Proofs 27.01 *[54 marks]*

1. Consider two consecutive positive integers, n and $n+1$. *[4 marks]*

   Show that the difference of their squares is equal to the sum of the two integers.

2a. Show that $(2n-1)^2 + (2n+1)^2 = 8n^2 + 2$, where $n \in \mathbb{Z}$. *[2 marks]*

2b. Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. *[3 marks]*

The first three terms of an arithmetic sequence are $u_1,\ 5u_1 - 8$ and $3u_1 + 8$.

3a. Show that $u_1 = 4$. *[2 marks]*

3b. Prove that the sum of the first n terms of this arithmetic sequence is a square number. *[4 marks]*

4a. Explain why any integer can be written in the form $4k$ or $4k+1$ or $4k+2$ or $4k+3$, where $k \in \mathbb{Z}$. *[2 marks]*

4b. Hence prove that the square of any integer can be written in the form $4t$ or $4t+1$, where $t \in \mathbb{Z}^+$. *[6 marks]*

Consider any three consecutive integers, $n-1$, n and $n+1$.

5a. Prove that the sum of these three integers is always divisible by 3. *[2 marks]*

5b. Prove that the sum of the squares of these three integers is never divisible by 3. *[4 marks]*

6. Prove by contradiction that $\log_2 5$ is an irrational number. *[6 marks]*

7. Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd. *[6 marks]*

8. Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots. *[5 marks]*

9a. Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. *[2 marks]*

9b. The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β. *[6 marks]*

Consider the equation $x^2 + mx + n = 0$, where $m,\ n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n.

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