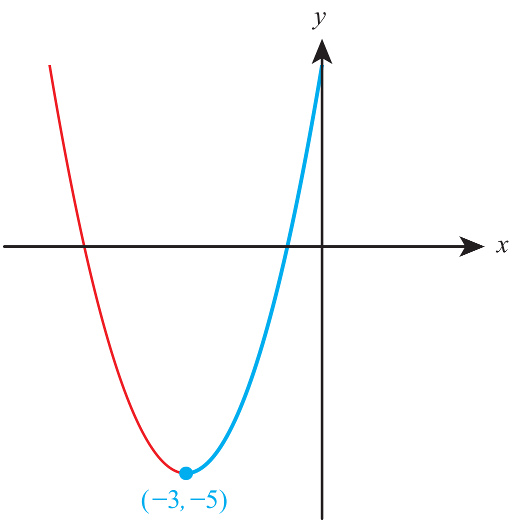
Find the smallest value of k such that the inverse function f^{-1} exists.

For f^{-1} to exist, f must be one-to-one, so find the turning point of f

$$f(x) = (x+3)^2 - 9 + 4$$
$$= (x+3)^2 - 5$$

The turning point of f is (–3, –5) so f is one-to-one for $x \geqslant -3$

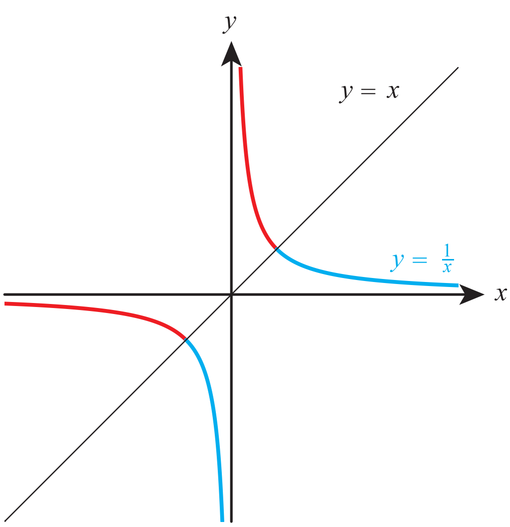
So, the smallest possible value of k is –3.



Self-inverse functions

We commented in Mathematics: analysis and approaches SL Chapter 16 that the function $f(x) = \frac{1}{x}$ is the same as its inverse, $f^{-1}(x) = \frac{1}{x}$. Functions such as this are said to **self-inverse**.

Since the inverse function is a reflection in the line $y = x$ of the original function, this means that self-inverse functions must be symmetric with respect to the line $y = x$.



KEY POINT 7.11

- A function f is said to be self-inverse if $f^{-1}(x) = f(x)$ for all x in the domain of f.
- The graph of a self-inverse function is symmetric in the line $y = x$.

WORKED EXAMPLE 7.17

Show that the function $f(x) = \frac{x}{x-1}$ is self-inverse.

Use the standard procedure for finding f^{-1} — Let $y = (x)$

$$y = \frac{x}{x-1}$$

$$xy - y = x$$

$$xy - x = y$$

$$x(y-1) = y$$

$$x = \frac{y}{y-1}$$

So,

$$f^{-1}(x) = \frac{x}{x-1}$$

Conclude by stating that f and its inverse are the same — $f(x) = f^{-1}(x)$ so f is self-inverse.

Exercise 7E

For questions 1 to 4, use the method demonstrated in Worked Example 7.14 to determine whether the given function is odd, even or neither.

1 a $f(x) = x^3 - 4x + 1$
  b $f(x) = x^4 - 3x^2 + 2$

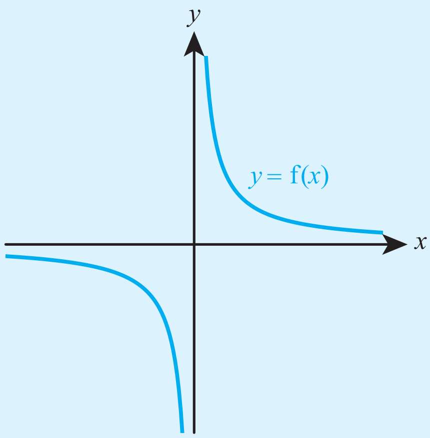
2 a $f(x) = 2x + \cos x$
  b $f(x) = 2x + \tan x$

3 a $f(x) = e^{x^3}$
  b $f(x) = e^{x^2}$

4 a $f(x) = |x| - 3$
  b $f(x) = |x - 3|$

For questions 5 to 8, use the method demonstrated in Worked Example 7.15 to determine from its graph whether the function is odd, even or neither.

5 a



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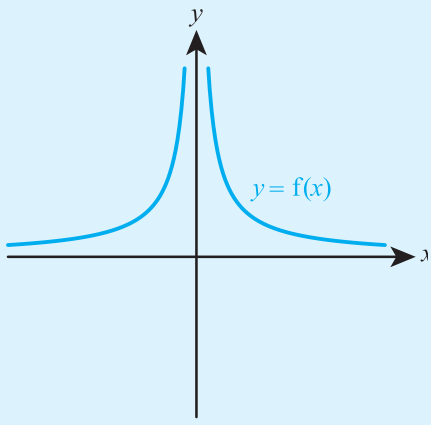
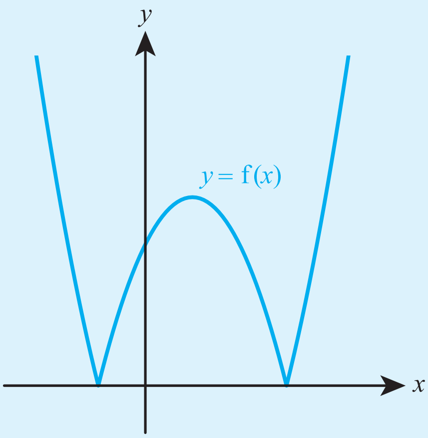


6 a



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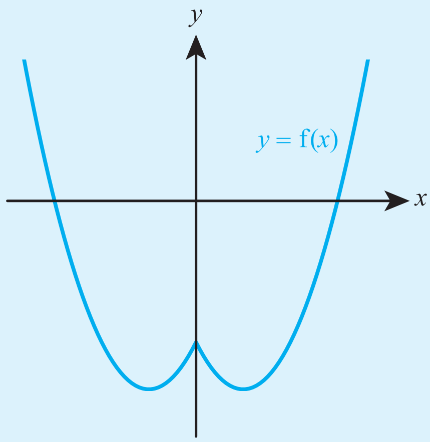
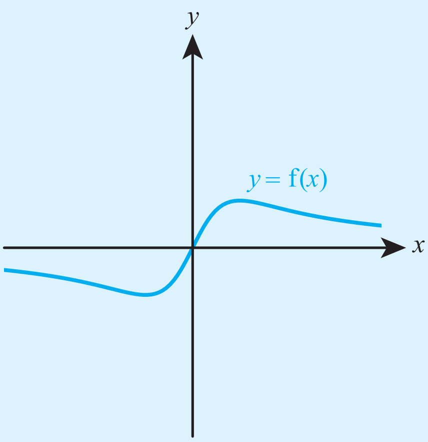
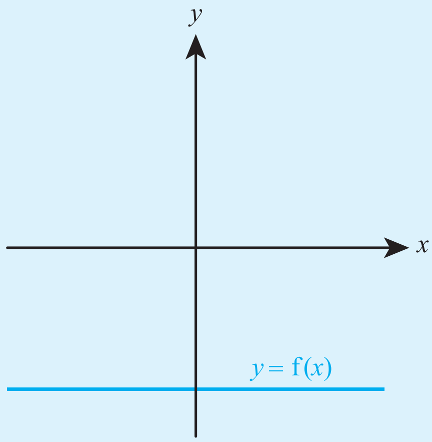


7 a



8 a



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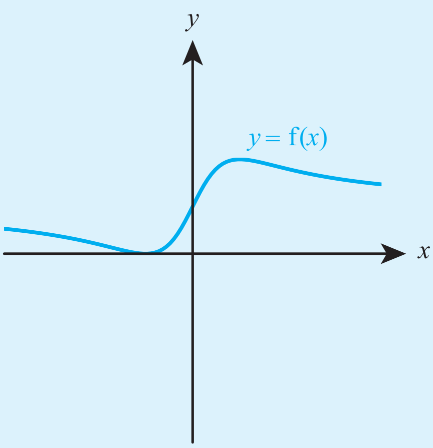

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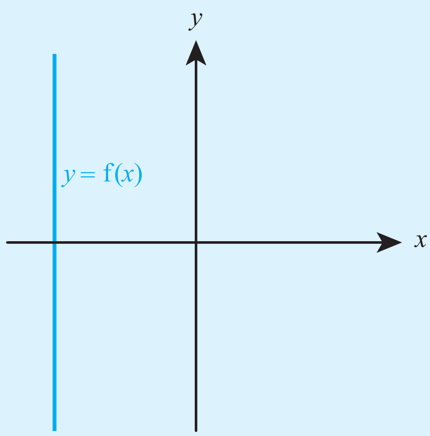
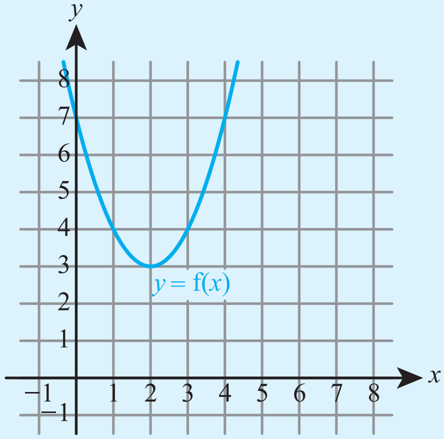


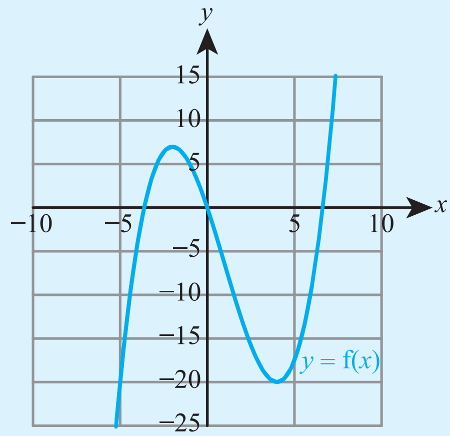
For questions 9 to 11, use the method demonstrated in Worked Example 7.16 to determine from the graph the largest possible domain of the given form for which the inverse function exists.

9



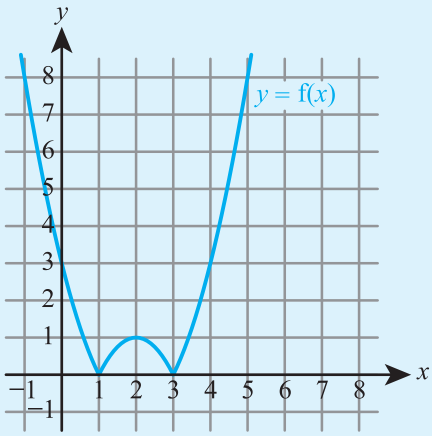
a $x \leqslant k$ **b** $x \geqslant k$

10



a $x \geqslant k$ **b** $x \leqslant k$

11



$|x^2 - 4x + 3|$

a $x \leqslant k$ **b** $x \geqslant k$

For questions 12 to 14, use the method demonstrated in Worked Example 7.15 to determine whether or not the function is self-inverse.

12 **a** $f(x) = \frac{1}{2x}$ **b** $f(x) = -\frac{5}{x}$

13 **a** $f(x) = 3 - 2x$ **b** $f(x) = 4 - x$

14 **a** $f(x) = \frac{x}{x+1}$ **b** $f(x) = \frac{3x+1}{2x-3}$

15 Determine algebraically whether the function $f(x) = \frac{x^3}{x^2 - 6}$ is odd, even or neither.

16 Determine algebraically whether the function $f(x) = \tan x + 3x^2$ is odd, even or neither.

17 Determine algebraically whether the function $f(x) = x \cos x - \sin x$ is odd, even or neither.

18 Let $f(x) = x^2 + 8x + 19, x \geqslant k$.

a Find the smallest value of k such that f^{-1} exists.

b For this value of k, find f^{-1} and its domain.

19 Let $f(x) = x^2 - 3x + 1$.

a Given that f^{-1} exists, find the largest possible domain of f of the form $x \leqslant k$.

b For the domain in part **a**, find f^{-1} and its domain.

20 **a** Show that the function $f(v) = \dfrac{3-2x}{2}$, $b \in \mathbb{R}$ is self-inverse.

b State the domain of f^{-1}.

21 The function f is even and the function g is odd.

a Prove that the function h defined by $h(x) = \dfrac{f(x)}{g(x)}$ is odd.

b Determine, with proof, whether the sum of an even and an odd function is odd, even or neither.

22 Prove that the only function that is both odd and even is $f(x) = 0$.

23 Determine whether the function $f(x) = |x-1| + |x+1|$ is even, odd or neither, fully justifying your answer.

24 The function f is defined by $f(x) = x^3 + 6x^2 + 9x - 2$, $-5 \leqslant x \leqslant 1$.

a Find the largest possible domain of the form $a \leqslant x \leqslant b$ for which f has an inverse function.

b For the domain in part **a**, find the domain of f^{-1}.

25 The function f is defined by $f(x) = x^4 - 8x^2 + 5$, $x \geqslant k$

a Find the smallest value of k such that f has an inverse function.

b For this value of k, find the domain of f^{-1}.

26 The function f is defined by $f(x) = e^x - 4x$, $x \leqslant k$.

a Find the largest value of k such that f^{-1} exists.

b For this value of k, find the domain of f^{-1}.

27 **a** Show that the function $f(x) = \dfrac{2x+1}{3x-2}$, $x \neq 2$ is self-inverse.

b Find the domain of f^{-1}.

28 Determine, with proof, the condition on $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ is self-inverse.

29 **a** For any function f, show that $f(x) + f(-x)$ is an even function.

b For any function f, show that $f(x) - f(-x)$ is an odd function.

c Hence show that any function can be expressed as the sum of an even and an odd function.

30 Find the value of c for which the function $f(x) = \dfrac{3-2x}{x+c}$ is self-inverse.

Checklist

- You should be able to sketch the graphs of functions of the form $f(x) = \dfrac{ax+b}{cx^2+dx+e}$ and $f(x) = \dfrac{ax^2+bx+c}{dx+e}$.
  - If $y = \dfrac{ax+b}{cx^2+dx+e}$, then
    - — the y-intercept is $\left(\dfrac{b}{e}, 0\right)$
    - — the x-intercept is $\left(0, -\dfrac{b}{a}\right)$
    - — the horizontal asymptote is at $y = 0$
    - — any vertical asymptotes occur at solutions of $cx^2 + dx + e = 0$.
  - If $y = \dfrac{ax^2+bx+c}{dx+e}$, then
    - — the y-intercept is $\left(0, \dfrac{c}{e}\right)$
    - — any x-intercepts occur at solutions of $ax^2 + dx + e = 0$
    - — the vertical asymptote is at $x = -\dfrac{e}{d}$
    - — there will be an oblique asymptote of the form $y = px + q$.
- You should be able to solve cubic inequalities.
- You should be able to solve other inequalities graphically using your GDC.

- You should be able to sketch graphs of the functions $y = |f(x)|$ and $y = f(|x|)$.
  - $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
  - To sketch the graph of $y = |f(x)|$, start with the graph of $y = f(x)$ and reflect in the x-axis any parts that are below the x-axis.
  - To sketch the graph of $y = f(|x|)$, start with the graph of $y = f(x)$ for $x \geq 0$ and reflect that in the y-axis.
- You should be able to solve modulus equations and inequalities.
- You should be able to sketch graphs of the form $y = \frac{1}{f(x)}$.

  To sketch the graph of $y = \frac{1}{f(x)}$ consider the following key features:

Feature of $y = f(x)$	**Feature of $y = \frac{1}{f(x)}$**
x-intercept at $(a, 0)$	$x = a$ is a vertical asymptote
y-intercept at $(0, b)$, $b \neq 0$	y-intercept at $\left(0, \frac{1}{b}\right)$
$x = a$ is a vertical asymptote	x-intercept at $(a, 0)$
$y = a$ is a horizontal asymptote, $a \neq 0$	$y = \frac{1}{a}$ is a horizontal asymptote
$y = 0$ is a horizontal asymptote	$y \to \infty$
$y \to \pm\infty$	$y = 0$ is a horizontal asymptote
(a, b) is a turning point, $b \neq 0$	$\left(a, \frac{1}{b}\right)$ is the opposite turning point

- You should be able to sketch graphs of the form $y = f(ax + b)$.
  When two horizontal transformations are applied, the order matters: $y = f(ax + b)$ is a horizontal translation by $-b$ followed by a horizontal stretch with scale factor $\frac{1}{a}$.
- You should be able to sketch graphs of the form $y = [f(x)]^2$.
  To sketch the graph of $y = [f(x)]^2$ consider the following key features:

Feature of $y = f(x)$	**Feature of $y = [f(x)]^2$**
$y < 0$	$y > 0$
x-intercept at $(a, 0)$	Local minimum at $(a, 0)$
y-intercept at $(0, b)$	y-intercept at $(0, b^2)$
$x = a$ is a vertical asymptote	$x = a$ is a vertical asymptote
$y = a$ is a horizontal asymptote	$y = a^2$ is a horizontal asymptote
$y \to \pm\infty$	$y \to \infty$

- You should be able to determine whether a function is odd, even or neither.
  - A function is
    - odd if $f(-x) = -f(x)$ for all x in the domain of f
    - even if $f(-x) = f(x)$ for all x in the domain of f.
  - The graph of
    - an odd function is symmetric with respect to the origin
    - an even function is symmetric with respect to the y-axis.
- You should be able to restrict the domain of a many-to-one function so that the inverse function exists.
- You should be able to determine whether a function is self-inverse.
  - A function is self-inverse if $f^{-1}(x) = f(x)$ for all x in the domain of f.
  - The graph of a self-inverse function is symmetric in the line $y = x$.

Mixed Practice

1 Let $f(x) = \dfrac{2x+1}{(3x-2)(x+2)}$.

a State the equation of the vertical asymptotes.
b Find the coordinates of the axis intercepts.
c Sketch the graph of $y = f(x)$.

2 Let $f(x) = x - 2 - \dfrac{8}{x-4}$.

a State the equation of
  i the vertical asymptote
  ii the oblique asymptote.
b Find the coordinates of the axis intercepts.
c Sketch the graph of $y = f(x)$.

3 Find the set of values of x for which $6x + x^2 - 2x^3 < 0$.

4 **a** Show that $(x + 2)$ is a factor of $x^3 - 3x^2 - 6x + 8$.
b Hence solve the inequality $x^3 - 1 \geqslant 3(x^2 + 2x - 3)$.

5 Solve the inequality $2x^4 - 5x^2 + x + 1 < 0$.

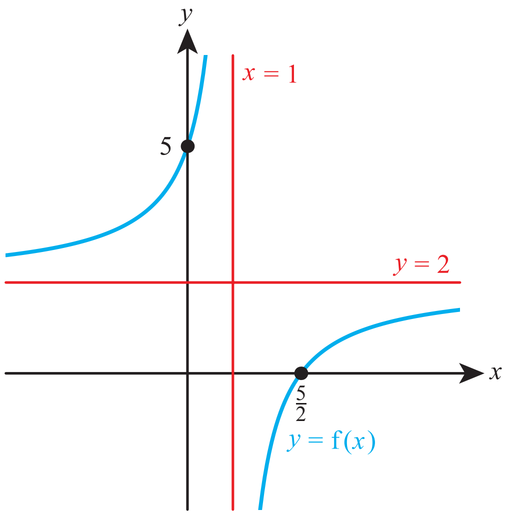
6 Solve the inequality $\ln x \leqslant e^{\sin x}$ for $0 < x \leqslant 10$.

7 **a** Sketch the graph of $y = |\cos 3x|$ for $0 \leqslant x \leqslant \pi$.
b Solve $|\cos 3x| = \dfrac{1}{2}$ for $0 \leqslant x \leqslant \pi$.

8 **a** On the same axes, sketch the graphs of $y = |4 + x|$ and $y = |5 - 3x|$, labelling any axis intercepts.
b Hence solve the inequality $|4 + x| \leqslant |5 - 3x|$.

9 **a** On the same axes, sketch the graphs of $y = |5x + 1|$ and $y = 3 - x$, labelling any axis intercepts.
b Hence solve the inequality $3 - x > |5x + 1|$.

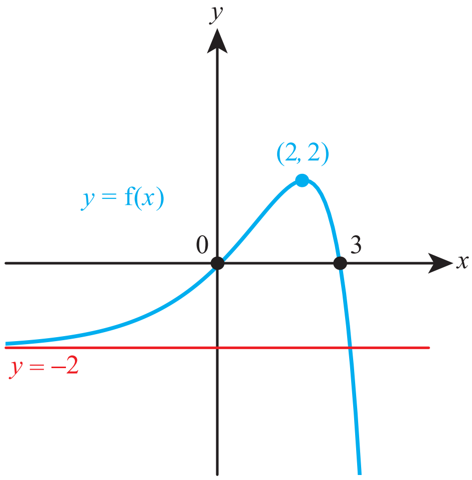
10 The graph of $y = f(x)$ is shown below.



Labelling any axis intercepts and asymptotes, on separate axes sketch the graph of

a $y = |f(x)|$
b $y = f(|x|)$.

11 The graph of $y = \mathrm{f}(x)$ is shown below.



Labelling any x-axis intercepts, turning points and asymptotes, on separate axes sketch the graph of

a $y = \dfrac{1}{\mathrm{f}(x)}$

b $y = [\mathrm{f}(x)]^2$

c $y = \mathrm{f}(2x - 1)$.

12 The function f is defined by $\mathrm{f}(x) = 3^x + 3^{-x}$. Deterine algebraically whether f is even, odd or neither.

13 The function f is defined by $\mathrm{f}(x) = -x^2 + 6x - 4,\ x \geqslant k$.

a Find the smallest value of k such that f has an inverse function.

b For this value of k, find $\mathrm{f}^{-1}(x)$ and state its domain.

14 The functions f and g are defined by $\mathrm{f}(x) = ax^2 + bx + c,\ x \in \mathbb{R}$ and $\mathrm{g}(x) = p \sin x + qx + r,\ x \in \mathbb{R}$ where a, b, c, p, q, r are real constants.

a Given that f is an even function, show that $b = 0$.

b Given that g is an odd function, find the value of r.

The functions h is both odd and even, with domain $\mathbb{R}$.

c Find $\mathrm{h}(x)$.

Mathematics HL May 2015 Paper 1 TZ1 Q5

15 **a** **i** Find the set of values of k for which the equation $kx^2 - 2(k + 1)x + 7 - 3k = 0$ has real roots.

ii Hence determine the range of the function $\mathrm{f}(x) = \dfrac{2x - 7}{x^2 - 2x - 3}$.

b Sketch the graph of $y = \mathrm{f}(x)$ labelling any vertical asymptotes.

16 **a** Sketch the graph of $y = |2|x| - 3|$. State the coordinates of any axis intercepts.

b Solve the equation $|2|x| - 3| = 2$.

17 The function f is defined by $\mathrm{f}(x) = (x - a)(x - b)$. On separate axes, sketch the graph of $y = \mathrm{f}(|x|)$ in the case where

a $0 < b < a$ **b** $b < 0 < a$ **c** $b < a < 0$.

18 **a** Describe a sequence of two transformations that map the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \mathrm{f}\left(\dfrac{x - 6}{3}\right)$.

b Describe a different sequence of two transformations that has the same effect as in part **a**.

19 Let $f(x) = x^2 - 3$.

a On the same axes, sketch the graphs of $y = |f(x)|$ and $y = \frac{1}{f(x)}$.

b Hence solve the inequality $|f(x)| \leqslant \frac{1}{f(x)}$.

20 Given $f(x) = |x + a| + |x + b|$, where $a, b \neq 0$, find the condition on a and b such that f is an even function.

21 The function f is defined by $f(x) = e^{2x} - 8e^x + 7$, $x \leqslant k$.

a Find the largest value of k such that f has an inverse function.

b For this value of k, find $f^{-1}(x)$ and state its domain.

22 The function f is defined by $f(x) = xe^{\frac{x}{2}}$, $x \geqslant k$.

a Find $f'(x)$ and $f''(x)$.

b Find the smallest value of k such that f has an inverse function.

c For this value of k, find the domain of f^{-1}.

23 Let $f(x) = \frac{3x}{x^2 + 1}$.

a **i** Show algebraically that f is an odd function.

ii What type of symmetry does this mean the graph of $y = f(x)$ must have?

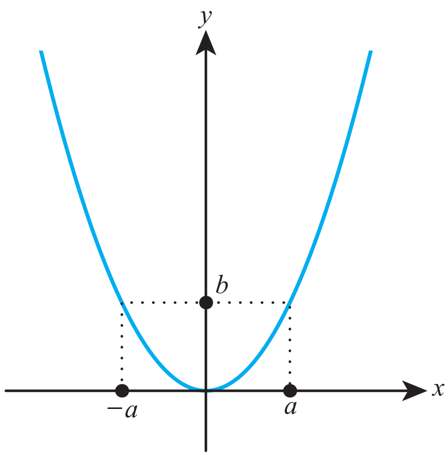
b **i** If the line $y = k$ intersects the curve, show that $4k^2 - 9 \leqslant 0$.

ii Hence find the coordinates of the turning points of the curve.

c Sketch the graph of $y = |f(x)|$.

d Solve the inequality $|f(x)| \geqslant |x|$.

24 The diagram below shows the graph of the function $y = f(x)$, defined for all $x \in \mathbb{R}$, where $b > a > 0$.



Consider the function $g(x) = \frac{1}{f(x - a) - b}$.

a Find the largest possible domain of the function g.

b Sketch the graph of $y = g(x)$. Indicate any asymptotes and local maxima or minima, and write down their equations and coordinates.

Mathematics HL May 2011 Paper 1 TZ1 Q10

25 The function f is defined by $f(x) = \dfrac{2x-1}{x+2}$, with domain $D = \{x : -1 \leqslant x \leqslant 8\}$.

a Express f(x) in the form $A + \dfrac{B}{x+2}$, where A and $B \in \mathbb{Z}$.

b Hence show that $f'(x) > 0$ on D.

c State the range of f.

d **i** Find an expression for $f^{-1}(x)$.

ii Sketch the graph of f(x), showing the points of intersection with both axes.

iii On the same diagram, sketch the graph of $y = f^{-1}(x)$.

e **i** On a different diagram, sketch the graph of $y = f(|x|)$ where $x \in D$.

ii Find all the solutions of the equation $f(|x|) = -\dfrac{1}{4}$.

Mathematics HL May 2013 Paper 1 TZ2 Q12

26 Let $f(x) = \dfrac{x^2+7x+10}{x+1}$.

a Find the equation of the oblique asymptote.

b By finding a condition on k such that $f(x) = k$ has real solutions, or otherwise, find the coordiantes of the turning points of f.

c On the same axes, sketch the grap of $y = f(x)$ and $y = 2x + 7$.

d Hence solve the inequality $\dfrac{x^2+7x+10}{x+1} < 2x+7$.

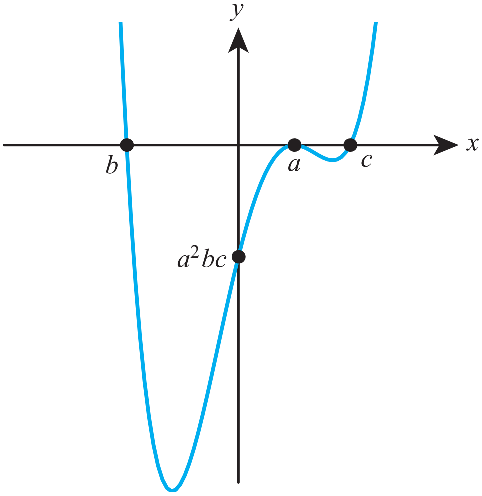
e On a separate set of axes, sketch the graph of $y = |f(x)|$, labelling the coordinates of all axis intercepts.

f State the complete set of values of c for which $|f(x)| = c$ has two solutions.

27 The function f is defined by $f(x) = \dfrac{ax^2+bx+c}{dx+e}$ and the function g is defined by $g(x) = \dfrac{1}{f(x)}$. f(x) has an oblique asymptote $y = x + 1$ and g(x) has vertical asymptotes $x = \dfrac{3}{2}$ and $x = -4$. Solve the equation $f(x) = g(x)$.

28 Find the value of c for which the function $f(x) = \dfrac{3x-5}{x+c}$ is self-inverse.

19



20 $-3,\ 2 \pm \mathrm{i}$

21 $a = 3,\ b = -42$

22 $\dfrac{3 \pm \mathrm{i}\sqrt{3}}{3}$

24 $(a,\ b) = \pm\left(\dfrac{5}{3},\ -\dfrac{4}{3}\right),\ \pm\left(-\dfrac{1}{3},\ \dfrac{8}{3}\right)$

25 $p - 2,\ q = 45$

26 a $-\dfrac{26}{9}$

b $9x^2 + 26x + 49 = 0$

27 a 2

b 22

28 a $-\dfrac{2}{5}, \dfrac{3}{5}$

b $\dfrac{1}{135}$

29 a -2

30 50

31 a $2 + \mathrm{i},\ 2 - \mathrm{i}$

b $a = -12,\ b = 15$

32 $a = -10,\ b = -18$

33 b $b = -8,\ \mathrm{e} = 12$

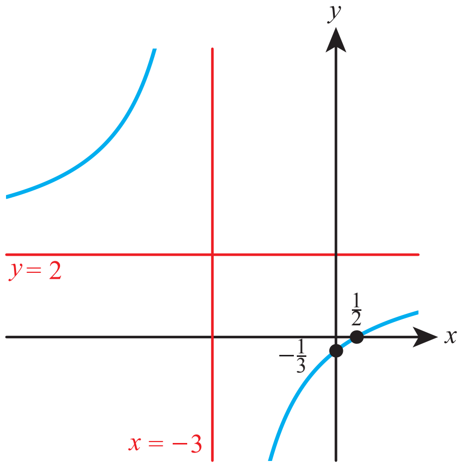
34 $c = 19,\ d = -6$

35 b ii -20

c -27

Chapter 7 Prior Knowledge

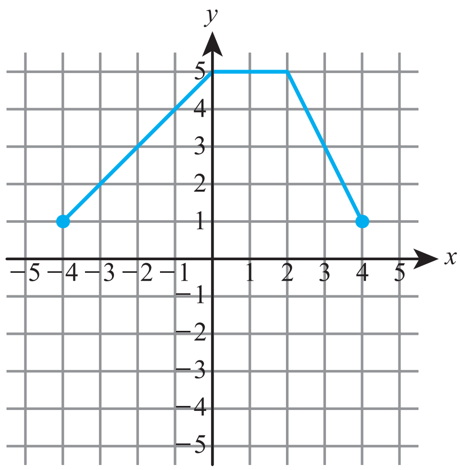
1



2 $-4 < x < 2$

3 1.40

4 a



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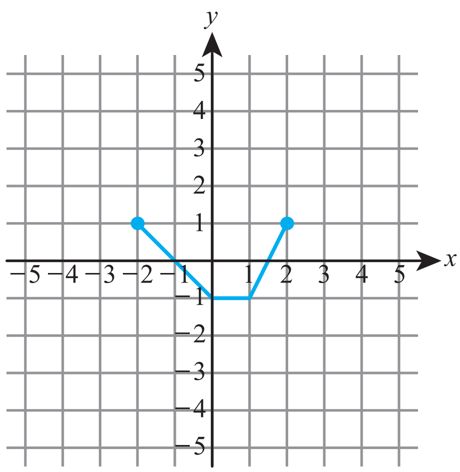
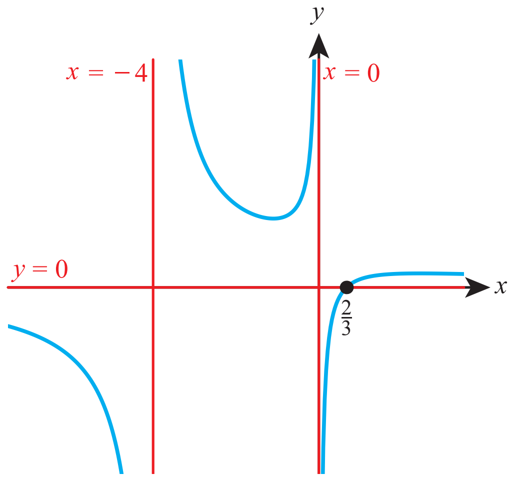


5 $\dfrac{3x + 1}{2 - x},\ x \neq 2$

Exercise 7A

1 a



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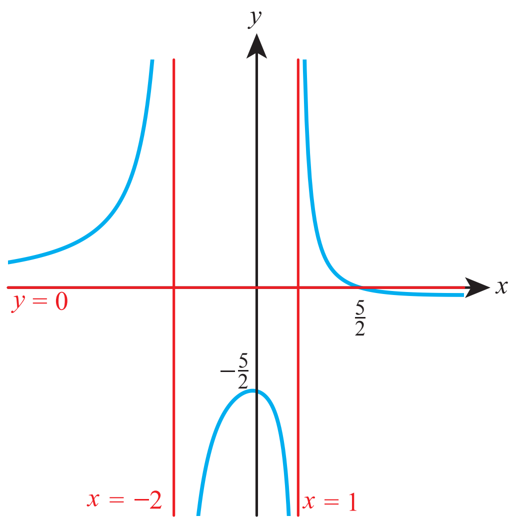
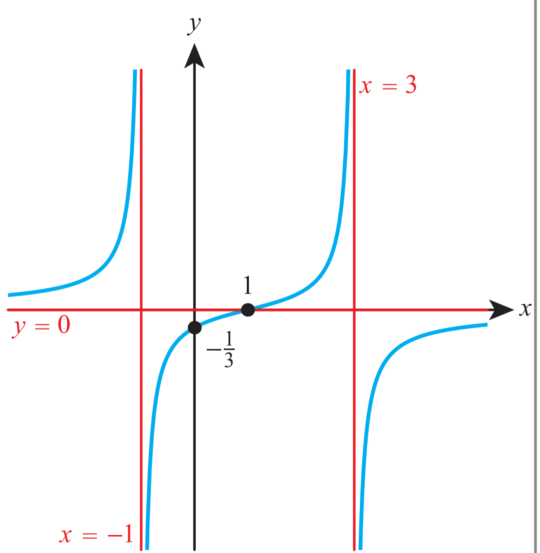
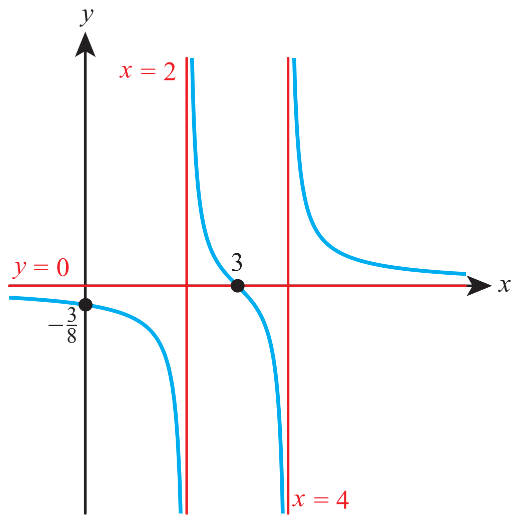


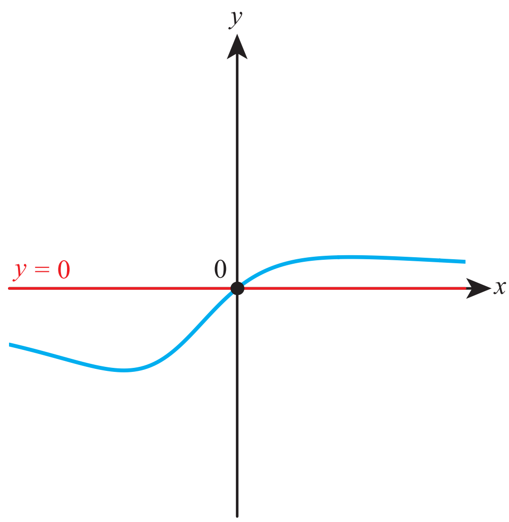
2 a



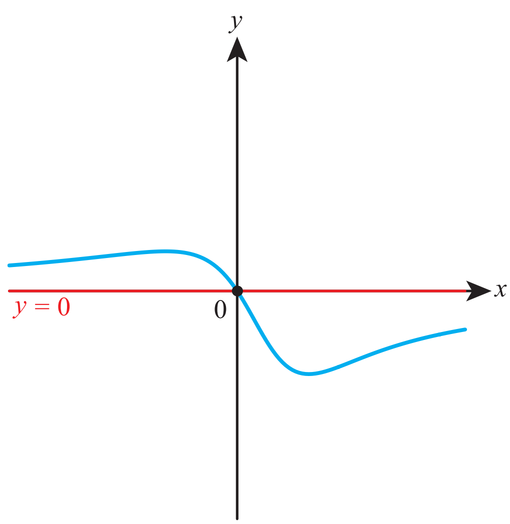
b



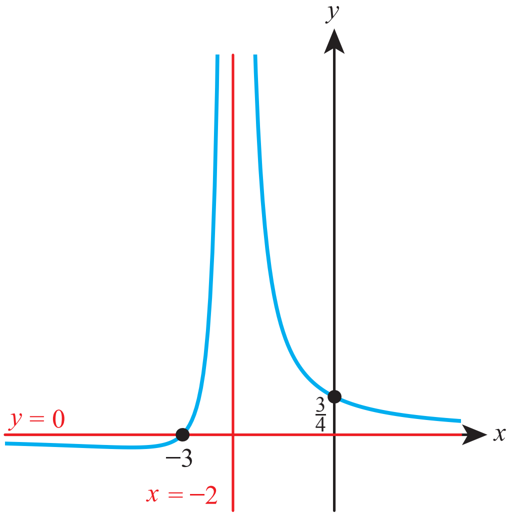
3 a



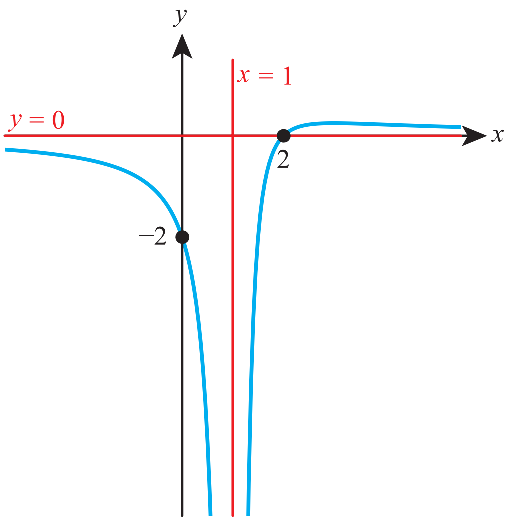
b



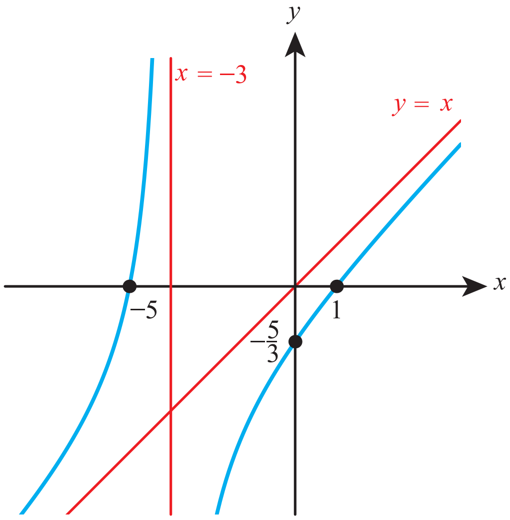
4 a


y
x
y = 0
−3
x = −2
3/4

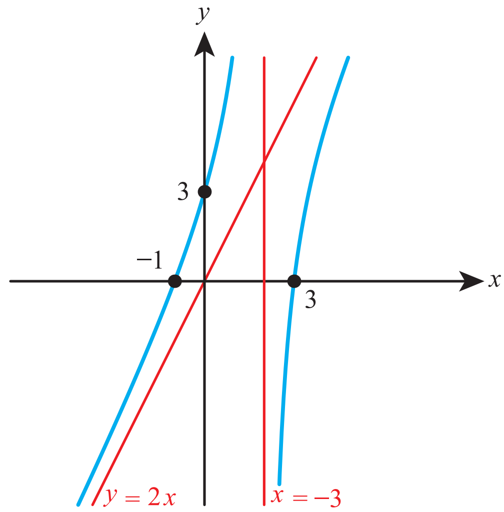
b


y
x
x = 1
y = 0
2
−2

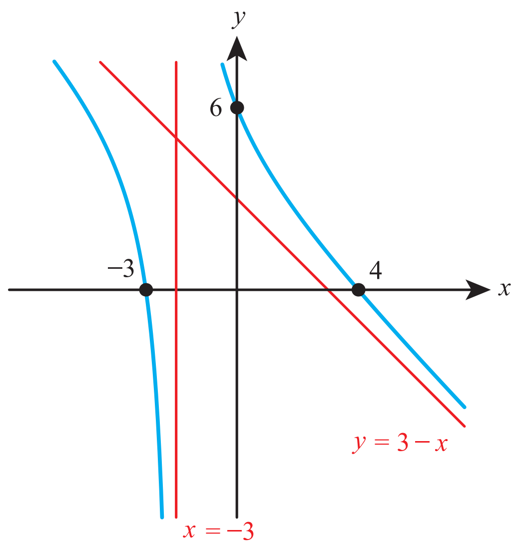
5 a


y
x
x = −3
y = x
−5
1
−5/3

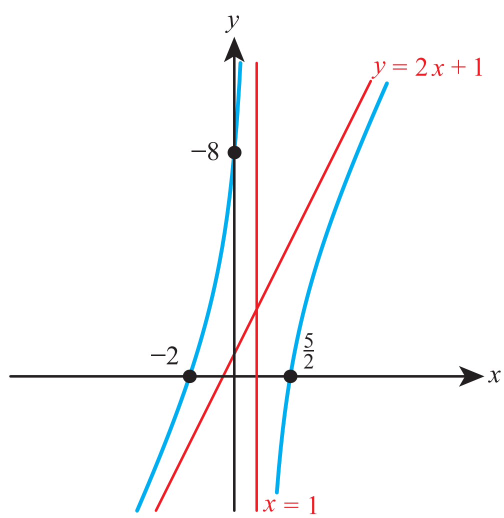
b


y
x
3
−1
3
y = 2x
x = −3

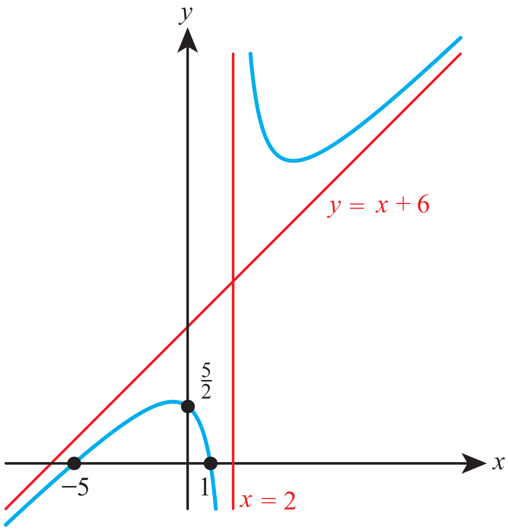
6 a


y
x
6
−3
4
y = 3 − x
x = −3

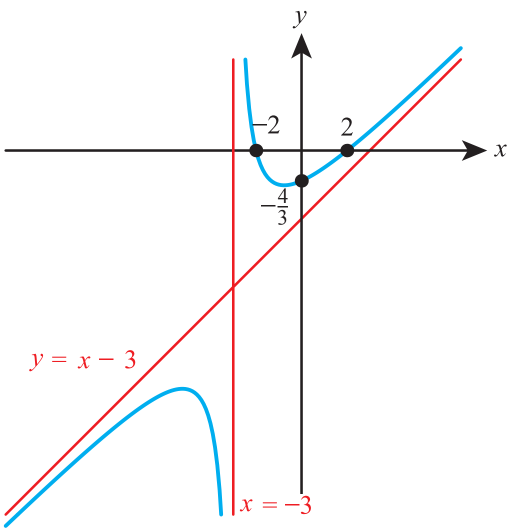
b


y
x
y = 2x + 1
−8
−2
5/2
x = 1

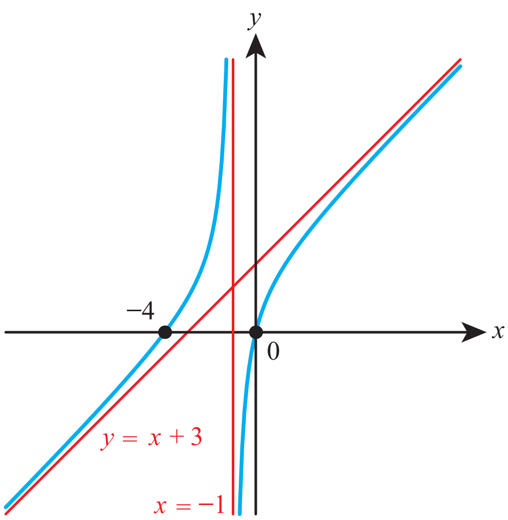
7 a



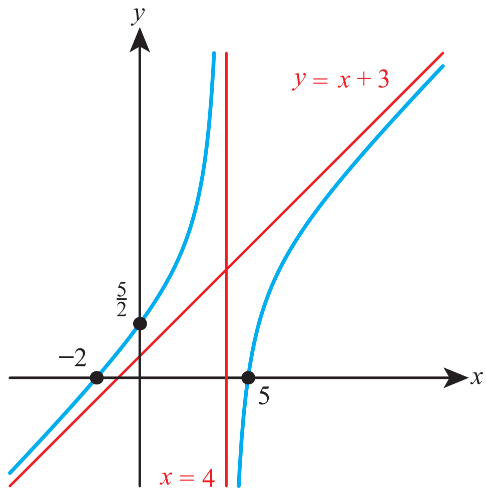
b



8 a

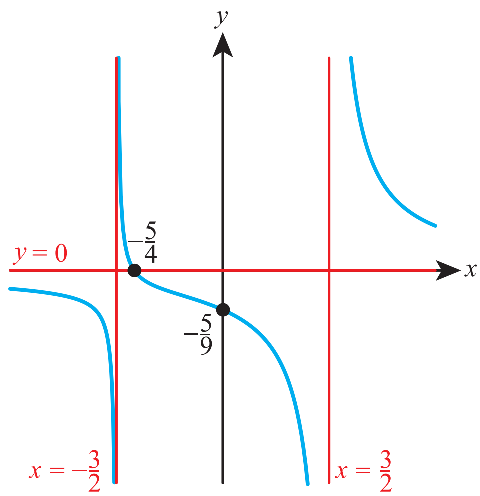


b



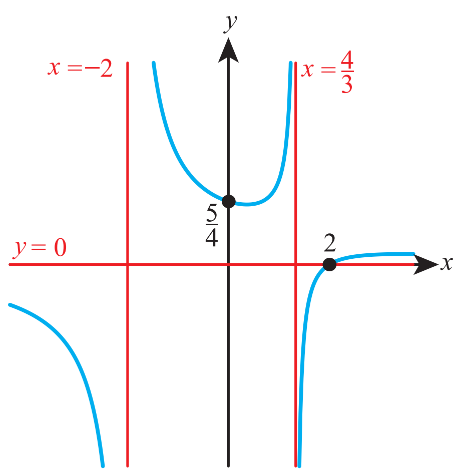
9 a $x = \pm\frac{3}{2}$

b



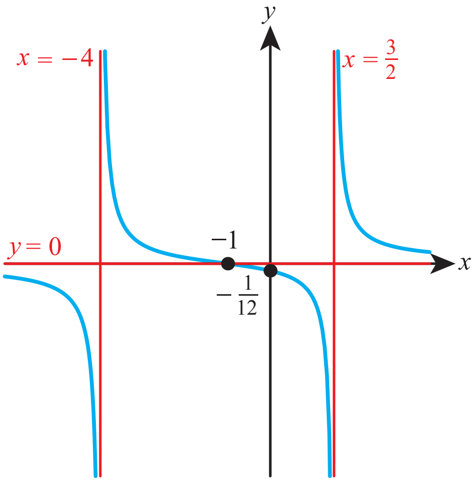
10 a $x = \frac{4}{3}, x = -2$

b

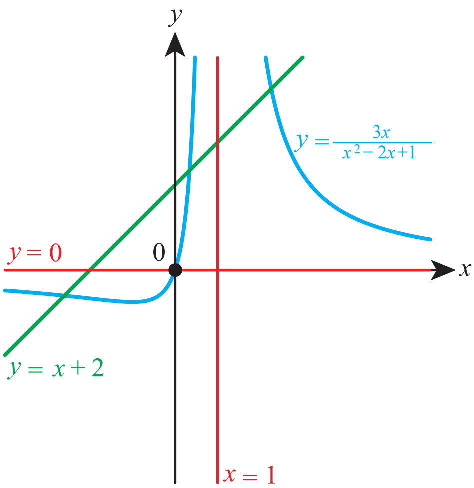


11 a $k = 5$ b $x = \frac{3}{2}$

c

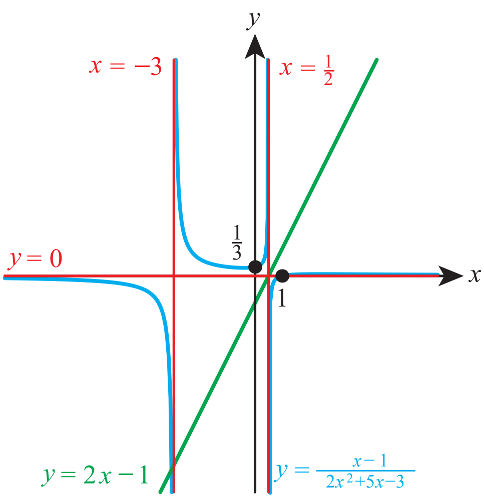


12 a



b Three

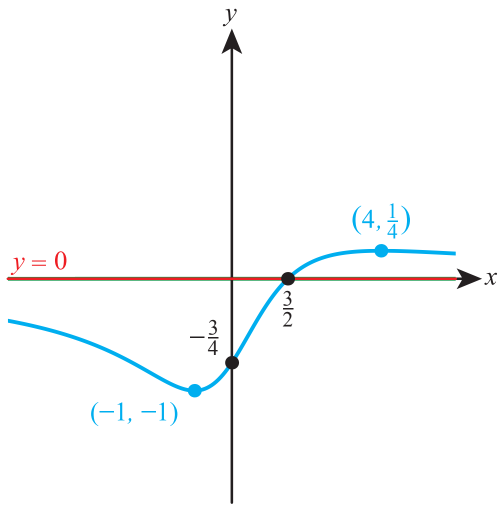
13 a



b One

14 a ii $(-1, -1), \left(4, \frac{1}{4}\right)$

b

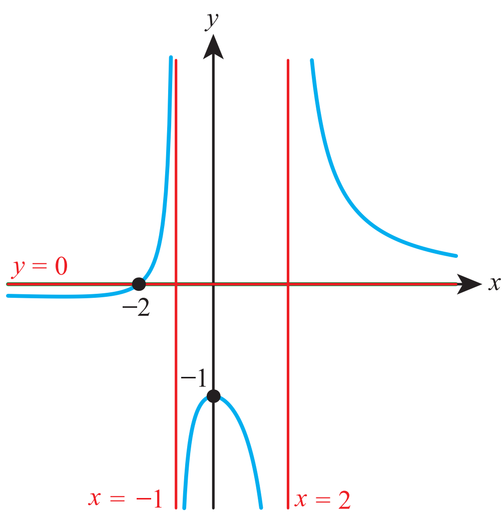


15 a i $k \geqslant -\frac{1}{9}$ or $k \leqslant -1$

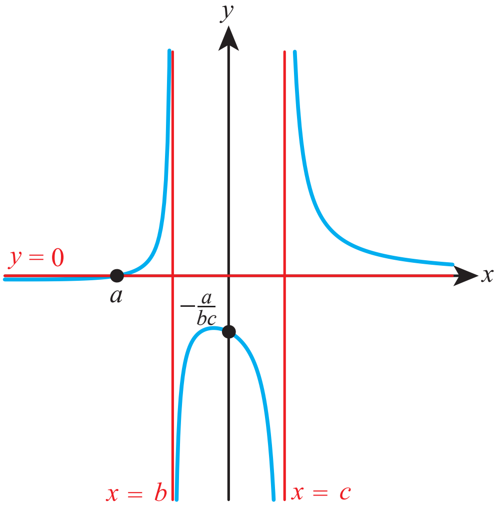
ii $f(x) \geqslant -\frac{1}{9}$ or $f(x) \leqslant -1$

b $x = 2$, $x = -1$, $(-2, 0)$, $(0, -1)$

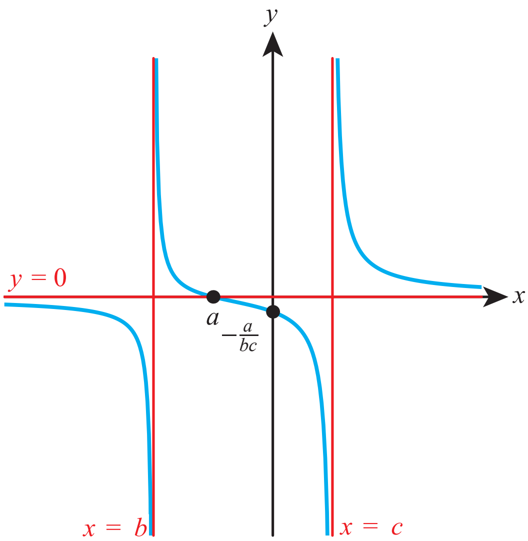
c



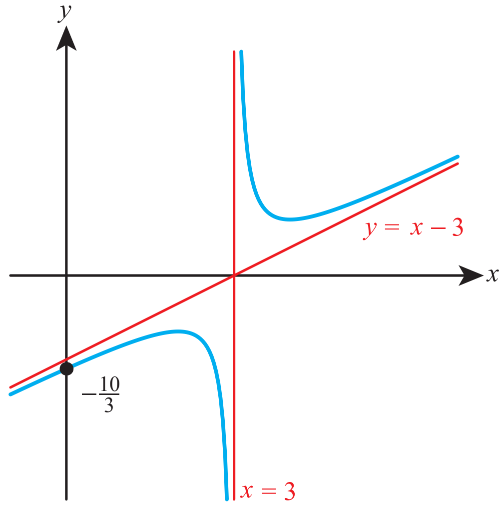
16 a



b



d



17 a $A = 1,\ B = -3$

b $(4, 2), (2, -2)$

c $\left(0, -\frac{10}{3}\right),\ x = 3$

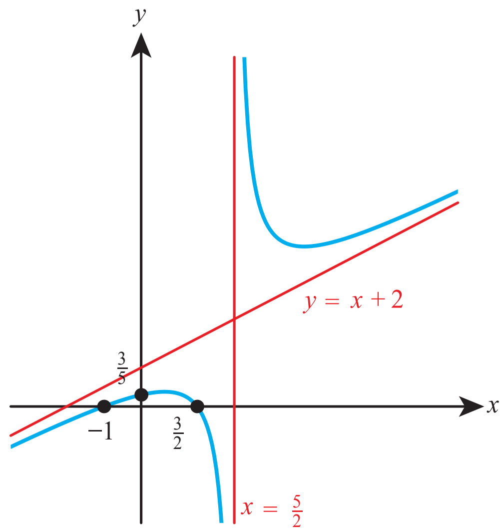
18 a $A = 1,\ B = 2,\ C = 7$

b $y = x + 2$

c $\mathrm{f}(x) \geqslant \frac{9 + 2\sqrt{14}}{2}$ or $\mathrm{f}(x) \leqslant \frac{9 - 2\sqrt{14}}{2}$

d $\left(0, \frac{3}{5}\right), \left(\frac{3}{2}, 0\right), (-1, 0),\ x = \frac{5}{2}$

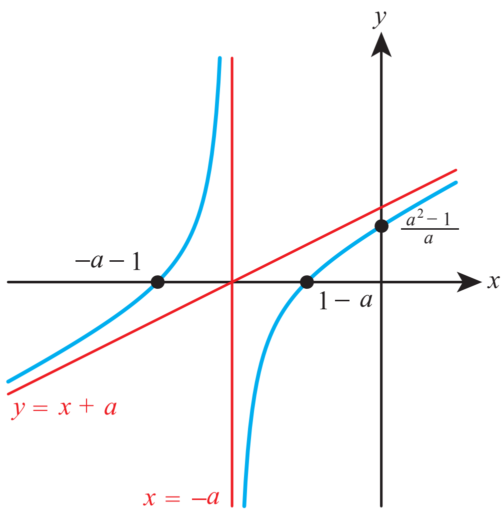
e



19 $-2 < c < 0$

20 a $y = x + a$

c



Exercise 7B

1 a $0 < x < 1$ or $x > 2$

b $-1 < x < 0$ or $x > 5$

2 a $x < 0$ or $2 < x < 4$

b $x < 0$ or $3 < x < 6$

3 a $x \leqslant -3$ or $2 \leqslant x \leqslant 5$

b $x \leqslant 1$ or $3 \leqslant x \leqslant 4$

4 a $x \leqslant -2$ or $3 \leqslant x \leqslant 4$

b $x \leqslant 1$ or $2 \leqslant x \leqslant 8$

5 a $-1 < x < 2$ or $x > 2$

b $x > 4$

6 a $x \leqslant 0.820$ b $x \leqslant -1$

7 a $x \leqslant 1.82$ b $x \leqslant 1.32$

8 a $0.728 < x < 11.9$ b $1.06 < x < 2.79$

9 a $-2.50 < x < -1.22$ or $0.220 < x < 1.50$

b $-3.98 < x < -2.06$

10 a $-0.901 \leqslant x \leqslant -0.468$ or $x \geqslant 0.081$

b $x \leqslant 1.22$ or $2.10 \leqslant x \leqslant 2.91$

11 $-2 < x < 0$ or $x > \frac{3}{2}$

12 b $x \leqslant -\frac{3}{2}$ or $-1 \leqslant x \leqslant 2$

13 b $x > \frac{1}{2}$

14 $a < x < b$ or $x > c$

15 $x < a$

16 $x \in [-2.27, -0.251]$

17 $-0.933 \leqslant x \leqslant -0.377$ or $0.371 \leqslant x \leqslant 1.76$ or $x \geqslant 2.18$

18 $x \in (-0.727, 1.48) \cup (13.7, \infty)$

19 $b = -5,\ c = 7,\ d = -1$

20 $a = -2,\ b = -7,\ c = 7,\ d = 15$

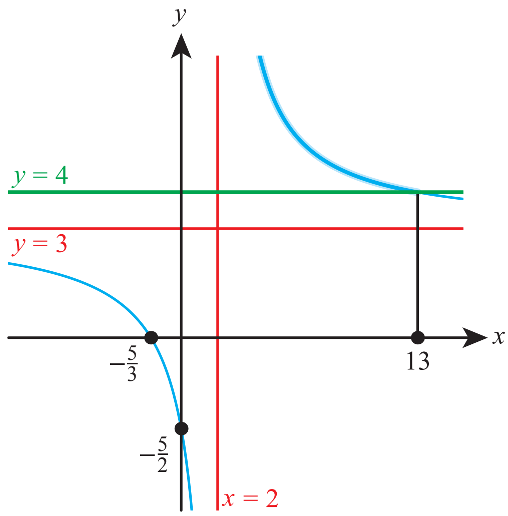
21 $-3 < x \leqslant -1$ or $2 < x \leqslant 2.27$

22 $-3.26 \leqslant x < -2$ or $-1.54 \leqslant x \leqslant 1.29$ or $x > \frac{4}{3}$

23 $x \in (6, \infty)$

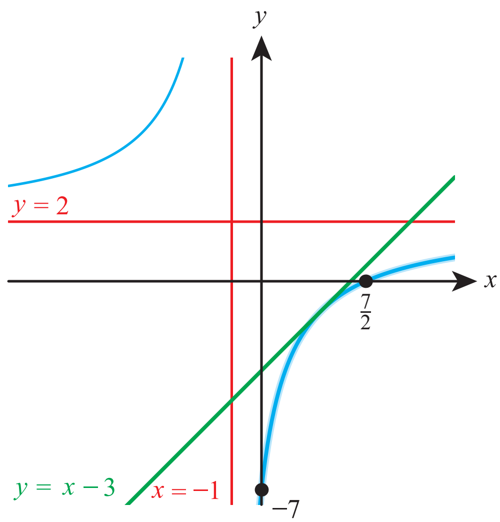
24 $x \in (-1, 1) \cup (3, \infty)$

25 a



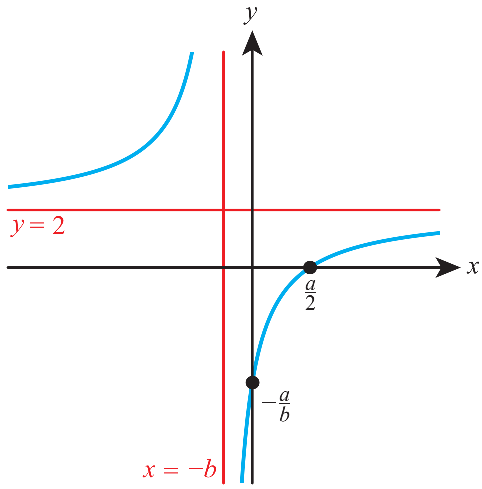
b $2 < x \leq 13$

26 a



b $x > -1,\ x \neq 2$

27 **a**

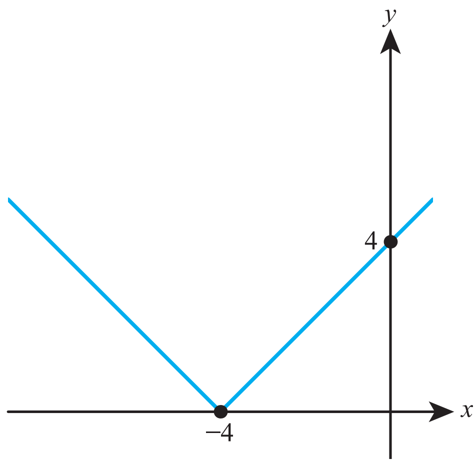


b $-a-3b<x<-b$

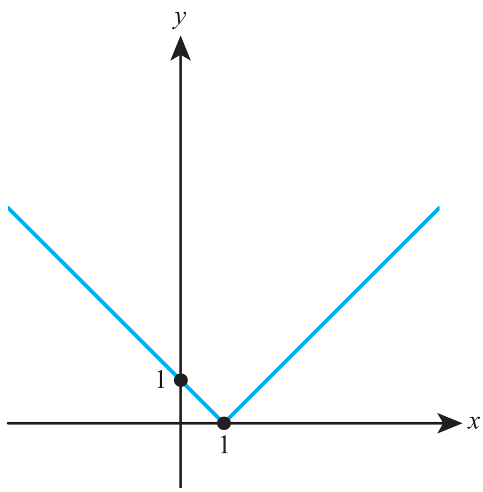
28 $p=2,\ q=-0.5,\ r=0.5$

Exercise 7C

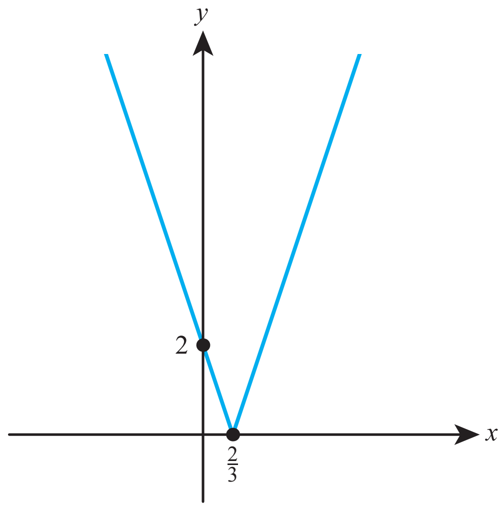
1 **a**



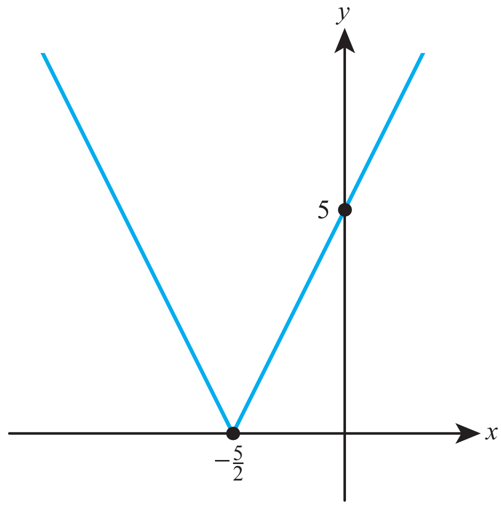
b



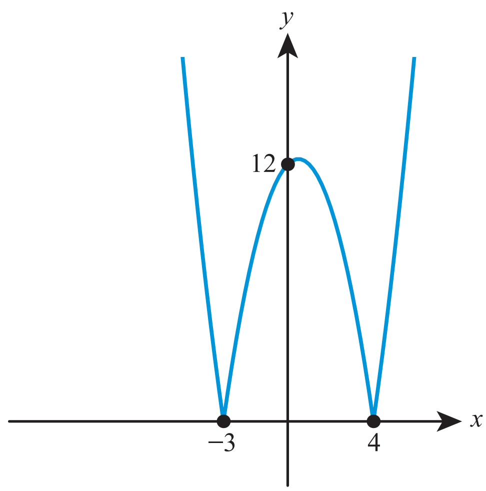
2 **a**



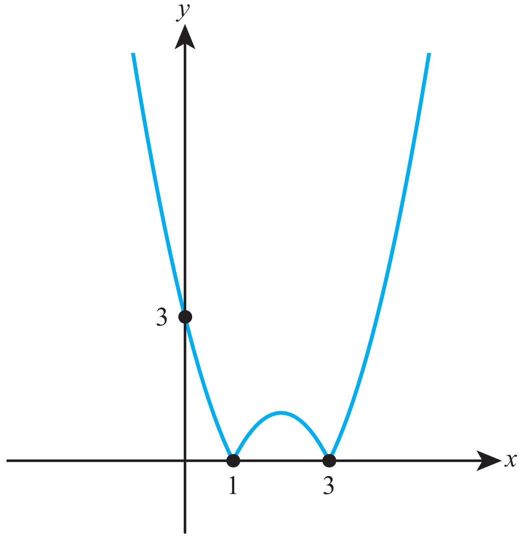
b



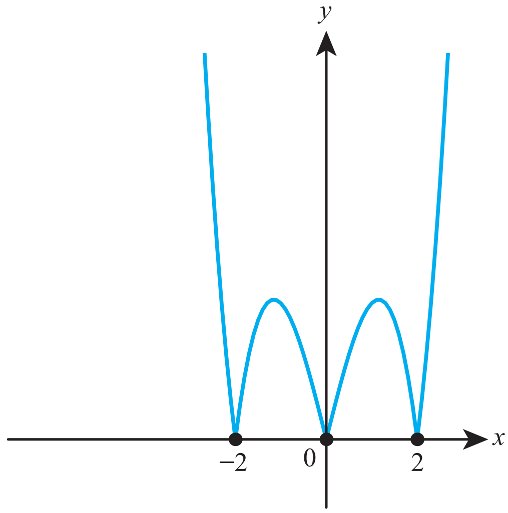
3 **a**



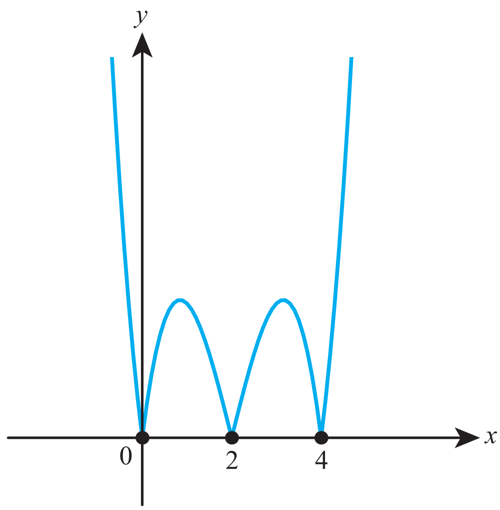
b



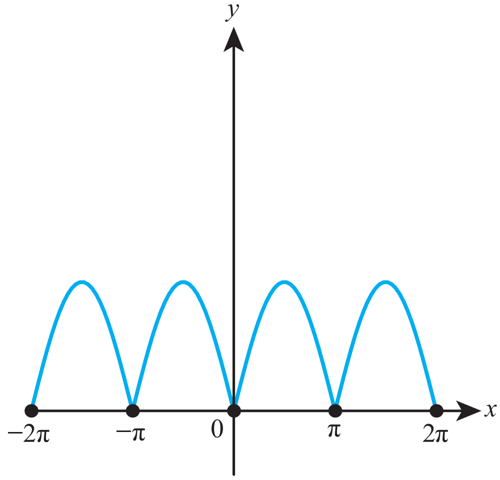
4 a



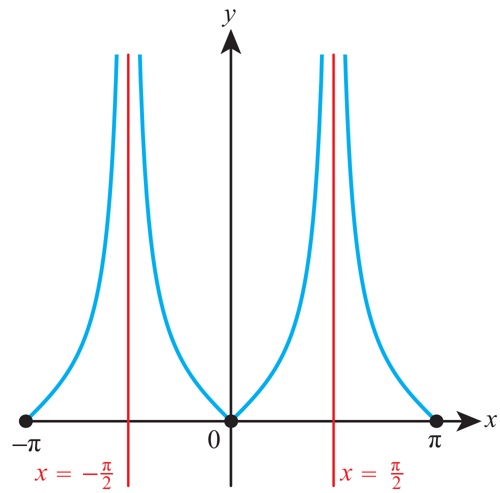
b



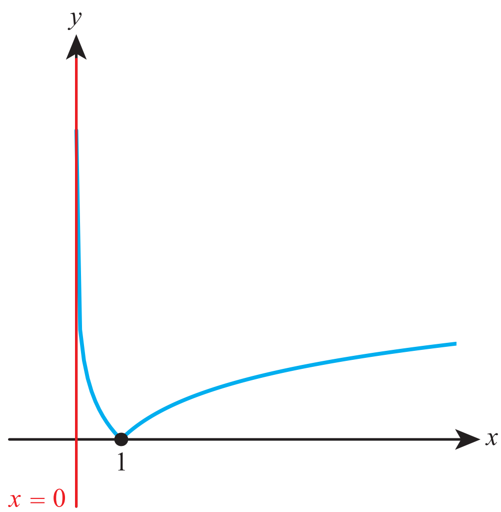
5 a



b



6 a



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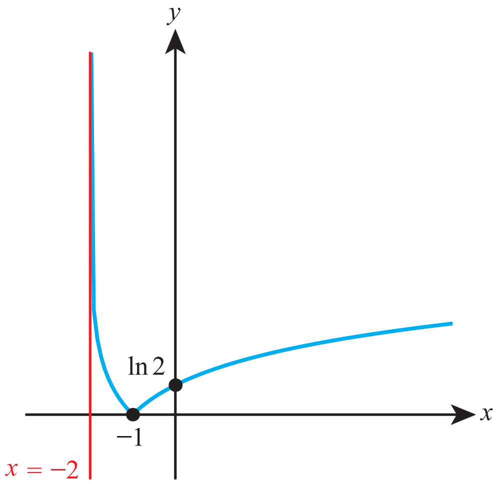
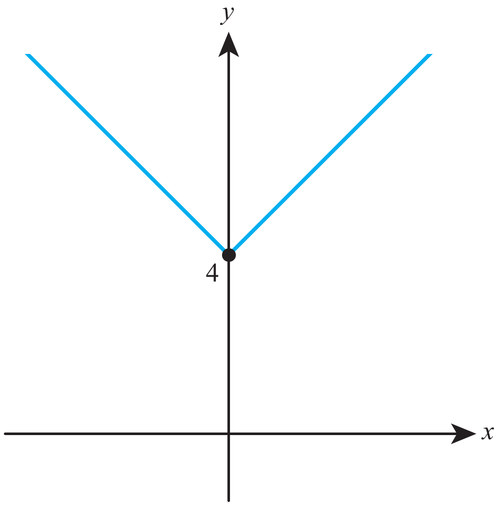


7 a



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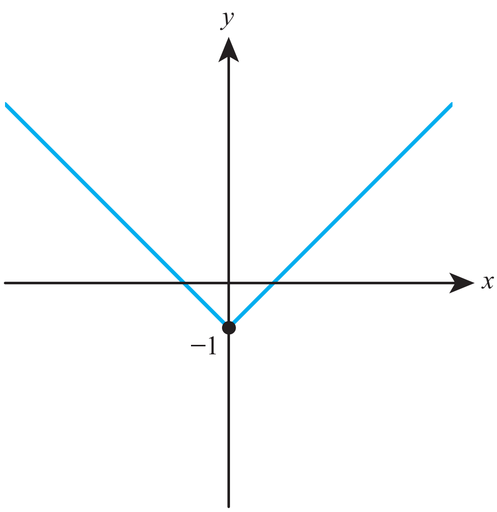
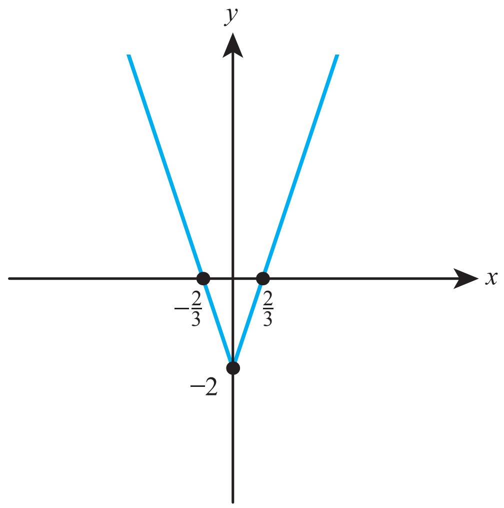
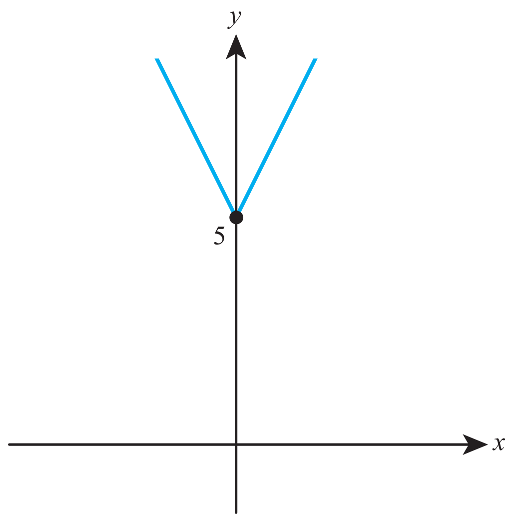


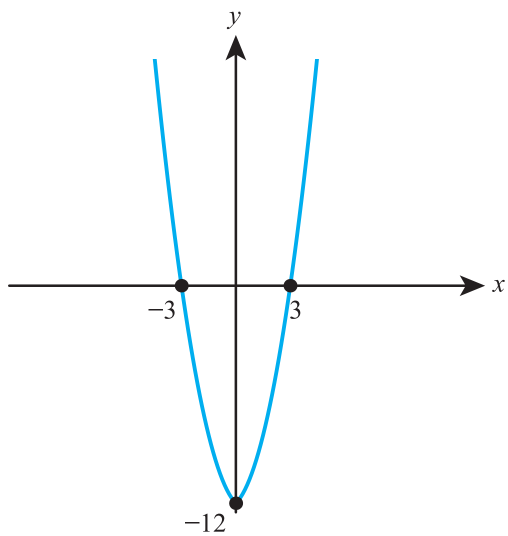
8 a



b



9 a



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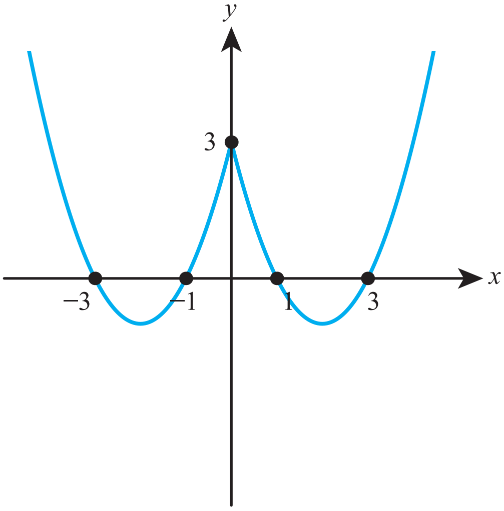
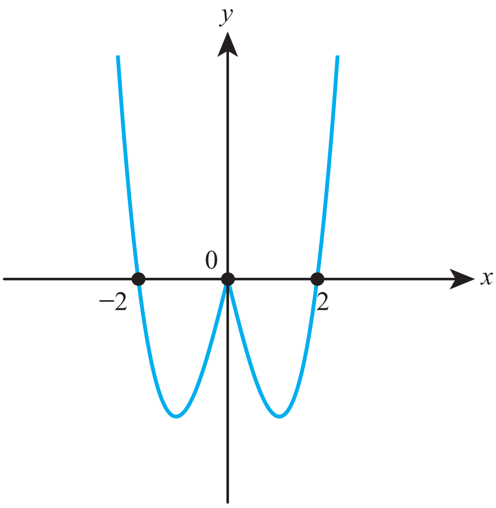
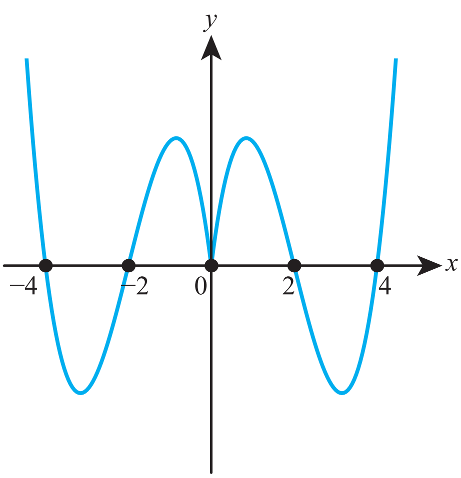


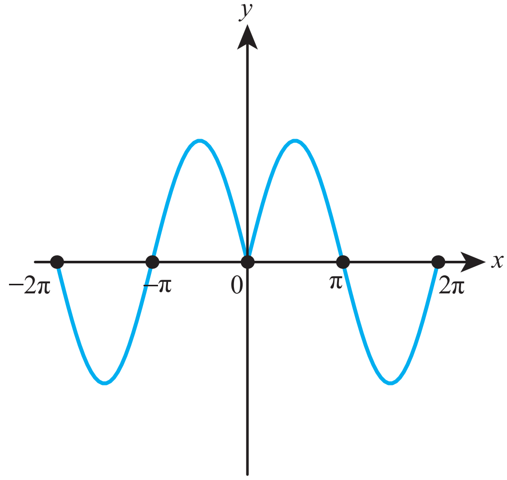
10 a



b



11 a



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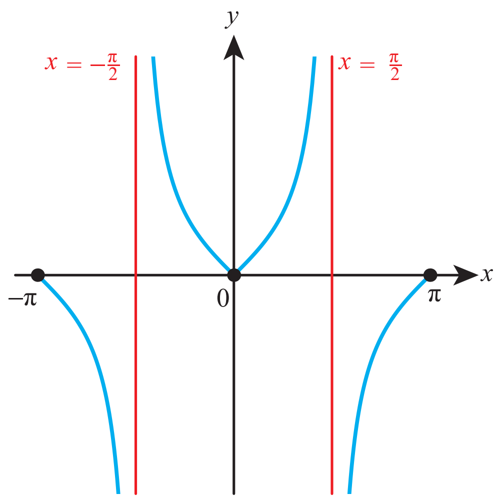
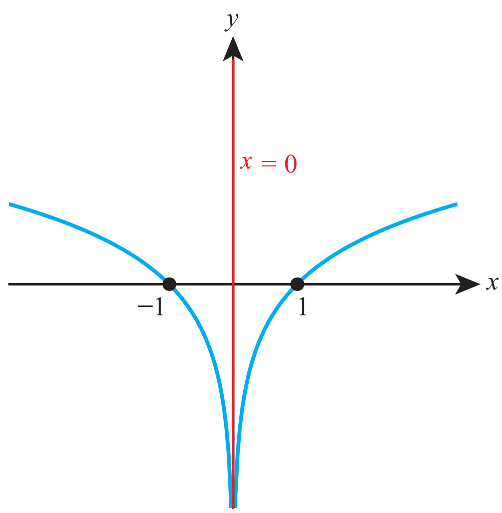
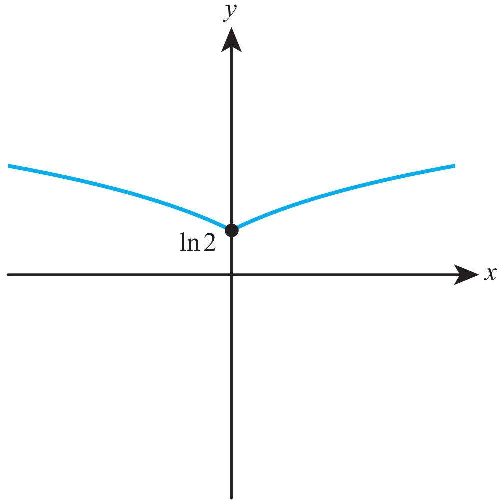


12 a



b



13 a $x = -2, 3$ **b** $x = -\frac{10}{3}, 2$

14 a $x = \frac{3}{4}, \frac{7}{2}$ **b** $x = -\frac{4}{3}, \frac{2}{5}$

15 a $x = -\frac{1}{3}$ **b** $x = -2, 3$

16 a $x = -4, -1, 0, 3$ **b** $x = 0, 2, 3, 5$

17 a $x = -2, 4, 6$ **b** $x = -3, -1$

18 a $x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$ **b** $x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

19 a $x < -2$ or $x > 3$ **b** $-\frac{10}{3} < x < 2$

20 a $\frac{3}{4} \leqslant x \leqslant \frac{7}{2}$ **b** $x \leqslant -\frac{4}{3}$ or $x \geqslant \frac{2}{5}$

21 a $x > -\frac{1}{3}$ **b** $-2 < x < 3$

22 a $x \leqslant -4$ or $-1 \leqslant x \leqslant 0$ or $x \geqslant 3$

b $0 \leqslant x \leqslant 2$ or $3 \leqslant x \leqslant 5$

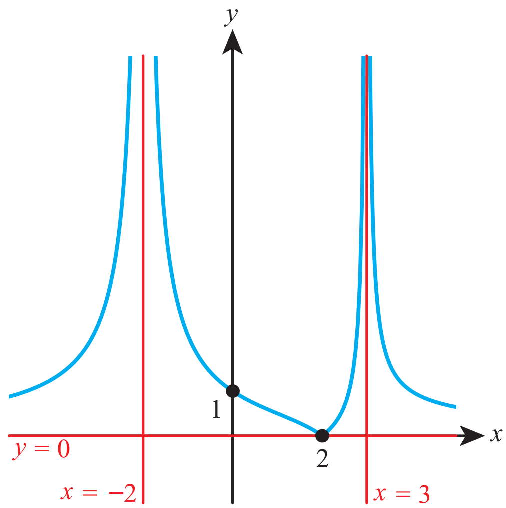
23 a $4 < x < 6$

b $x < -3$ or $x > -1$

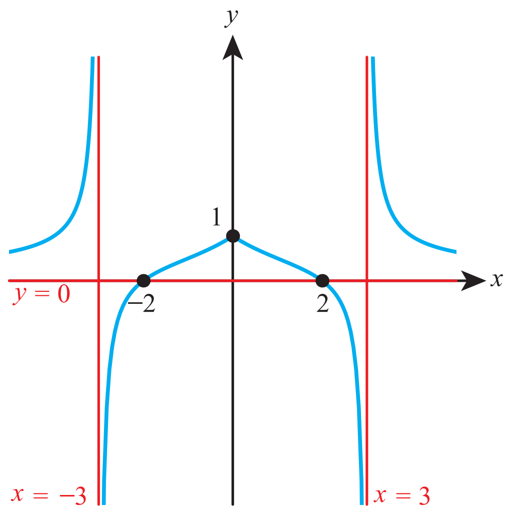
24 a $-\frac{2\pi}{3} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{2\pi}{3}$

b $-\frac{3\pi}{4} < x < -\frac{\pi}{4}$ or $\frac{\pi}{4} < x < \frac{3\pi}{4}$

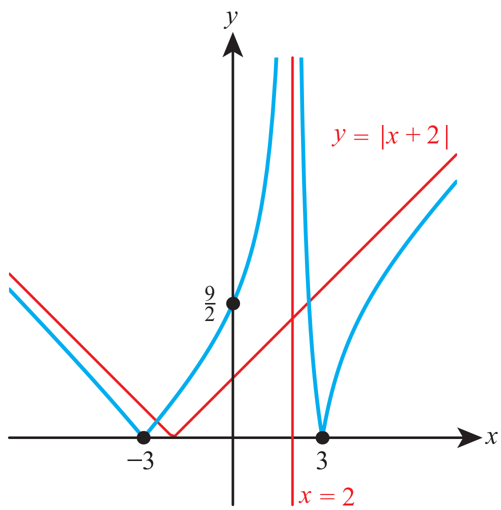
25 a



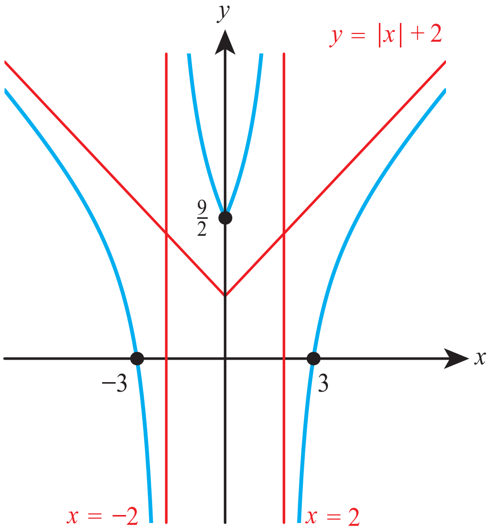
b



26 a



b



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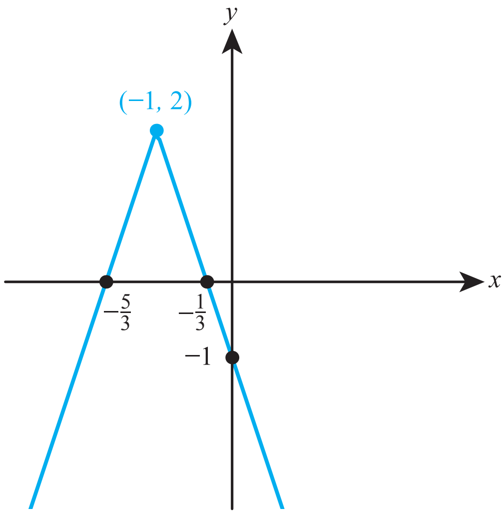

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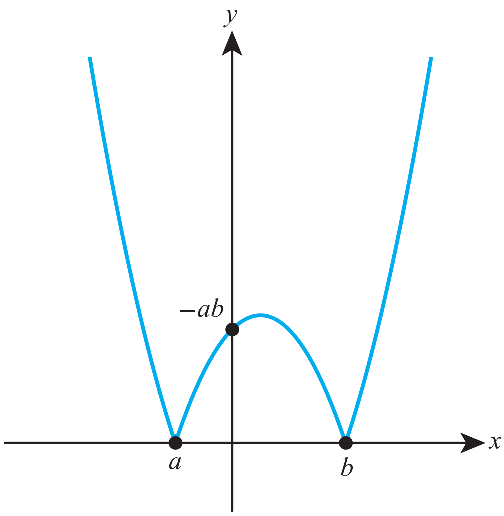
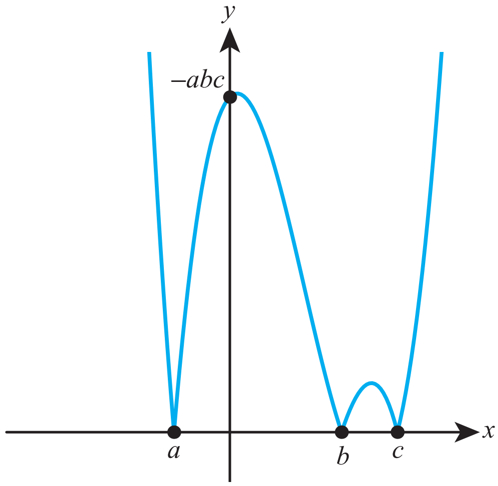


29

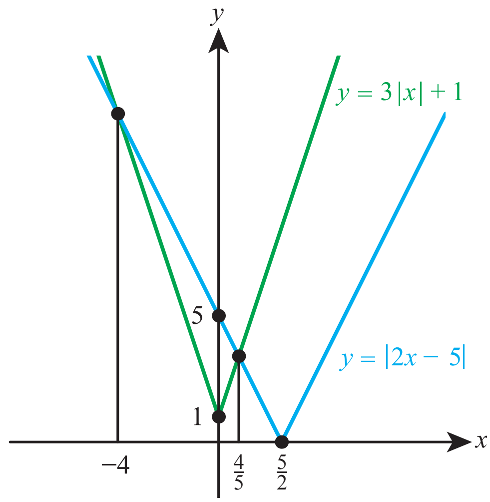


30 $-1.28 \leqslant x \leqslant 0.720$

31 $x \in (-\infty, -4.80) \cup (-3.32, 4.80)$

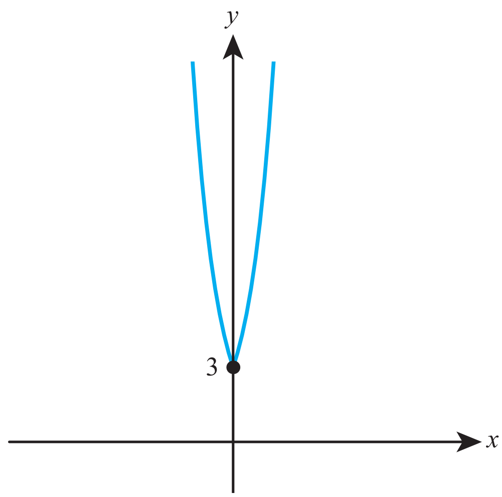
32 $x < -0.146$ or $0.180 < x < 0.967$

33 **a**



b $x < -4$ or $x > \frac{4}{5}$

34 **a**



b $x \in \left]-\infty, -\ln\frac{5}{3}\right] \cup \left[\ln\frac{5}{3}, \infty\right[$

35 $a = -2,\ b = 3,\ c = 5$

36 $x < -2$

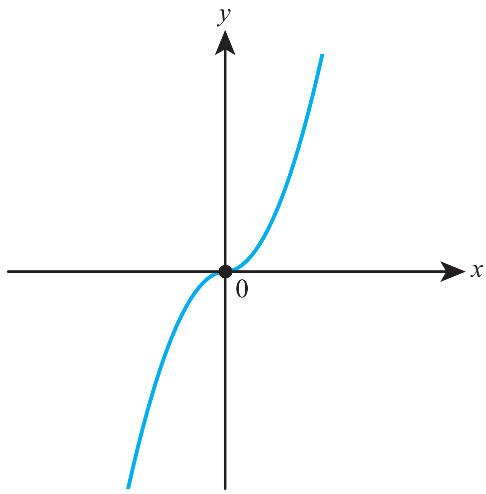
37 $x = -2,\ 2 - \sqrt{6}$

38 $x < \frac{5-\sqrt{17}}{2}$ or $2 < x < 3$ or $x > \frac{5+\sqrt{17}}{2}$

39 $x = 4,\ -\frac{4}{3}$

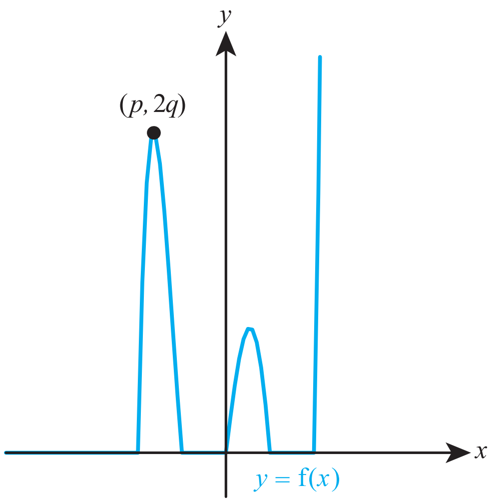
40 $x = 1,\ -\frac{1}{3}$

41 a



b $x = \pm k,\ 0$

42



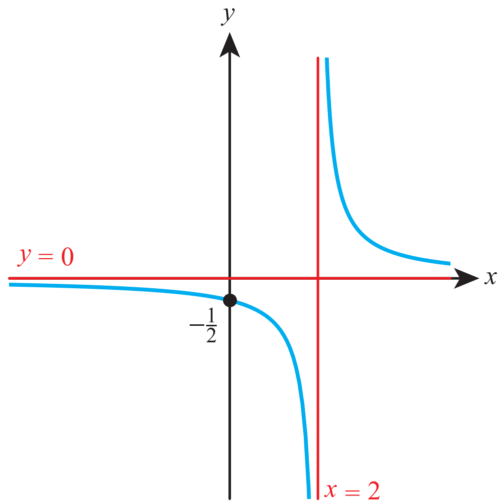
43 $\frac{a^2}{2}$

44 $0 < k < 11$

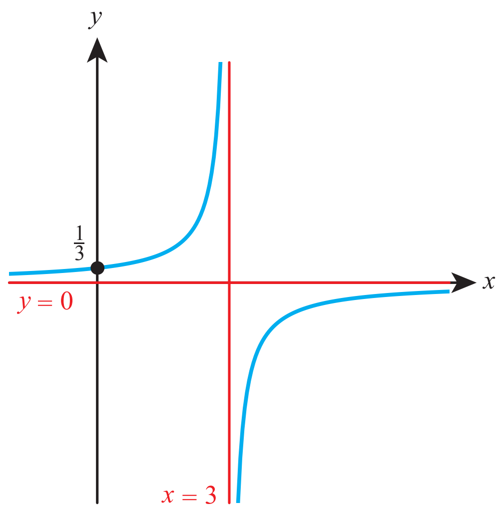
45 $12 < k < 20$

Exercise 7D

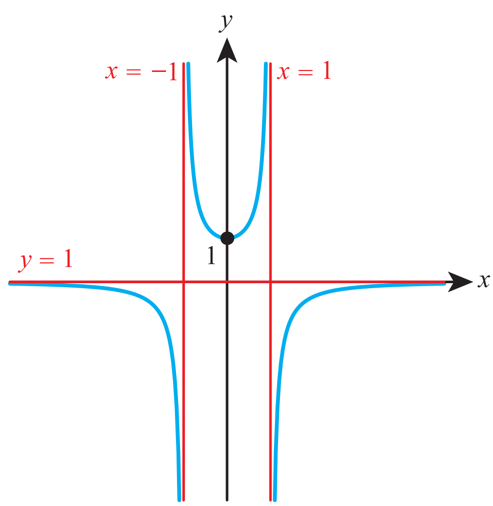
1 a



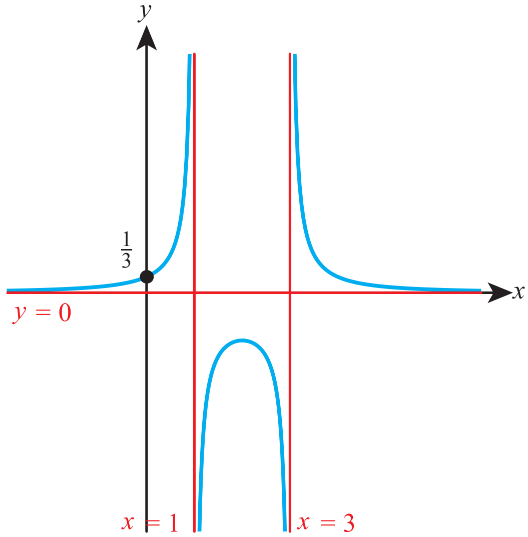
b



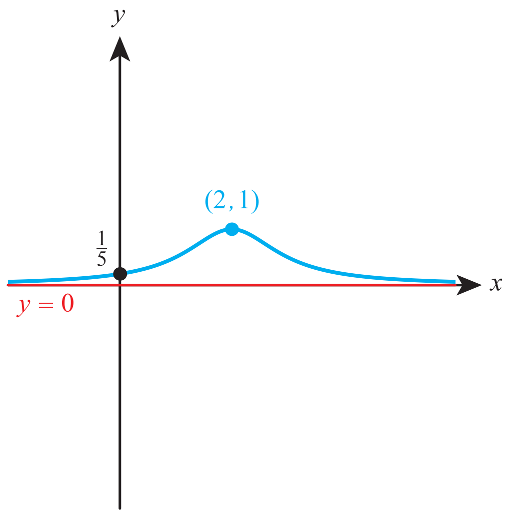
2 a



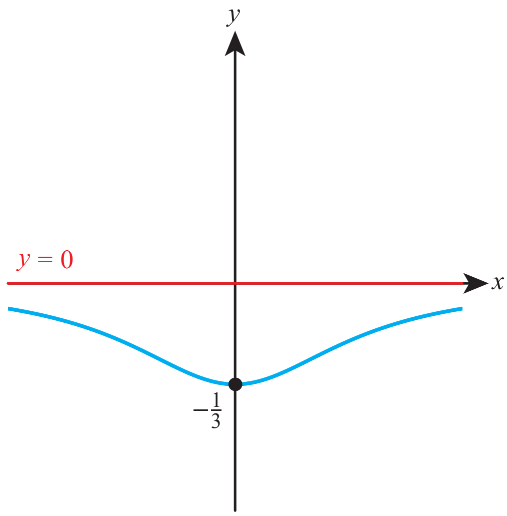
b


y
x
$\frac{1}{3}$
$y = 0$
$x = 1$
$x = 3$

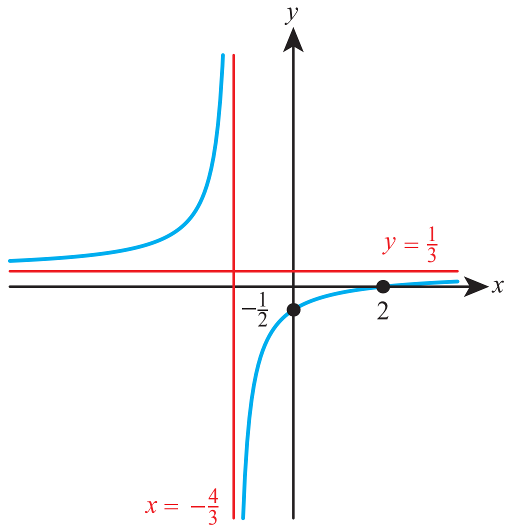
3 a


y
x
(2, 1)
$\frac{1}{5}$
$y = 0$

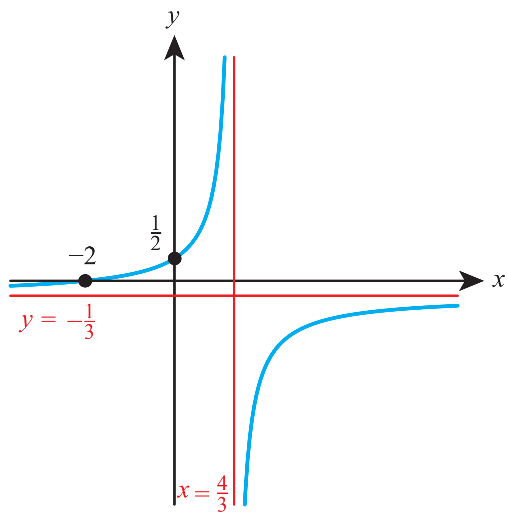
b


y
x
$y = 0$
$-\frac{1}{3}$

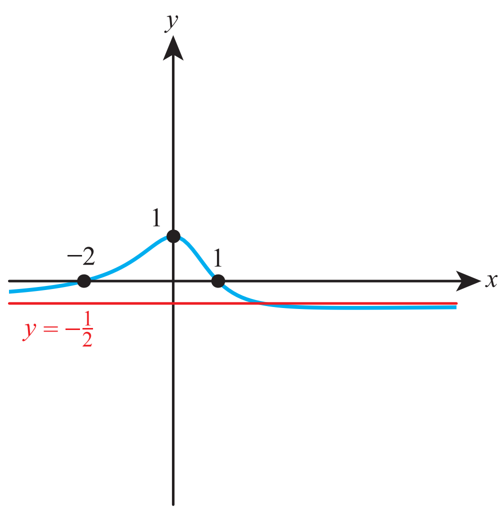
4 a


y
x
$y = \frac{1}{3}$
$-\frac{1}{2}$
2
$x = -\frac{4}{3}$

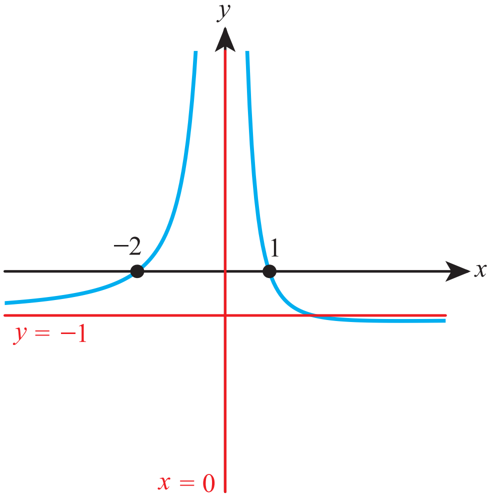
b


y
x
-2
$\frac{1}{2}$
$y = -\frac{1}{3}$
$x = \frac{4}{3}$

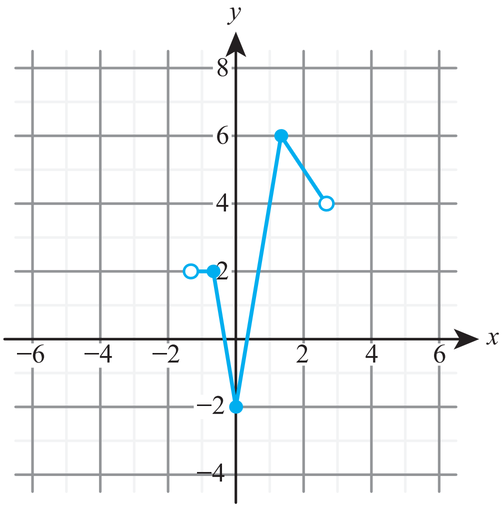
5 a


y
x
1
-2
1
$y = -\frac{1}{2}$

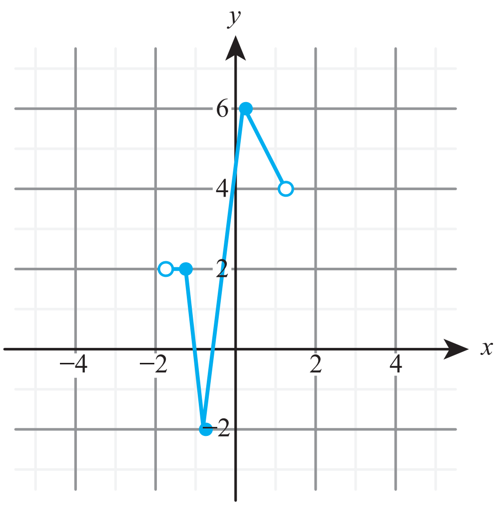
b


y
x
−2
1
y = −1
x = 0

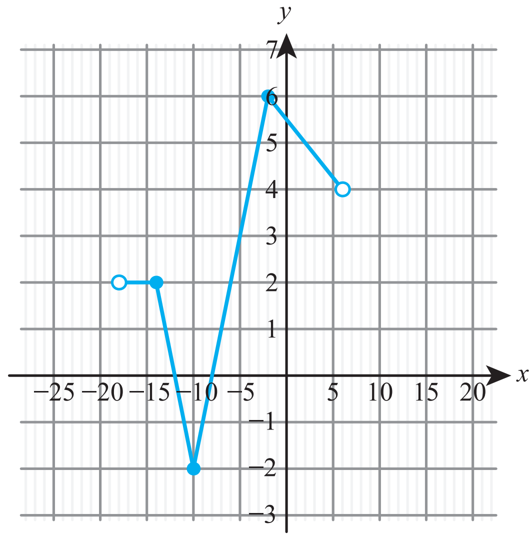
6 a


y
x
8
6
4
2
−2
−4
−6
−4
−2
2
4
6

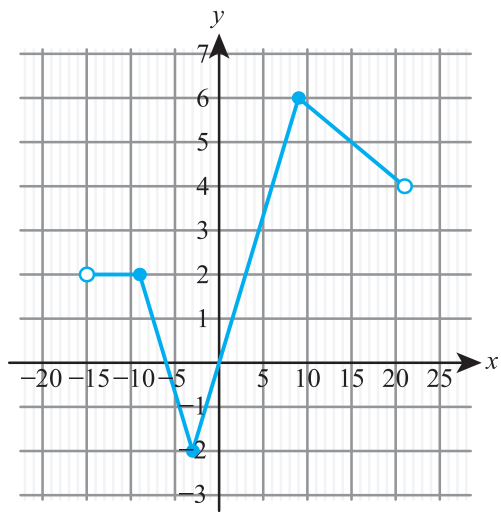
b


y
x
6
4
2
−2
−4
−2
2
4

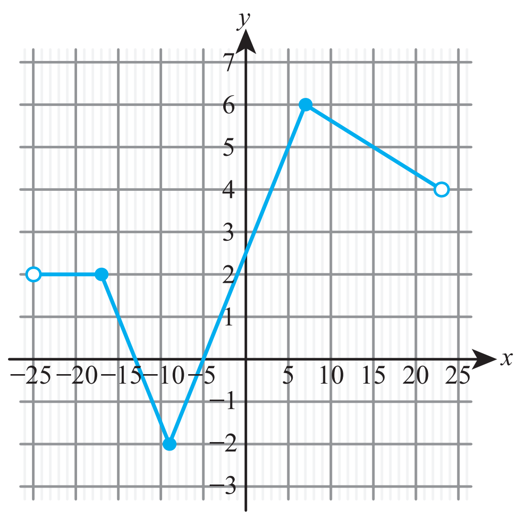
7 a


y
x
7
6
5
4
3
2
1
−1
−2
−3
−25 −20 −15 −10 −5
5 10 15 20

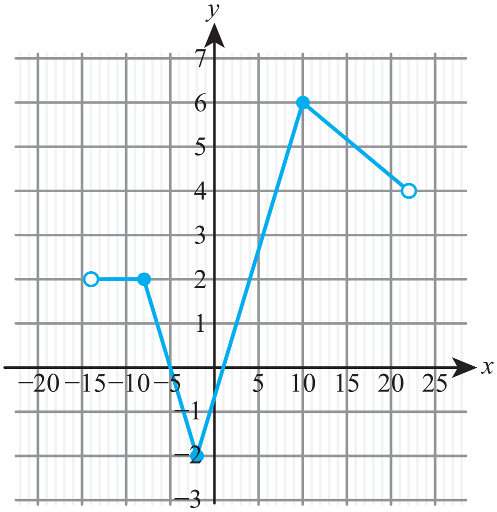
b


y
x
7
6
5
4
3
2
1
−1
−2
−3
−20 −15 −10 −5
5 10 15 20 25

8 a


y
x
7
6
5
4
3
2
1
−1
−2
−3
−25 −20 −15 −10 −5
5 10 15 20 25

b



9 a Translation right by 2 followed by horizontal stretch with scale factor 4

b Translation left by 1 followed by horizontal stretch with scale factor 3

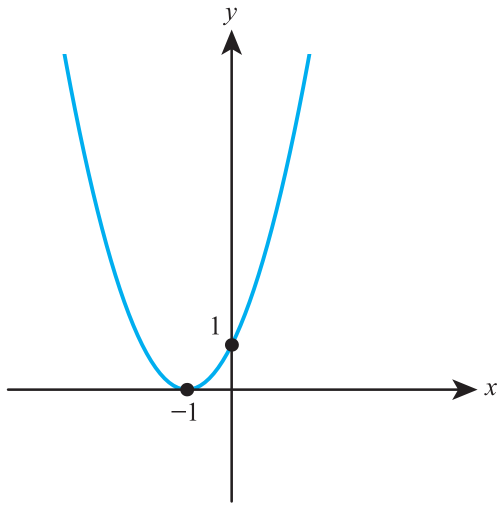
10 a Translation right by 1 followed by horizontal stretch with scale factor $\frac{1}{3}$

b Translation left by 3 followed by horizontal stretch with scale factor $\frac{1}{2}$

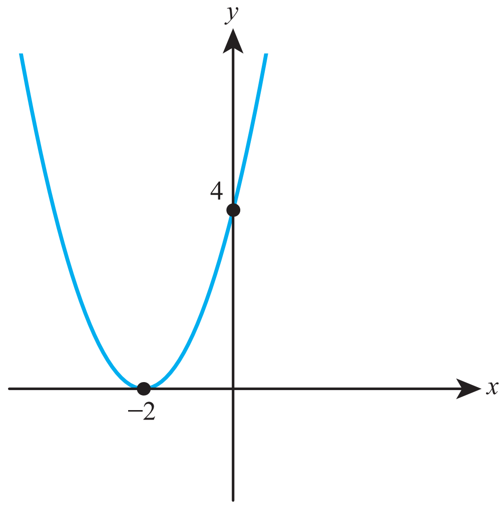
11 a Translation right by 3 followed by reflection in the y-axis or just translate right by 2

b Translation left by 2 followed by reflection in the y-axis or just translate right by 2

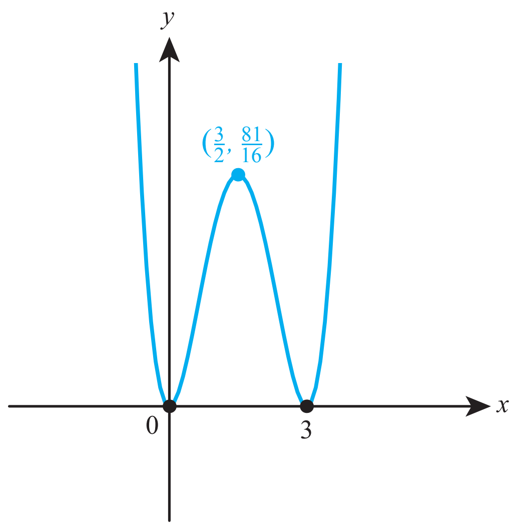
12 a



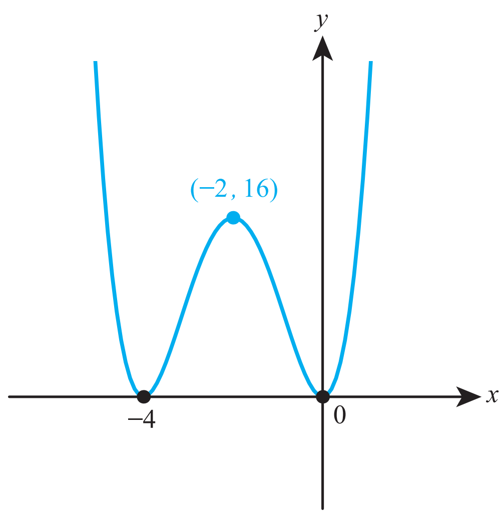
b



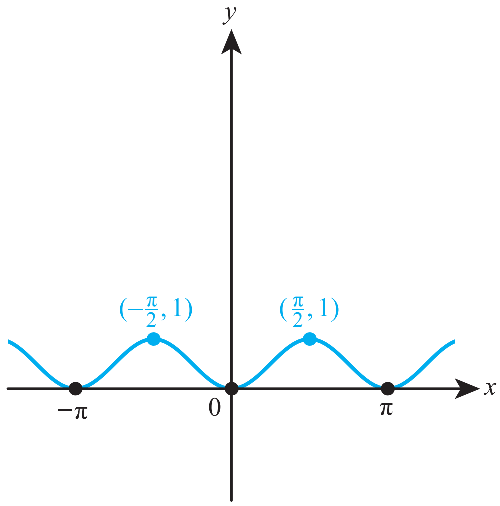
13 a



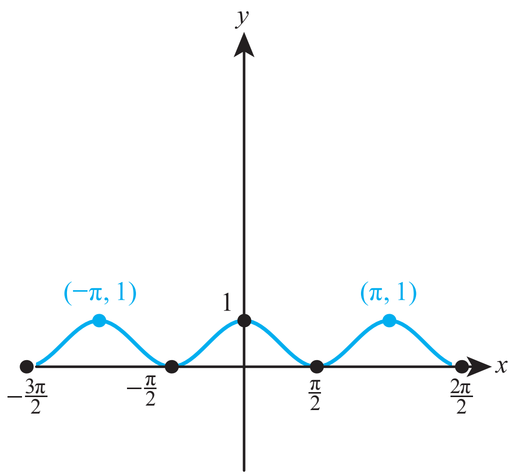
b



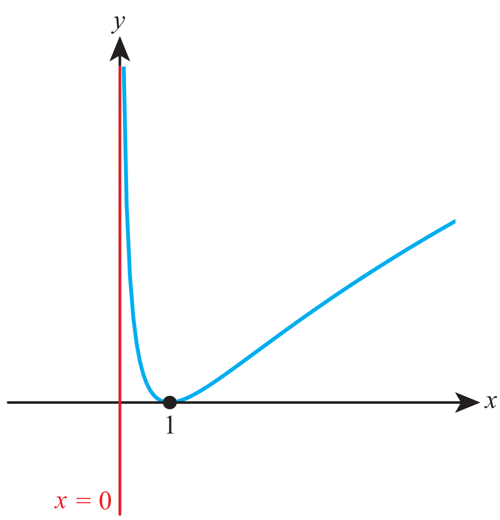
14 a



b



15 a



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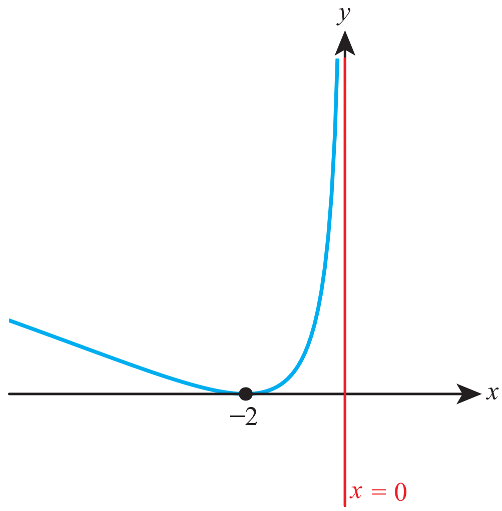
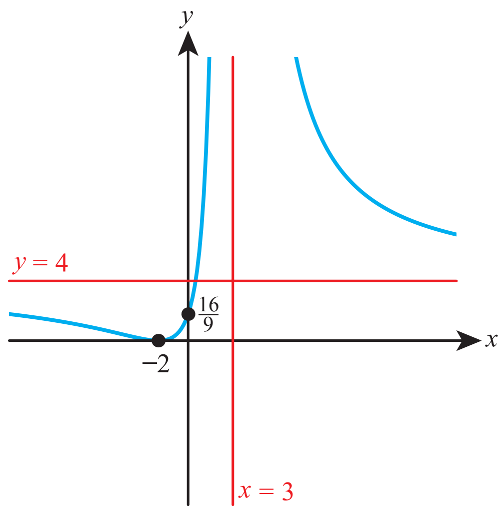


16 a



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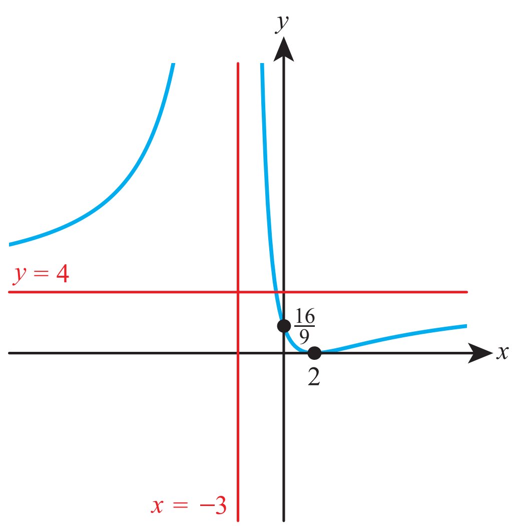

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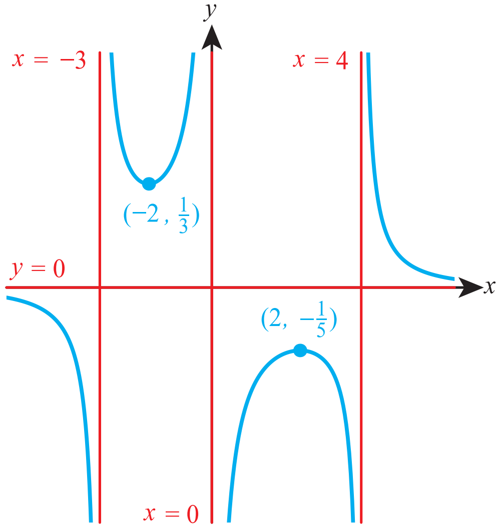

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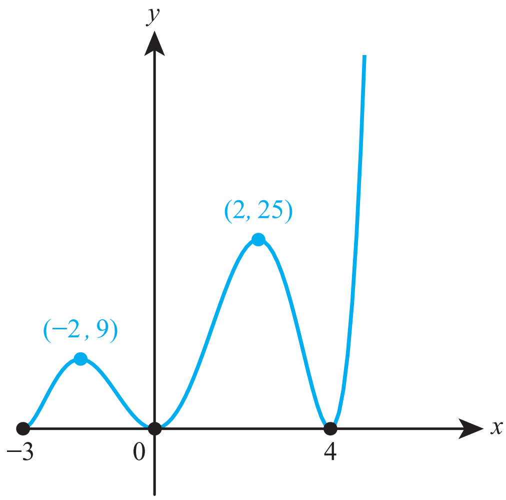
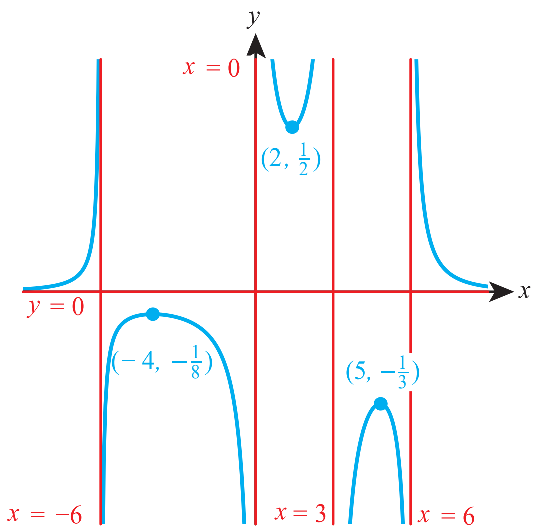


18 a



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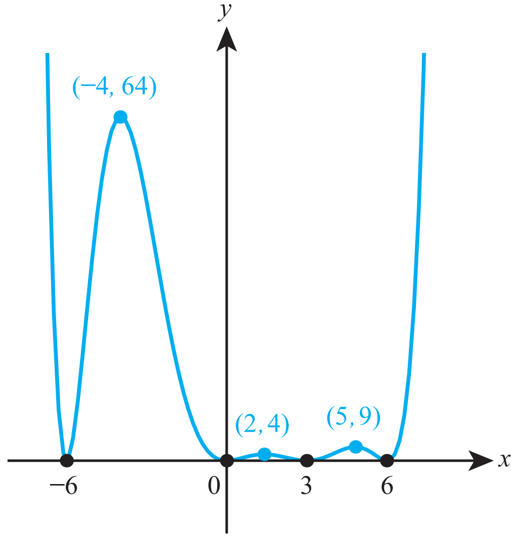
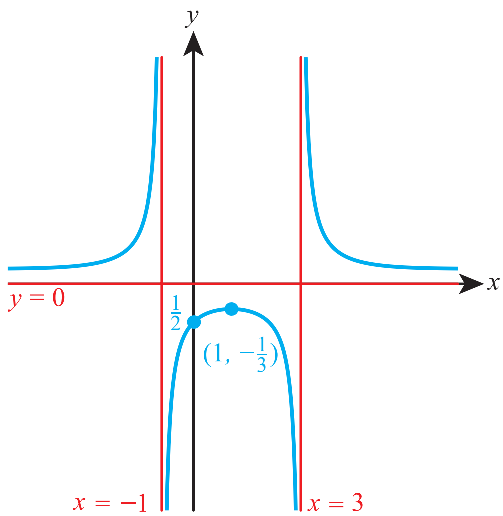


19 a



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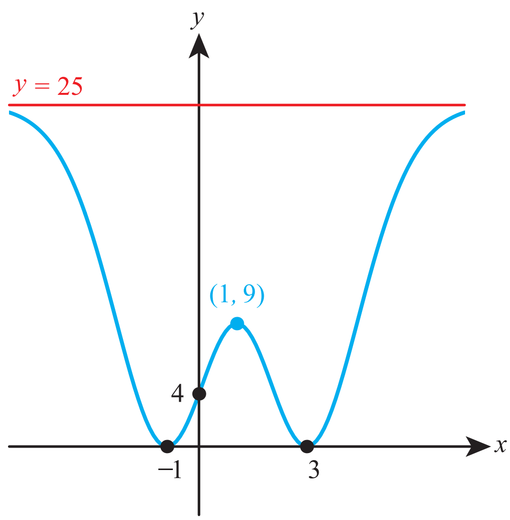


20 **a** $\left(3, -\frac{1}{4}\right)$ **b** $(3, 16)$ **c** $(2, -4)$

21 Translation right by $\frac{\pi}{4}$ followed by horizontal stretch with scale factor $\frac{1}{3}$

22 Translation left by 3 followed by horizontal stretch with scale factor $\frac{5}{2}$

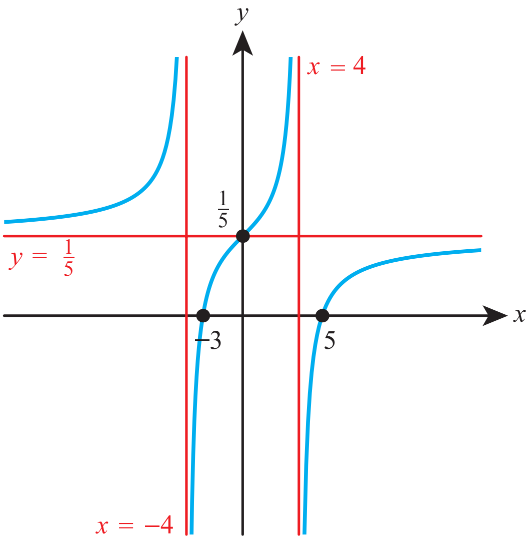
23 $y = 12x^2 - 16x + 4$

24 $y = 2\mathrm{f}(2x - 4) + 6$

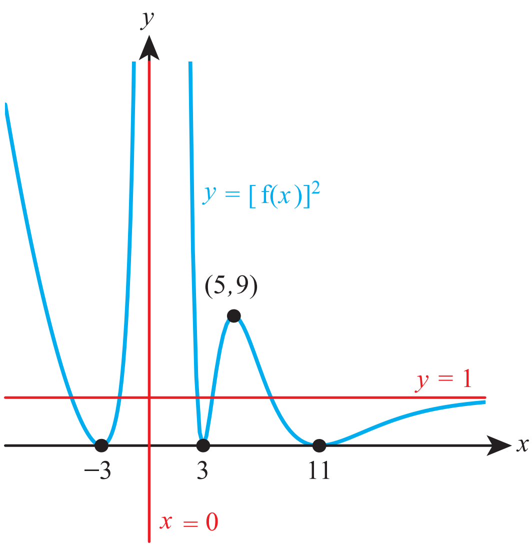
25 $y = \frac{1}{3}\mathrm{f}\left(\frac{x+1}{2}\right) - 4$

26 **a** $y = \mathrm{f}(-x + 5)$ **b** $y = \mathrm{f}(-x - 5)$

27



28



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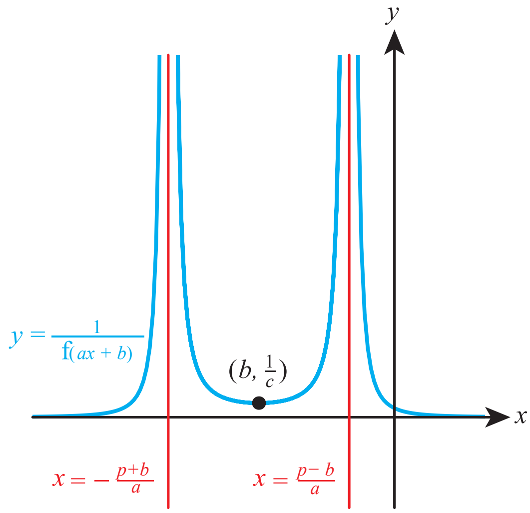

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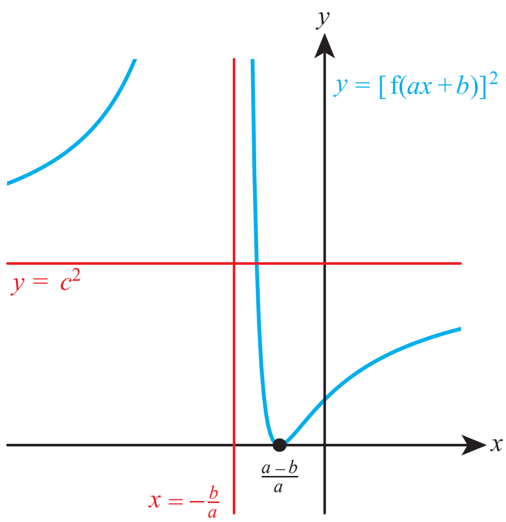


31 $a = 4,\ b = \frac{1}{2},\ c = -\frac{\pi}{3}$

32 $a = -1,\ b = 3,\ c = \frac{\pi}{2}$

33 Horizontal stretch with scale factor $\frac{1}{5}$ followed by translation right by $\frac{2}{5}$

34 $a = 9,\ b = 6,\ c = -10$

35 $a = \frac{1}{4},\ b = -2,\ c = 0$

36 Translation right by 1 followed by horizontal stretch with scale factor $\frac{2}{3}$

37 $y = \tan(-3x)$

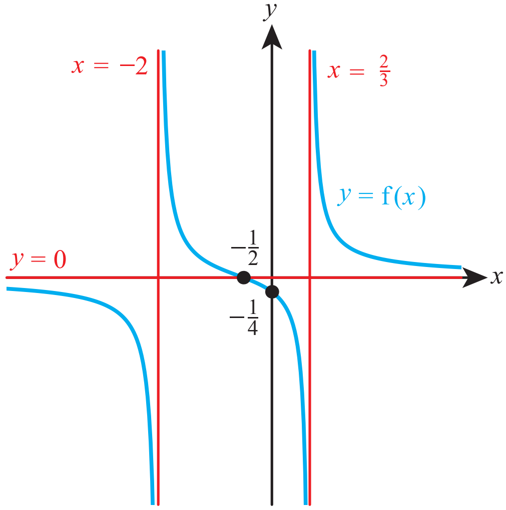
38 $c = -\log_2 5$

Exercise 7E

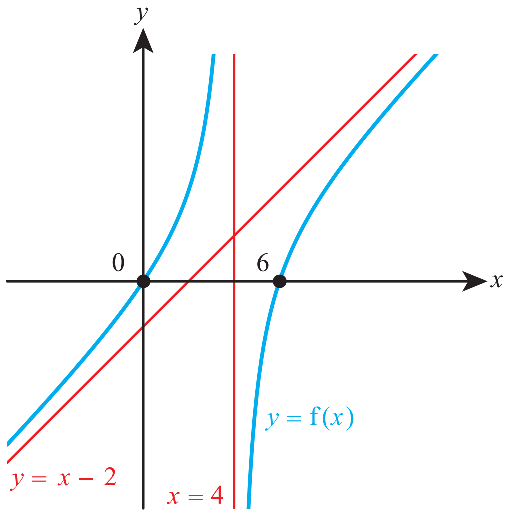
1 a Neither b Even
2 a Neither b Odd
3 a Neither b Even
4 a Even b Neither
5 a Odd b Even
6 a Neither b Even
7 a Odd b Neither
8 a Even b Neither
9 a $x \leqslant 2$ b $x \geqslant 2$
10 a $x \geqslant 4$ b $x \leqslant -2$
11 a $x \leqslant 1$ b $x \geqslant 3$
12 a Yes b Yes
13 a No b Yes
14 a No b Yes
15 Odd
16 Neither
17 Odd
18 a $k = -4$
b $f^{-1}(x) = -4 + \sqrt{x-3}, x \geqslant 3$
19 a $x \leqslant \frac{3}{2}$
b $f^{-1}(x) = \frac{3}{2} - \sqrt{x + \frac{5}{4}}, x \geqslant -\frac{5}{4}$
20 b $x \in \mathbb{R}$
21 b Neither
23 Even
24 a $-3 \leqslant x \leqslant -1$
b $-6 \leqslant x \leqslant -2$
25 a $k = 2$
b $x \geqslant -11$
26 a $x \leqslant \ln 4$
b $x \leqslant 4 - 4\ln 4$
27 b $x \neq \frac{2}{3}$
28 $a = -1$
30 $c = 2$

Chapter 7 Mixed Practice

1 a $x = \frac{2}{3}, x = -2$
b $\left(-\frac{1}{2}, 0\right)$ and $\left(0, -\frac{1}{4}\right)$
c

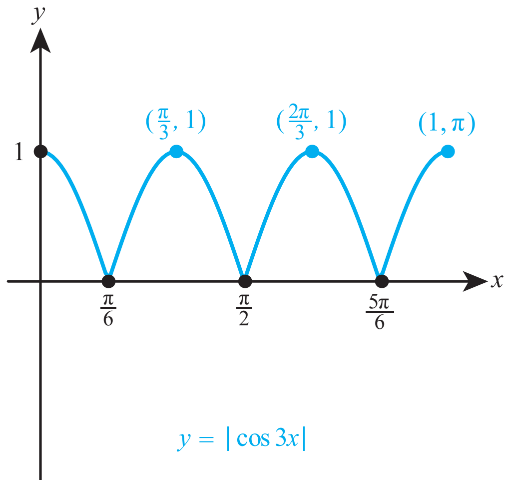


2 a i $x = 4$ ii $y = x - 2$
b (0, 0) and (6, 0)
c



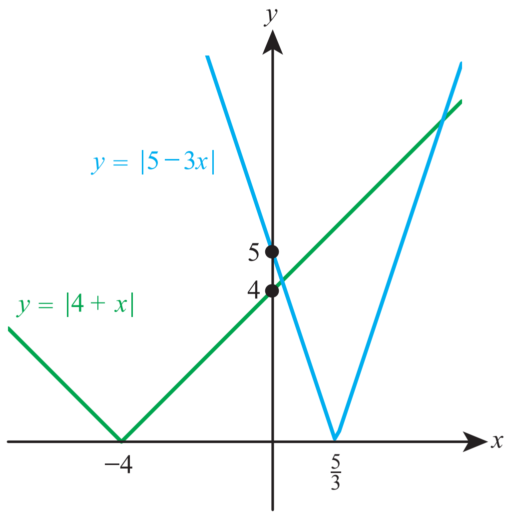
3 $\left]-\frac{3}{2}, 0\right[\cup \left]2, \infty\right[$
4 b $-2 \leqslant x \leqslant 1$ or $x \geqslant 4$
5 $-1.62 \leqslant x \leqslant -0.366$ or $0.618 \leqslant x \leqslant 1.37$
6 $0 < x \leqslant 3.04$ or $7.01 \leqslant x \leqslant 8.56$

7 a



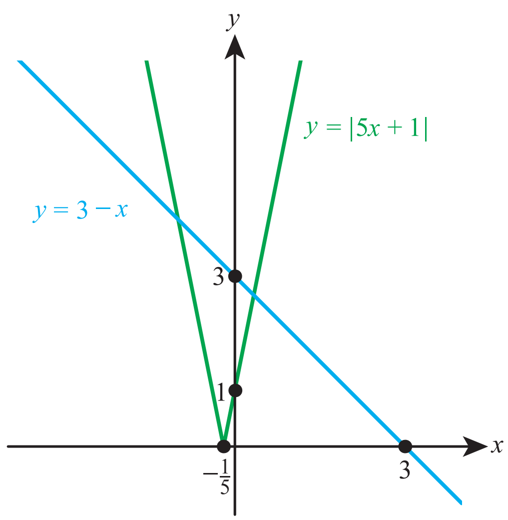
b $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$

8 a



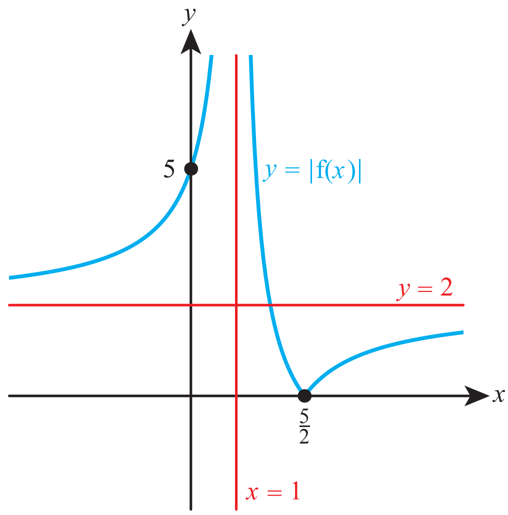
b $x \leqslant \frac{1}{4}$ or $x \geqslant \frac{9}{2}$

9 a

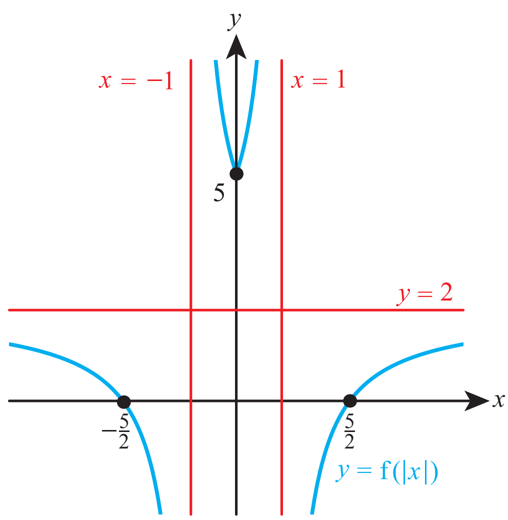


b $-1 < x < \frac{1}{3}$

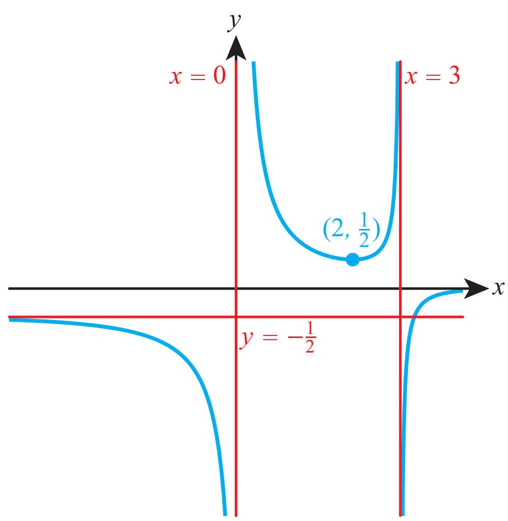
10 a



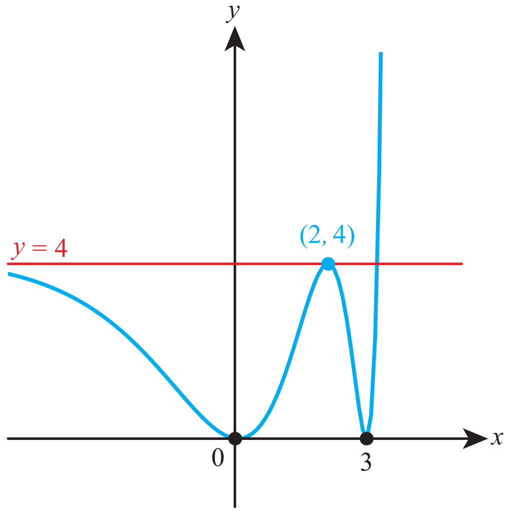
b



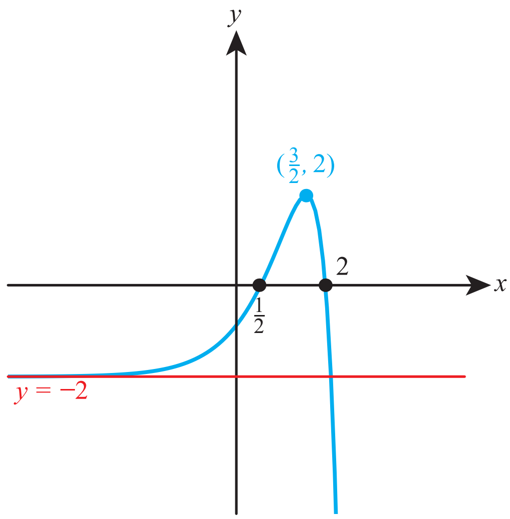
11 a



b



c



12 Even

13 **a** $k = 3$

b $f^{-1}(x) = 3 + \sqrt{5 - x},\ x \leqslant 5$

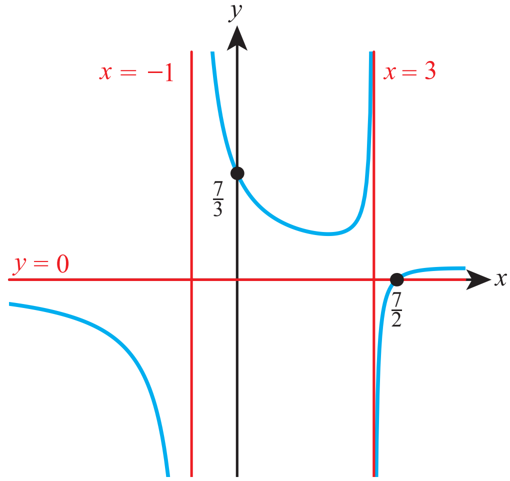
14 **b** $r = 0$

c $k(x) = 0$

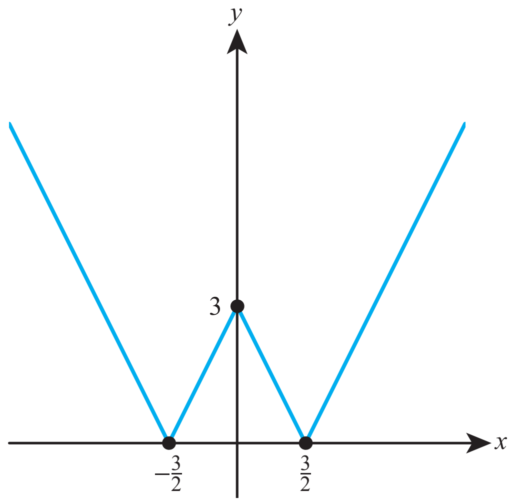
15 **a** **i** $k \leqslant \frac{1}{4}$ or $k \geqslant 1$

ii $f(x) \leqslant \frac{1}{4}$ or $f(x) \geqslant 1$

b



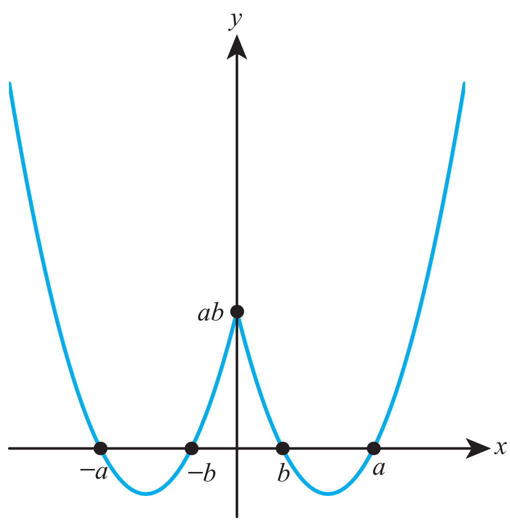
16 **a**



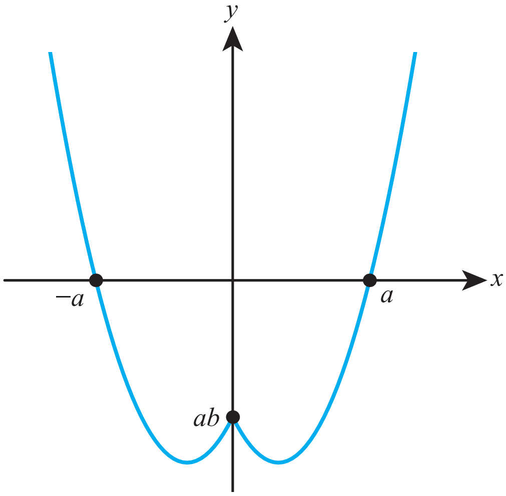
$\left(-\frac{3}{2}, 0\right), \left(\frac{3}{2}, 0\right), (0, 3)$

b $x = \pm\frac{1}{2}, \pm\frac{5}{2}$

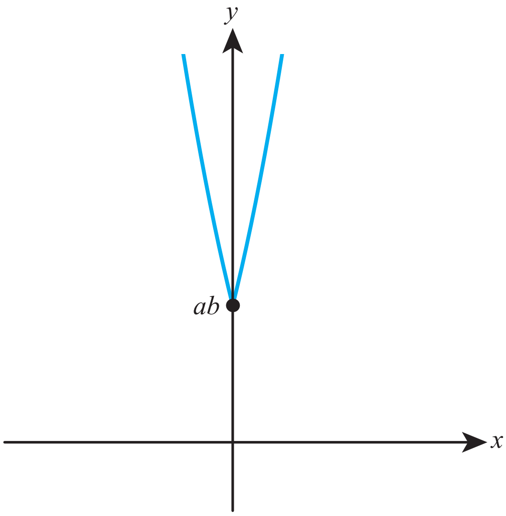
17 **a**



b



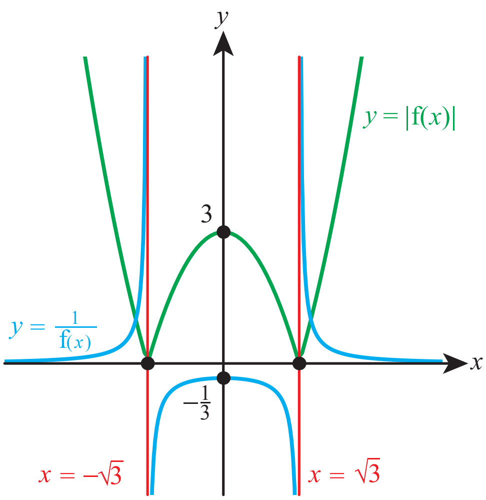
c



18 a Horizontal stretch with scale factor 3 followed by horizontal translation by +6

b Horizontal translation by +2 followed by horizontal stretch with scale factor 3

19 a



b $-2 \le x < -\sqrt{3}$ or $\sqrt{3} < x \le 2$

20 $a = -b$

21 a $k = \ln 4$

b $f^{-1}(x) = \ln(4 - \sqrt{x+9}),\ -9 \leqslant x < 7$

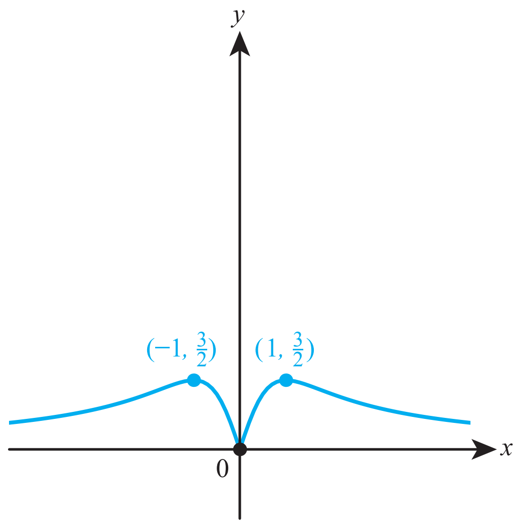
22 a $f'(x) = e^{\frac{x}{2}} + \frac{x}{2}e^{\frac{x}{2}},\ f''(x) = e^{\frac{x}{2}} + \frac{x}{4}e^{\frac{x}{2}}$

b $k = -2$ c $x \geqslant -2e^{-1}$

23 a ii Rotation 180° around the origin

b ii $\left(1, \frac{3}{2}\right), \left(-1, -\frac{3}{2}\right)$

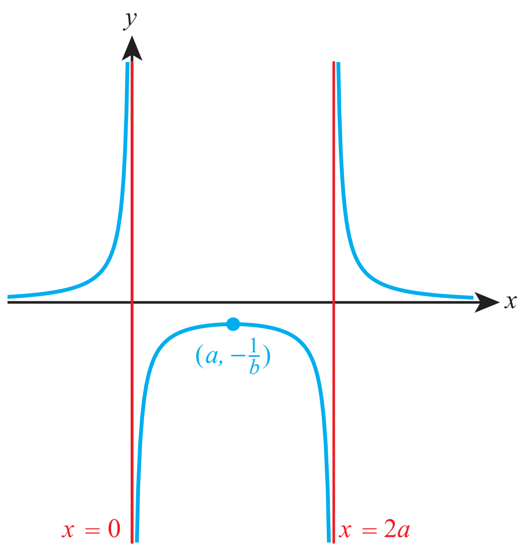
c



d $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

24 a $x \neq 0,\ 2a$

b

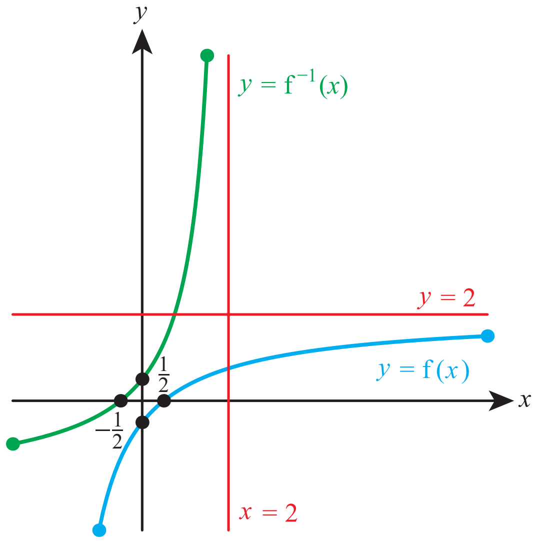


25 a $2-\dfrac{5}{x+2}$

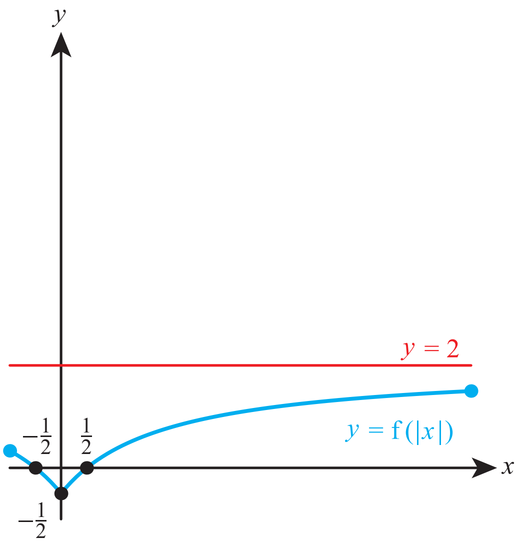
c $-3 \leqslant f(x) \leqslant 1.5$

d i $f^{-1}(x)=\dfrac{2x+1}{2-x}$

ii, iii



e i

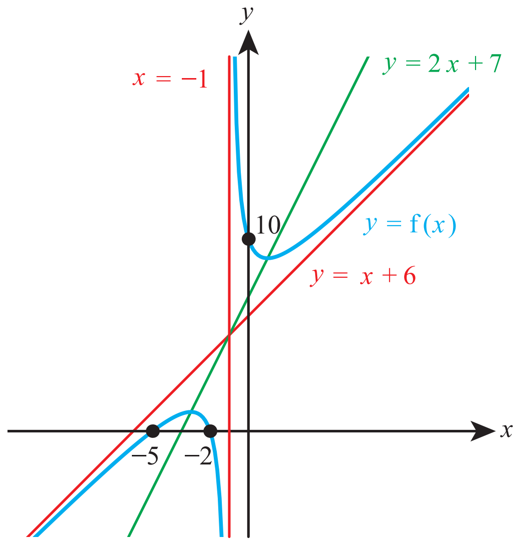


ii $x=\pm\dfrac{2}{9}$

26 a $y = x + 6$

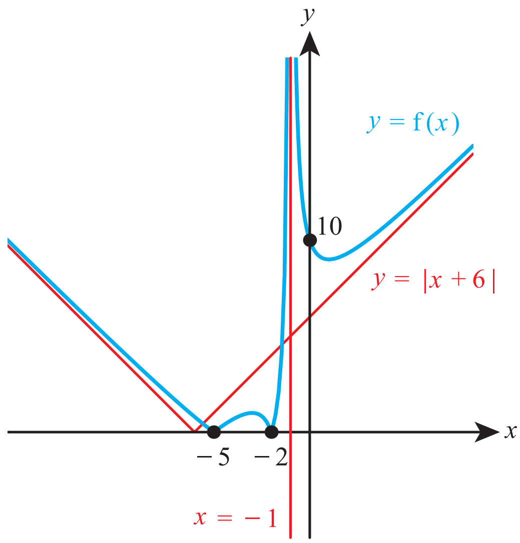
b (−3, 1) and (1, 9)

c



d $-3 < x < -1$ or $x > 1$

e



f $c = 0$ or $1 < c < 9$

27 $x = -4.5, -3.59, 1, 2.09$

28 $c = -3$