Further trig review *[107 marks]*

1. Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$. *[5 marks]*

Markscheme

determines $\frac{\pi}{4}$ (or 45°) as the first quadrant (reference) angle ***(A1)***

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ ***(M1)***

Note: Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4},\frac{7\pi}{4}(,\ldots)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}\Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected ***(R1)***

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\left(=\frac{7\pi}{4}\right)$ ***A1***

$x=\frac{17\pi}{6}$ (must be in radians) ***A1***

[5 marks]

Let $\sin\theta=\frac{\sqrt{5}}{3}$, where θ is acute.

2a. Find $\cos\theta$. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

evidence of valid approach ***(M1)***

eg right triangle, $\cos^2\theta = 1 - \sin^2\theta$

correct working ***(A1)***

eg missing side is 2, $\sqrt{1 - \left(\frac{\sqrt{5}}{3}\right)^2}$

$\cos\theta = \frac{2}{3}$ ***A1 N2***

[3 marks]

2b. Find $\cos 2\theta$. *[2 marks]*

Markscheme

correct substitution into formula for $\cos 2\theta$ ***(A1)***

eg $2 \times \left(\frac{2}{3}\right)^2 - 1,\ 1 - 2\left(\frac{\sqrt{5}}{3}\right)^2,\ \left(\frac{2}{3}\right)^2 - \left(\frac{\sqrt{5}}{3}\right)^2$

$\cos 2\theta = -\frac{1}{9}$ ***A1 N2***

[2 marks]

3a. Show that $\log_9(\cos 2x + 2) = \log_3\sqrt{\cos 2x + 2}$. *[3 marks]*

Markscheme

attempting to use the change of base rule ***M1***

$\log_9(\cos 2x + 2) = \frac{\log_3(\cos 2x+2)}{\log_3 9}$ ***A1***

$= \frac{1}{2}\log_3(\cos 2x + 2)$ ***A1***

$= \log_3\sqrt{\cos 2x + 2}$ ***AG***

[3 marks]

3b. Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. *[5 marks]*

Markscheme

$\log_3(2\sin x) = \log_3\sqrt{\cos 2x + 2}$

$2\sin x = \sqrt{\cos 2x + 2}$ ***M1***

$4\sin^2 x = \cos 2x + 2$ (or equivalent) ***A1***

use of $\cos 2x = 1 - 2\sin^2 x$ ***(M1)***

$6\sin^2 x = 3$

$\sin x = (\pm)\frac{1}{\sqrt{2}}$ ***A1***

$x = \frac{\pi}{4}$ ***A1***

Note: Award ***A0*** if solutions other than $x = \frac{\pi}{4}$ are included.

[5 marks]

Consider the functions $f\left(x\right) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

4a. Find $(f \circ g)(x)$. *[2 marks]*

Markscheme

$(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

4b. Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. *[5 marks]*

Markscheme

$\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognising to use $\tan$ or $\cot$ ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) ***(A1)***

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) =\right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6},\ \frac{7\pi}{6}$

$x = \frac{\pi}{12},\ \frac{7\pi}{12}$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.
Award ***A1A0*** for correct answers in degrees.
Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

5a. Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. *[2 marks]*

Markscheme

Note: Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS $= 2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x) =$ RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

5b. Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. *[6 marks]*

Markscheme

attempt to factorise ***M1***

$(\cos x - \sin x)(2\sin x + 1) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \frac{\sin x}{\cos x} = \tan x = 1$ OR $\sin x = -\frac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\frac{\pi}{4}$ OR $\frac{\pi}{6}$ (accept $-\frac{\pi}{6}, \ \frac{7\pi}{6}$)

Note: This ***(M1)(A1)*** is independent of the previous ***M1A1***.

$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \ \frac{\pi}{4}, \ \frac{5\pi}{4}$ ***A2***

Note: Award ***A1*** for any two correct (radian) answers.
Award ***A1A0*** if additional values given with the four correct (radian) answers.
Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

6a. Show that the equation $2\cos^2 x + 5\sin x = 4$ may be written in the form $2\sin^2 x - 5\sin x + 2 = 0$. *[1 mark]*

Markscheme

METHOD 1

correct substitution of $\cos^2 x = 1 - \sin^2 x$ ***A1***

$2(1 - \sin^2 x) + 5\sin x = 4$

$2\sin^2 x - 5\sin x + 2 = 0$ ***AG***

METHOD 2

correct substitution using double-angle identities ***A1***

$(2\cos^2 x - 1) + 5\sin x = 3$

$1 - 2\sin^2 x - 5\sin x = 3$

$2\sin^2 x - 5\sin x + 2 = 0$ ***AG***

[1 mark]

6b. Hence, solve the equation $2\cos^2 x + 5\sin x = 4, 0 \leq x \leq 2\pi$. *[5 marks]*

Markscheme

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{4} \left(= \frac{5 \pm 3}{4}\right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6}, \ \frac{5\pi}{6}$ ***A1A1***

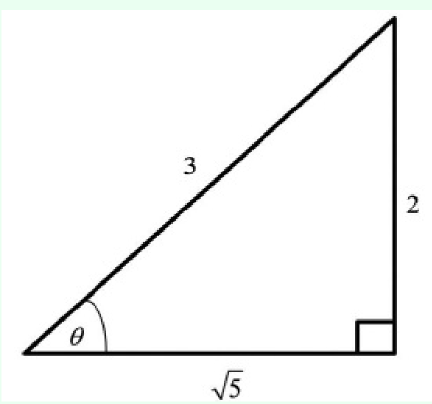
[5 marks]

7. It is given that $\operatorname{cosec} \theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot \theta$. *[4 marks]*

Markscheme

METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot \theta < 0$ (since $\operatorname{cosec} \theta > 0$ puts θ in the second quadrant) ***R1***

$\cot \theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot \theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given

as a final answer.

METHOD 2

Attempt to use $1+\cot^2\theta = \text{cosec}^2\theta$ ***M1***

$1+\cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\text{cosec}\,\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\text{cosec}\,\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

8. Solve the equation $2\cos^2 x + 5\sin x = 4, 0 \le x \le 2\pi$. *[7 marks]*

Markscheme

attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4\times 2\times 2}}{4} \left(= \frac{5\pm 3}{4}\right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6},\ \frac{5\pi}{6}$ ***A1A1***

[7 marks]

9. Let $f(x) = 4\cos\left(\frac{x}{2}\right) + 1$, for $0 \leqslant x \leqslant 6\pi$. Find the values of x for which $f(x) > 2\sqrt{2} + 1$. *[8 marks]*

Markscheme

METHOD 1 – FINDING INTERVALS FOR *x*

$4\cos\left(\frac{x}{2}\right) + 1 > 2\sqrt{2} + 1$

correct working ***(A1)***

eg $4\cos\left(\frac{x}{2}\right) = 2\sqrt{2},\ \cos\left(\frac{x}{2}\right) > \frac{\sqrt{2}}{2}$

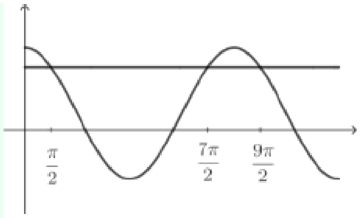
recognizing $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ ***(A1)***

one additional correct value for $\frac{x}{2}$ (ignoring domain and equation/inequalities) ***(A1)***

eg $-\frac{\pi}{4},\ \frac{7\pi}{4},\ 315^\circ,\ \frac{9\pi}{4},\ -45^\circ,\ \frac{15\pi}{4}$

three correct values for x ***A1A1***

eg $\frac{\pi}{2},\ \frac{7\pi}{2},\ \frac{9\pi}{2}$

valid approach to find intervals ***(M1)*** *eg* 

correct intervals (must be in radians) ***A1A1*** ***N2***

$0 \leqslant x < \frac{\pi}{2},\ \frac{7\pi}{2} < x < \frac{9\pi}{2}$

Note: If working shown, award ***A1A0*** if inclusion/exclusion of endpoints is incorrect. If no working shown award ***N1***.
If working shown, award ***A1A0*** if both correct intervals are given, **and** additional intervals are given. If no working shown award ***N1***.
Award ***A0A0*** if inclusion/exclusion of endpoints are incorrect **and** additional intervals are given.

METHOD 2 – FINDING INTERVALS FOR $\frac{x}{2}$

$4\cos\left(\frac{x}{2}\right)+1 > 2\sqrt{2}+1$

correct working ***(A1)***

eg $4\cos\left(\frac{x}{2}\right)= 2\sqrt{2},\ \cos\left(\frac{x}{2}\right)> \frac{\sqrt{2}}{2}$

recognizing $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ ***(A1)***

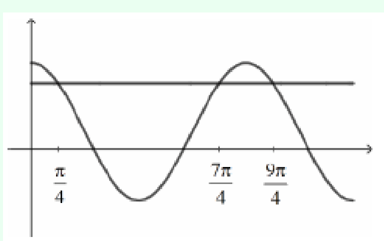
one additional correct value for $\frac{x}{2}$ (ignoring domain and equation/inequalities) ***(A1)***

eg $-\frac{\pi}{4},\ \frac{7\pi}{4},\ 315^{\circ},\ \frac{9\pi}{4},\ -45^{\circ},\ \frac{15\pi}{4}$

three correct values for $\frac{x}{2}$ ***A1***

eg $\frac{\pi}{4},\ \frac{7\pi}{4},\ \frac{9\pi}{4}$

valid approach to find intervals ***(M1)***

eg 

one correct interval for $\frac{x}{2}$ ***A1***

eg $0 \leqslant \frac{x}{2} < \frac{\pi}{4},\ \frac{7\pi}{4} < \frac{x}{2} < \frac{9\pi}{4}$

correct intervals (must be in radians) ***A1A1*** ***N2***

$0 \leqslant x < \frac{\pi}{2},\ \frac{7\pi}{2} < x < \frac{9\pi}{2}$

Note: If working shown, award ***A1A0*** if inclusion/exclusion of endpoints is

incorrect. If no working shown award ***N1***.
If working shown, award ***A1A0*** if both correct intervals are given, **and** additional intervals are given. If no working shown award ***N1***.
Award ***A0A0*** if inclusion/exclusion of endpoints are incorrect **and** additional intervals are given.

[8 marks]

10. A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$. *[7 marks]*

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$.

Markscheme

attempt to use $\cos(2A + B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) ***M1***

attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A \left(= \sqrt{1 - \frac{4}{9}}\right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B \left(= \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3}\right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A \left(= 2\cos^2 A - 1\right) = -\frac{1}{9}$ ***A1***

$\sin 2A \left(= 2 \sin A \cos A\right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A + B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

11. Given that $\sin x = \frac{1}{3}$, where $0 < x < \frac{\pi}{2}$, find the value of $\cos 4x$. *[6 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

correct substitution into formula for $\cos\left(2x\right)$ or $\sin\left(2x\right)$ ***(A1)***

eg $1-2\left(\frac{1}{3}\right)^2,\ 2\left(\frac{\sqrt{8}}{3}\right)^2-1,\ 2\left(\frac{1}{3}\right)\left(\frac{\sqrt{8}}{3}\right),\ \left(\frac{\sqrt{8}}{3}\right)^2-\left(\frac{1}{3}\right)^2$

$\cos\left(2x\right)=\frac{7}{9}$ or $\sin\left(2x\right)=\frac{2\sqrt{8}}{9}\ \left(=\frac{\sqrt{32}}{9}=\frac{4\sqrt{2}}{9}\right)$ (may be seen in substitution) ***A2***

recognizing $4x$ is double angle of $2x$ (seen anywhere) ***(M1)***

eg $\cos\left(2\left(2x\right)\right),\ 2\cos^2\left(2\theta\right)-1,\ 1-2\sin^2\left(2\theta\right),\ \cos^2\left(2\theta\right)-\sin^2\left(2\theta\right)$

correct substitution of **their** value of $\cos\left(2x\right)$ and/or $\sin\left(2x\right)$ into formula for $\cos\left(4x\right)$ ***(A1)***

eg $2\left(\frac{7}{9}\right)^2-1,\ \frac{98}{81}-1,\ 1-2\left(\frac{2\sqrt{8}}{9}\right)^2,\ 1-\frac{64}{81},\ \left(\frac{7}{9}\right)^2-\left(\frac{2\sqrt{8}}{9}\right)^2,\ \frac{49}{81}-\frac{32}{81}$

$\cos\left(4x\right)=\frac{17}{81}$ ***A1 N2***

METHOD 2

recognizing $4x$ is double angle of $2x$ (seen anywhere) ***(M1)***

eg $\cos\left(2\left(2x\right)\right)$

double angle identity for $2x$ ***(M1)***

eg $2\cos^2\left(2\theta\right)-1,\ 1-2\sin^2\left(2x\right),\ \cos^2\left(2\theta\right)-\sin^2\left(2\theta\right)$

correct expression for $\cos\left(4x\right)$ in terms of $\sin x$ and/or $\cos x$ ***(A1)***

eg $2\left(1-2\sin^2\theta\right)^2-1,\ 1-2\left(2\sin x\cos x\right)^2,$
$\left(1-2\sin^2\theta\right)^2-\left(2\sin\theta\cos\theta\right)^2$

correct substitution for $\sin x$ and/or $\cos x$ ***A1***

eg $2\left(1-2\left(\frac{1}{3}\right)^2\right)^2-1,\ 2\left(1-4\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^4\right)-1,\ 1-2\left(2\times\frac{1}{3}\times\frac{\sqrt{8}}{3}\right)^2$

correct working ***(A1)***

eg $2\left(\frac{49}{81}\right)-1,\ 1-2\left(\frac{32}{81}\right),\ \frac{49}{81}-\frac{32}{81}$

$\cos\left(4x\right)=\frac{17}{81}$ ***A1 N2***

[6 marks]

12. Let $f(x) = \tan(x + \pi)\cos\left(x - \frac{\pi}{2}\right)$ where $0 < x < \frac{\pi}{2}$. *[5 marks]*

Express $f(x)$ in terms of $\sin x$ and $\cos x$.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\tan(x + \pi) = \tan x \left(= \frac{\sin x}{\cos x}\right)$ ***(M1)A1***

$\cos\left(x - \frac{\pi}{2}\right) = \sin x$ ***(M1)A1***

Note: The two ***M1***s can be awarded for observation or for expanding.

$\tan(x + \pi) = \cos\left(x - \frac{\pi}{2}\right) = \frac{\sin^2 x}{\cos x}$ ***A1***

[5 marks]

13. Solve $\log_2(2\sin x) + \log_2(\cos x) = -1$, for $2\pi < x < \frac{5\pi}{2}$. *[7 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

correct application of $\log a + \log b = \log ab$ ***(A1)***

eg $\log_2(2\sin x\cos x),\ \log 2 + \log(\sin x) + \log(\cos x)$

correct equation without logs ***A1***

eg $2\sin x\cos x = 2^{-1},\ \sin x\cos x = \frac{1}{4},\ \sin 2x = \frac{1}{2}$

recognizing double-angle identity (seen anywhere) ***A1***

eg $\log(\sin 2x),\ 2\sin x\cos x = \sin 2x,\ \sin 2x = \frac{1}{2}$

evaluating $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}\ (30°)$ ***(A1)***

correct working ***A1***

eg $x = \frac{\pi}{12} + 2\pi,\ 2x = \frac{25\pi}{6},\ \frac{29\pi}{6},\ 750°,\ 870°,\ x = \frac{\pi}{12}$ **and** $x = \frac{5\pi}{12}$, one correct final answer

$x = \frac{25\pi}{12},\ \frac{29\pi}{12}$ (do not accept additional values) ***A2*** ***N0***

[7 marks]

14. Solve the equation $\sec^2 x + 2\tan x = 0,\ 0 \leqslant x \leqslant 2\pi$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

use of $\sec^2 x = \tan^2 x + 1$ ***M1***

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)^2 = 0$ ***(M1)***

$\tan x = -1$ ***A1***

$x = \frac{3\pi}{4},\ \frac{7\pi}{4}$ ***A1A1***

METHOD 2

$\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x} = 0$ ***M1***

$1 + 2\sin x\cos x = 0$

$\sin 2x = -1$ ***M1A1***

$2x = \frac{3\pi}{2},\ \frac{7\pi}{2}$

$x = \frac{3\pi}{4},\ \frac{7\pi}{4}$ ***A1A1***

Note: Award ***A1A0*** if extra solutions given or if solutions given in degrees (or both).

[5 marks]

15a. Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. *[1 mark]*

Markscheme

stating the relationship between $\cot$ and $\tan$ and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \frac{1}{\tan 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

15b. Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)\,x - 1 = 0$. *[7 marks]*

Markscheme

METHOD 1

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

LHS = $\tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0$(= RHS) ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

LHS = $\cot^2\theta - 2\cot\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \frac{1}{\tan^2\theta} - \left(\frac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\theta} + 1 - 1 = 0$(= RHS) ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \frac{1}{\tan\theta}$

$= \frac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

= −1 ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and −1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

15c. Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. *[5 marks]*

Markscheme

METHOD 1

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

15d. Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$. *[6 marks]*

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$.

Markscheme

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2-\sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]