Homework 19.01 *[56 marks]*

The functions f and g are defined such that $f\left(x\right)=\frac{x+3}{4}$ and $g\left(x\right)=8x+5$.

1a. Show that $\left(g\circ f\right)\left(x\right)=2x+11$. *[2 marks]*

Markscheme

attempt to form composition ***M1***

correct substitution $g\left(\frac{x+3}{4}\right)=8\left(\frac{x+3}{4}\right)+5$ ***A1***

$\left(g\circ f\right)\left(x\right)=2x+11$ ***AG***

[2 marks]

1b. Given that $\left(g\circ f\right)^{-1}\left(a\right)=4$, find the value of a. *[3 marks]*

Markscheme

attempt to substitute 4 (seen anywhere) ***(M1)***

correct equation $a=2\times 4+11$ ***(A1)***

a = 19 ***A1***

[3 marks]

2. The functions f and g are defined for $x\in\mathbb{R}$ by $f(x)=x-2$ and $g(x)=ax+b$, where $a,\ b\in\mathbb{R}$. *[6 marks]*

Given that $(f\circ g)(2)=-3$ and $(g\circ f)(1)=5$, find the value of a and the value of b.

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(f \circ g)(x) = ax + b - 2$ **(M1)**

$(f \circ g)(2) = -3 \Rightarrow 2a + b - 2 = -3 (2a + b = -1)$ **A1**

$(g \circ f)(x) = a(x - 2) + b$ **(M1)**

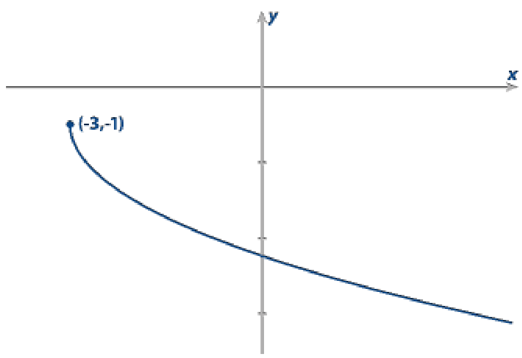
$(g \circ f)(1) = 5 \Rightarrow -a + b = 5$ **A1**

a valid attempt to solve their two linear equations for a and b **M1**

so $a = -2$ and $b = 3$ **A1**

[6 marks]

The following diagram shows the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$.



3a. Describe a sequence of transformations that transforms the graph of $y = \sqrt{x}$ for $x \geq 0$ to the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$. *[3 marks]*

Markscheme

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for example,

a reflection in the x-axis (in the line $y = 0$) **A1**

a horizontal translation (shift) 3 units to the left **A1**

a vertical translation (shift) down by 1 unit **A1**

Note: Award **A1** for each correct transformation applied in a correct position in the sequence. Do not accept use of the "move" for a translation.

Note: Award **A1A1A1** for a correct alternative sequence of transformations. For example,

a vertical translation (shift) down by 1 unit, followed by a horizontal translation (shift) 3 units to the left and then a reflection in the line $y = -1$.

[3 marks]

A function f is defined by $f(x) = -1 - \sqrt{x+3}$ for $x \geq -3$.

3b. State the range of f. *[1 mark]*

Markscheme

range is $f(x) \leq -1$ **A1**

Note: Correct alternative notations include $]-\infty, -1]$, $(-\infty, -1]$ or $y \leq -1$.

[1 mark]

3c. Find an expression for $f^{-1}(x)$, stating its domain. *[5 marks]*

Markscheme

$-1-\sqrt{y+3}=x$ **M1**

Note: Award **M1** for interchanging x and y (can be done at a later stage).

$\sqrt{y+3}=-x-1(=-(x+1))$ **A1**

$y+3=(x+1)^2$ **A1**

so $f^{-1}(x)=(x+1)^2-3\left(f^{-1}(x)=x^2+2x-2\right)$ **A1**

domain is $x\le -1$ **A1**

Note: Correct alternative notations include $]-\infty,-1]$ or $(-\infty,-1]$.

[5 marks]

3d. Find the coordinates of the point(s) where the graphs of $y=f(x)$ and $y=f^{-1}(x)$ intersect. *[5 marks]*

Markscheme

the point of intersection lies on the line $y = x$

EITHER

$(x+1)^2 - 3 = x$ **M1**

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

OR

$-1 - \sqrt{x+3} = x$ **M1**

$\left(-1 - \sqrt{x+3}\right)^2 = x^2 \Rightarrow 2\sqrt{x+3} + x + 4 = x^2$

substitutes $2\sqrt{x+3} = -2(x+1)$ to obtain $-2(x+1) + x + 4 = x^2$

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

THEN

$x = -2, 1$ **A1**

as $x \le -1$, the only solution is $x = -2$ **R1**

so the coordinates of the point of intersection are $(-2,\ -2)$ **A1**

Note: Award **R0A1** if $(-2,\ -2)$ is stated without a valid reason given for rejecting $(1,\ 1)$.

[5 marks]

The following table shows values of $f(x)$ and $g(x)$ for different values of x.

Both f and g are one-to-one functions.

x	-2	0	3	4
$f(x)$	8	4	0	-3
$g(x)$	-5	-2	4	0

4a. Find $g(0)$. *[1 mark]*

Markscheme

$g(0) = -2$ ***A1***

[1 mark]

4b. Find $(f \circ g)(0)$. *[2 marks]*

Markscheme

evidence of using composite function ***(M1)***

$f(g(0))$ OR $f(-2)$

$(f \circ g)(0) = 8$ ***A1***

[2 marks]

4c. Find the value of x such that $f(x) = 0$. *[2 marks]*

Markscheme

$x = 3$ ***A2***

[2 marks]

A function f is defined by $f(x)= \frac{2x-1}{x+1}$, where $x \in \mathbb{R},\ x \neq -1$.

The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

5a. Write down the equation of the vertical asymptote. *[1 mark]*

Markscheme

$x = -1$ ***A1***

[1 mark]

5b. Write down the equation of the horizontal asymptote. *[1 mark]*

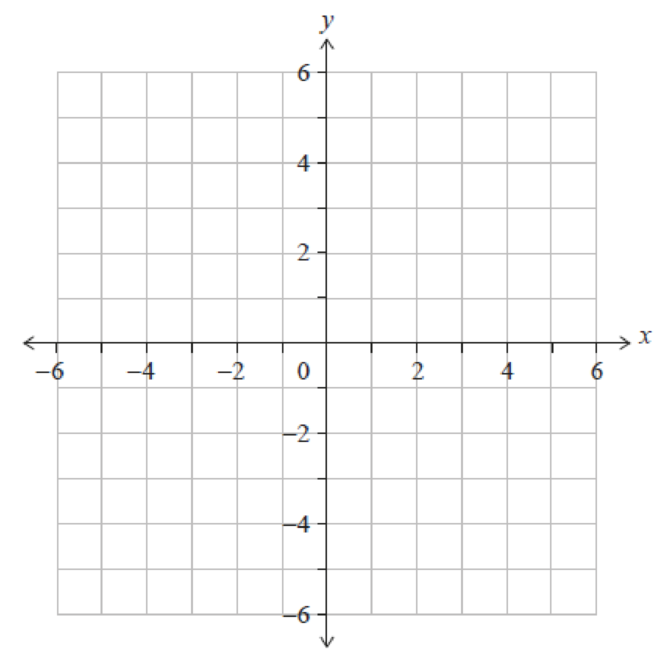
Markscheme

$y = 2$ ***A1***

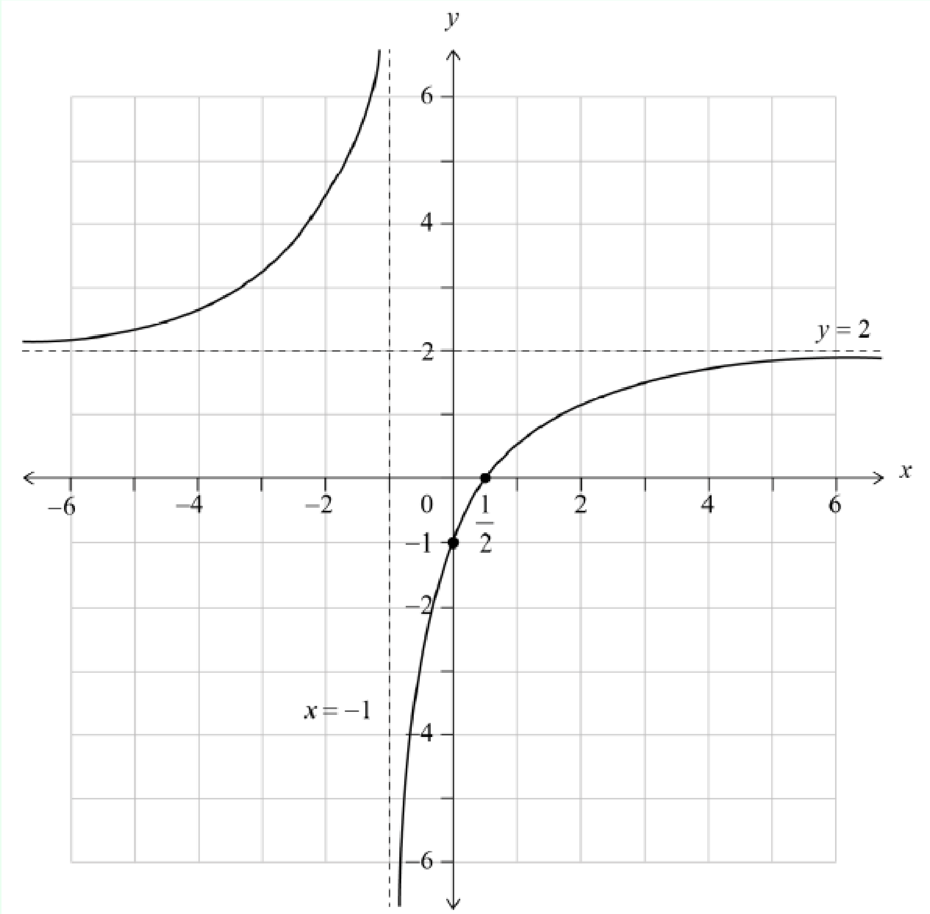
[1 mark]

5c. On the set of axes below, sketch the graph of $y = f(x)$. *[3 marks]*

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.



Markscheme



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

Note: The equations of the asymptotes are not required on the graph provided there is a clear indication of asymptotic behaviour at $x = -1$ and $y = 2$ (or at their FT asymptotes from part (a)).

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$ ***A1A1***

[3 marks]

5d. Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. *[1 mark]*

Markscheme

$x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2},\ \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

The function f is defined by $f(x) = \frac{2x+4}{3-x}$, where $x \in \mathbb{R},\ x \neq 3$.

Write down the equation of

6a. the vertical asymptote of the graph of f. *[1 mark]*

Markscheme

$x = 3$ ***A1***

[1 mark]

6b. the horizontal asymptote of the graph of f. *[1 mark]*

Markscheme

$y = -2$ ***A1***

[1 mark]

Find the coordinates where the graph of f crosses

6c. the x-axis. *[1 mark]*

Markscheme

$(-2,\ 0)$ (accept $x=-2$) ***A1***

[1 mark]

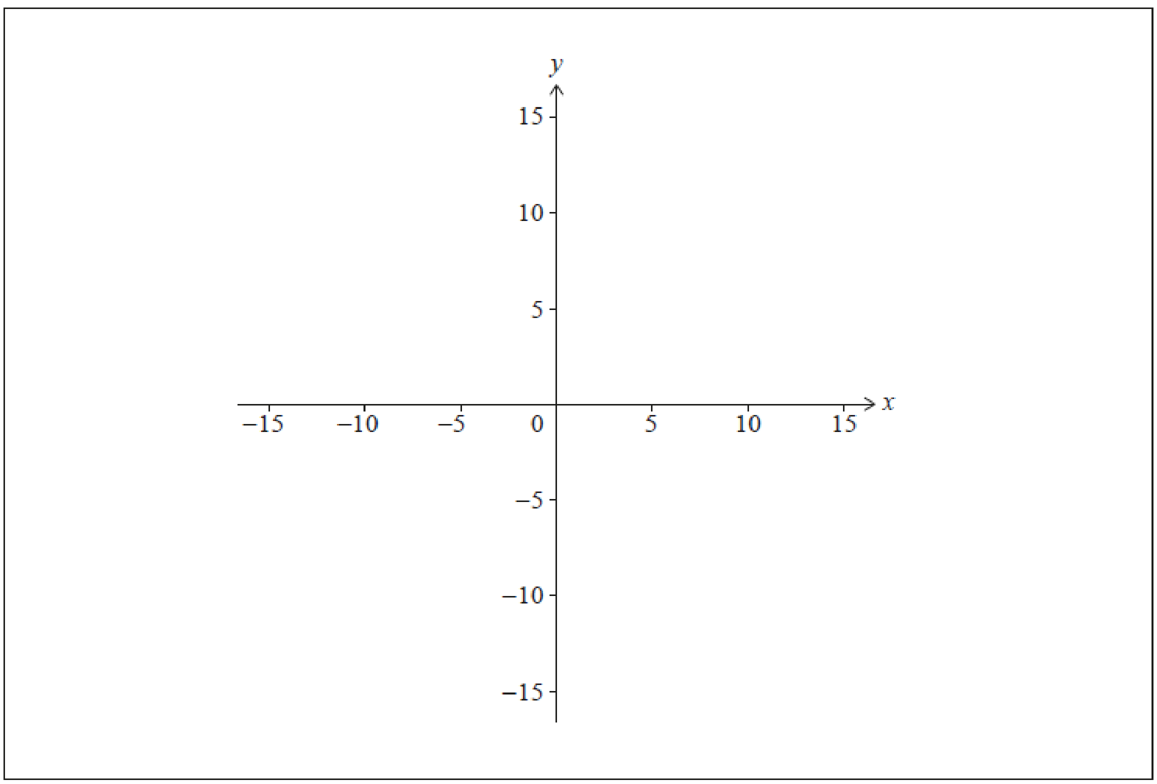
6d. the y-axis. *[1 mark]*

Markscheme

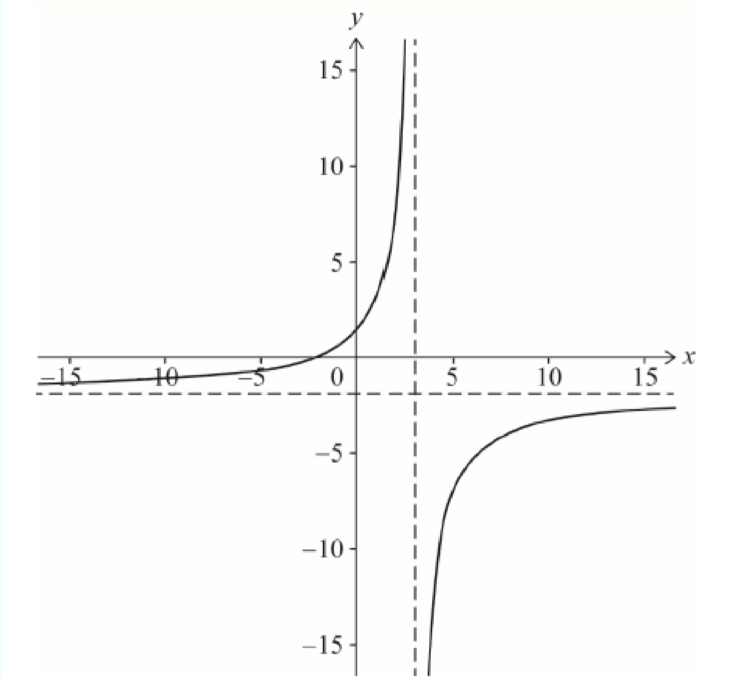
$\left(0,\ \frac{4}{3}\right)$ (accept $y=\frac{4}{3}$ and $f(0)=\frac{4}{3}$) ***A1***

[1 mark]

6e. Sketch the graph of f on the axes below. *[1 mark]*



Markscheme



A1

Note: Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

7a. Find the range of f. *[2 marks]*

Markscheme

attempting to find the vertex ***(M1)***

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ ***A1***

[2 marks]

7b. Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. *[4 marks]*

Markscheme

METHOD 1

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \left(= -\left(6(x-1)^2 - 5\right) + c\right)$ ***(A1)***

EITHER

relating to the range of f OR attempting to find $g(-5)$ ***(M1)***

$5 + c \leq 0$ ***(A1)***

OR

attempting to find the discriminant of $(g \circ f)(x)$ ***(M1)***

$144 + 24(c-1) \leq 0 (120 + 24c \leq 0)$ ***(A1)***

THEN

$c \leq -5$ ***A1***

METHOD 2

vertical reflection followed by vertical shift ***(M1)***

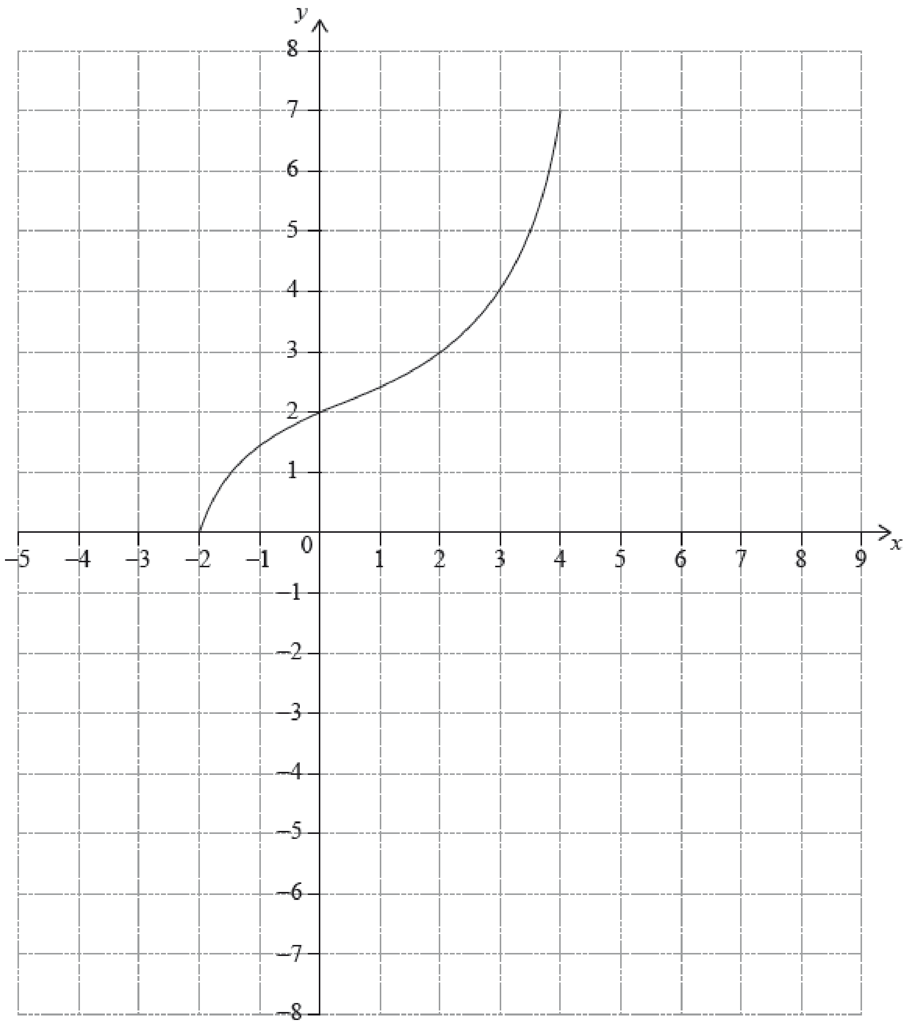
new vertex is $(1,\ 5+c)$ ***(A1)***

$5 + c \leq 0$ ***(A1)***

$c \leq -5$ ***A1***

[4 marks]

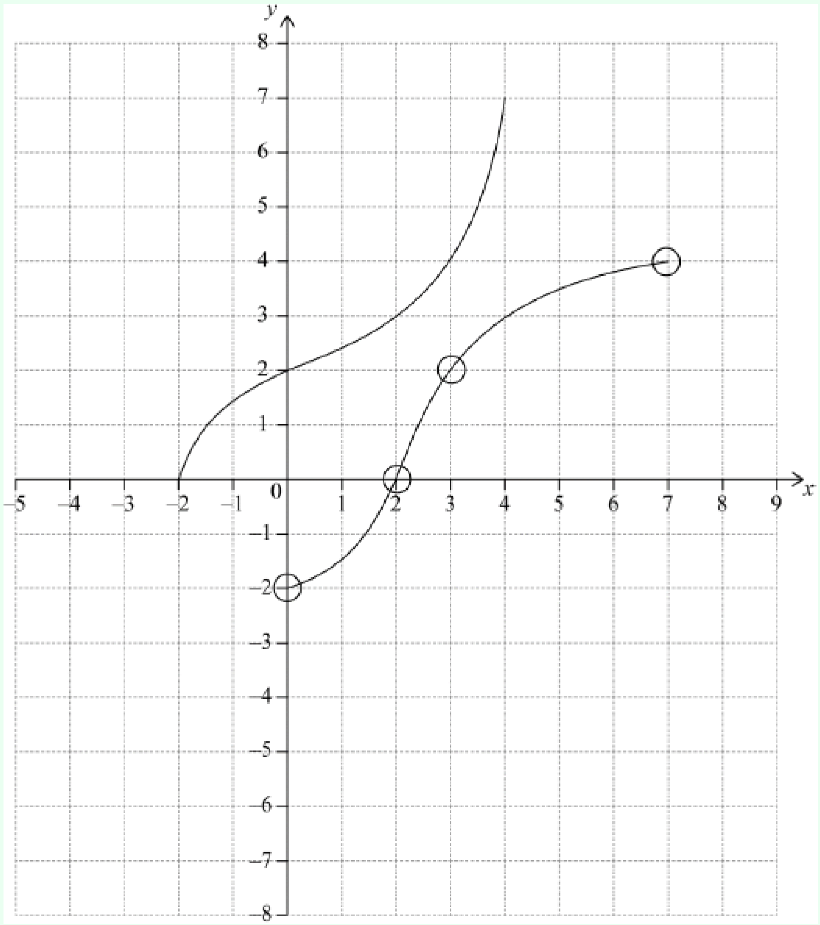
The following diagram shows the graph of a function f, with domain $-2 \leqslant x \leqslant 4$.



The points $(-2, 0)$ and $(4, 7)$ lie on the graph of f.

8. On the grid, sketch the graph of f^{-1}. *[3 marks]*

Markscheme



A1A1A1 ***N3***

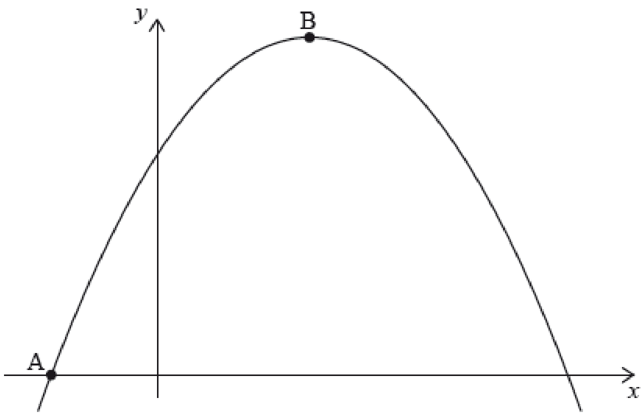
Notes: Award ***A1*** for both end points within circles,

A1 for images of $(2,\ 3)$ and $(0,\ 2)$ within circles,

A1 for approximately correct reflection in $y = x$, concave up then concave down shape (do not accept line segments).

[3 marks]

The graph of the quadratic function $f(x) = c + bx - x^2$ intersects the x-axis at the point $\mathrm{A}(-1, 0)$ and has its vertex at the point $\mathrm{B}(3, 16)$.



9a. Write down the equation of the axis of symmetry for this graph. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$x = 3$ ***(A1)(A1)*** ***(C2)***

Note: Award ***(A1)*** for $x =$ constant, ***(A1)*** for the constant being 3.

The answer must be an equation.

[2 marks]

9b. Find the value of b. *[2 marks]*

Markscheme

$\frac{-b}{2(-1)} = 3$ ***(M1)***

Note: Award ***(M1)*** for correct substitution into axis of symmetry formula.

OR

$b - 2x = 0$ ***(M1)***

Note: Award ***(M1)*** for correctly differentiating and equating to zero.

OR

$c + b(-1) - (-1)^2 = 0$ (or equivalent)

$c + b(3) - (3)^2 = 16$ (or equivalent) ***(M1)***

Note: Award ***(M1)*** for correct substitution of $(-1,\ 0)$ and $(3,\ 16)$ in the original quadratic function.

$(b =)\ 6$ ***(A1)*****(ft)** ***(C2)***

Note: Follow through from part (a).

[2 marks]

9c. Write down the range of $f(x)$. *[2 marks]*

Markscheme

$(-\infty,\ 16]$ **OR** $]-\infty,\ 16]$ ***(A1)(A1)***

Note: Award ***(A1)*** for two correct interval endpoints, ***(A1)*** for left endpoint excluded **and** right endpoint included.

[2 marks]