Proofs 27.01 *[54 marks]*

1. Consider two consecutive positive integers, n and $n+1$. *[4 marks]*

Show that the difference of their squares is equal to the sum of the two integers.

Markscheme

attempt to subtract squares of integers ***(M1)***

$(n+1)^2 - n^2$

EITHER

correct order of subtraction and correct expansion of $(n+1)^2$, seen anywhere ***A1A1***

$= n^2 + 2n + 1 - n^2 \left(= 2n + 1\right)$

OR

correct order of subtraction and correct factorization of difference of squares ***A1A1***

$= (n+1-n)(n+1+n)(= 2n+1)$

THEN

$= n + n + 1 = \text{RHS}$ ***A1***

Note: Do not award final ***A1*** unless all previous working is correct.

which is the sum of n and $n+1$ ***AG***

Note: If expansion and order of subtraction are correct, award full marks for candidates who find the sum of the integers as $2n+1$ and then show that the difference of the squares (subtracted in the correct order) is $2n+1$.

[4 marks]

2a. Show that $(2n-1)^2 + (2n+1)^2 = 8n^2 + 2$, where $n \in \mathbb{Z}$. *[2 marks]*

Markscheme

attempting to expand the LHS ***(M1)***

LHS $= \left(4n^2 - 4n + 1\right) + \left(4n^2 + 4n + 1\right)$ ***A1***

$= 8n^2 + 2$ (= RHS) ***AG***

[2 marks]

2b. Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. *[3 marks]*

Markscheme

METHOD 1

recognition that $2n - 1$ and $2n + 1$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$8n^2 + 2 = 2\left(4n^2 + 1\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

METHOD 2

recognition, *eg* that n and $n + 2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$n^2 + (n + 2)^2 = 2\left(n^2 + 2n + 2\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

The first three terms of an arithmetic sequence are $u_1,\ 5u_1 - 8$ and $3u_1 + 8$.

3a. Show that $u_1 = 4$. *[2 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

EITHER

uses $u_2 - u_1 = u_3 - u_2$ **(M1)**

$(5u_1 - 8) - u_1 = (3u_1 + 8) - (5u_1 - 8)$

$6u_1 = 24$ **A1**

OR

uses $u_2 = \frac{u_1 + u_3}{2}$ **(M1)**

$5u_1 - 8 = \frac{u_1 + (3u_1 + 8)}{2}$

$3u_1 = 12$ **A1**

THEN

so $u_1 = 4$ **AG**

[2 marks]

3b. Prove that the sum of the first n terms of this arithmetic sequence is a square number. *[4 marks]*

Markscheme

$d = 8$ **(A1)**

uses $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ **M1**

$S_n = \frac{n}{2}(8 + 8(n-1))$ **A1**

$= 4n^2$

$= (2n)^2$ **A1**

Note: The final **A1** can be awarded for clearly explaining that $4n^2$ is a square number.

so sum of the first n terms is a square number **AG**

[4 marks]

4a. Explain why any integer can be written in the form $4k$ or $4k+1$ or $4k+2$ or $4k+3$, where $k \in \mathbb{Z}$. *[2 marks]*

Markscheme

Upon division by 4 ***M1***

any integer leaves a remainder of 0, 1, 2 or 3. ***R1***

Hence, any integer can be written in the form $4k$ or $4k+1$ or $4k+2$ or $4k+3$, where $k \in \mathbb{Z}$ ***AG***

[2 marks]

4b. Hence prove that the square of any integer can be written in the form $4t$ or $4t+1$, where $t \in \mathbb{Z}^+$. *[6 marks]*

Markscheme

$(4k)^2 = 16k^2 = 4t$ ***M1A1***

$(4k+1)^2 = 16k^2 + 8k + 1 = 4t + 1$ ***M1A1***

$(4k+2)^2 = 16k^2 + 16k + 4 = 4t$ ***A1***

$(4k+3)^2 = 16k^2 + 24k + 9 = 4t + 1$ ***A1***

Hence, the square of any integer can be written in the form $4t$ or $4t+1$, where $t \in \mathbb{Z}^+$. ***AG***

[6 marks]

Consider any three consecutive integers, $n-1$, n and $n+1$.

5a. Prove that the sum of these three integers is always divisible by 3. *[2 marks]*

Markscheme

$(n-1)+n+(n+1)$ ***(A1)***

$=3n$ ***A1***

which is always divisible by 3 ***AG***

[2 marks]

5b. Prove that the sum of the squares of these three integers is never divisible by 3. *[4 marks]*

Markscheme

$(n-1)^2+n^2+(n+1)^2 \quad \left(=n^2-2n+1+n^2+n^2+2n+1\right)$ ***A1***

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept n^2-1 or n^2+1) ***(M1)***

$=3n^2+2$ ***A1***

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 ***R1***

$3n^2$ is divisible by 3 and so $3n^2+2$ is never divisible by 3

OR the first term is divisible by 3, the second is not

OR $3\left(n^2+\frac{2}{3}\right)$ OR $\frac{3n^2+2}{3}=n^2+\frac{2}{3}$

hence the sum of the squares is never divisible by 3 ***AG***

[4 marks]

6. Prove by contradiction that $\log_2 5$ is an irrational number. *[6 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

assume there exist $p,\ q \in \mathbb{N}$ where $q \geq 1$ such that $\log_2 5 = \frac{p}{q}$ **M1A1**

Note: Award **M1** for attempting to write the negation of the statement as an assumption. Award **A1** for a correctly stated assumption.

$\log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}}$ **A1**

$5^q = 2^p$ **A1**

EITHER

5 is a factor of 5^q but not a factor of 2^p **R1**

OR

2 is a factor of 2^p but not a factor of 5^q **R1**

OR

5^q is odd and 2^p is even **R1**

THEN

no $p,\ q \in \mathbb{N}$ (where $q \geq 1$) satisfy the equation $5^q = 2^p$ and this is a contradiction **R1**

so $\log_2 5$ is an irrational number **AG**

[6 marks]

7. Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd. *[6 marks]*

Markscheme

Assume that a and b are both odd. ***M1***

Note: Award ***M0*** for statements such as "let a and b be both odd".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m + 1$ and $b = 2n + 1$ ***A1***

$a^2 + b^2 \equiv (2m+1)^2 + (2n+1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ ***A1***

$= 4\left(m^2 + m + n^2 + n\right) + 2$ ***(A1)***

($4\left(m^2 + m + n^2 + n\right)$ is always divisible by 4) but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2 + b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

Note: Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

8. Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots. *[5 marks]*

Markscheme

METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.

Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

Note: Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

OR

$1 = 2\left(-\alpha^3 - 3\alpha\right)$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so α cannot be an integer ***R1***

Note: Accept '$\Rightarrow 1$ is even which gives a contradiction'.

OR

$\frac{1}{2} = -\alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$-\alpha^3 - 3\alpha$ is is not an integer $\left(= \frac{1}{2}\right)$ and so α cannot be an integer ***R1***

Note: Accept ' $-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ which gives a contradiction'.

OR

$\alpha = -\frac{1}{2\left(\alpha^2+3\right)}$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2\left(\alpha^2+3\right)} \in \mathbb{Z}$ ***R1***

$-\frac{1}{2\left(\alpha^2+3\right)}$ is not an integer and so α cannot be an integer ***R1***

Note: Accept $-\frac{1}{2\left(\alpha^2+3\right)}$ is not an integer which gives a contradiction'.

THEN

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

METHOD 2

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

let $f(x) = 2x^3 + 6x + 1$ (and $f(\alpha) = 0$)

$f'(x) = 6x^2 + 6 > 0$ for all $x \in \mathbb{R} \Rightarrow f$ is a (strictly) increasing function ***M1A1***

$f(0) = 1$ and $f(-1) = -7$ ***R1***

thus $f(x) = 0$ has only one real root between -1 and 0, which gives a contradiction

(or therefore, contradicting the assumption that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Z}$), ***R1***

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

9a. Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. *[2 marks]*

Markscheme

METHOD 1

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to expand $(p+q)^3$ ***M1***

$p^3 + 3p^2q + 3pq^2 + q^3$

$(p+q)^3 - 3pq(p+q) \equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3pq(p+q)$

$\equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3p^2q - 3pq^2$ ***A1***

$\equiv p^3 + q^3$ ***AG***

Note: Condone the use of equals signs throughout.

METHOD 2

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to factorise $(p+q)^3 - 3pq(p+q)$ ***M1***

$\equiv (p+q)\left((p+q)^2 - 3pq\right) \ \left(\equiv (p+q)\left(p^2 - pq + q^2\right)\right)$

$\equiv p^3 - p^2q + pq^2 + p^2q - pq^2 + q^3$ ***A1***

$\equiv p^3 + q^3$ ***AG***

Note: Condone the use of equals signs throughout.

METHOD 3

$p^3 + q^3 \equiv (p+q)^3 - 3pq(p+q)$

attempts to factorise $p^3 + q^3$ ***M1***

$\equiv (p+q)\left(p^2 - pq + q^2\right)$

$\equiv (p+q)\left((p+q)^2 - 3pq\right)$ ***A1***

$\equiv (p+q)^3 - 3pq(p+q)$ ***AG***

Note: Condone the use of equals signs throughout.

[2 marks]

9b. The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β. *[6 marks]*

Consider the equation $x^2 + mx + n = 0$, where $m,\ n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n.

Markscheme

Note: Award a maximum of ***A1M0A0A1M0A0*** for $m = -95$ and $n = 8$ found by using $\alpha,\ \beta = \frac{5 \pm \sqrt{17}}{4}$ $(\alpha,\ \beta = 0.219\ldots,\ 2.28\ldots)$.
Condone, as appropriate, solutions that state but clearly do not use the values of α and β.
Special case: Award a maximum of ***A1M1A0A1M0A0*** for $m = -95$ and $n = 8$ obtained by solving simultaneously for α and β from product of roots and sum of roots equations.

product of roots of $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

$\alpha\beta = \frac{1}{2}$ (seen anywhere) ***A1***

considers $\left(\frac{1}{\alpha^3}\right)\left(\frac{1}{\beta^3}\right)$ by stating $\frac{1}{(\alpha\beta)^3}(= n)$ ***M1***

Note: Award ***M1*** for attempting to substitute their value of $\alpha\beta$ into $\frac{1}{(\alpha\beta)^3}$.

$\frac{1}{(\alpha\beta)^3} = \frac{1}{\left(\frac{1}{2}\right)^3}$

$n = 8$ ***A1***

sum of roots of $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

$\alpha + \beta = \frac{5}{2}$ (seen anywhere) ***A1***

considers $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$ by stating
$\frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$ $\left(\left(\frac{\alpha+\beta}{\alpha\beta}\right)^3 - \frac{3(\alpha+\beta)}{(\alpha\beta)^2}\right)(= -m)$ ***M1***

Note: Award ***M1*** for attempting to substitute their values of $\alpha + b$ and $\alpha\beta$ into their expression. Award ***M0*** for use of $(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ only.

$= \frac{\left(\frac{5}{2}\right)^3 - \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\frac{1}{8}}$ $(= 125 - 30 = 95)$

$m = -95$ ***A1***

$\left(x^2 - 95x + 8 = 0\right)$

[6 marks]