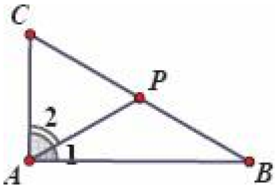
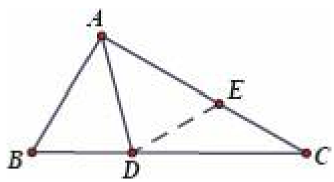
Solutions to Exercises

Chapter 1

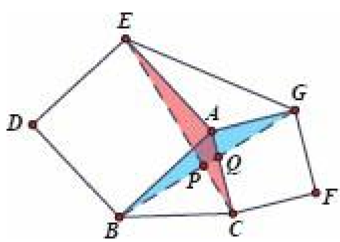
1.1 Since $AP = BP$, we have $\angle 1 = \angle B$. Now $\angle 2 = 90° - \angle 1 = 90° - \angle B = \angle C$, which implies $AP = CP$. The conclusion follows.



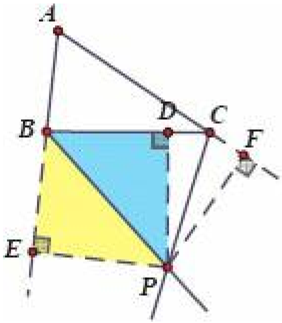
1.2 Choose E on AC such that $AB = AE$ Since AD bisects $\angle BAC$, one sees that $\Delta ABD \cong \Delta AED$ (S.A.S.). Hence, $BD = DE$ and $\angle AED = \angle ABD = 2\angle C$.
Since $\angle AED = \angle C + \angle CDE$, we conclude that $\angle C = \angle CDE$, i.e., $CE = DE$.
Now $CE = DE = BD$. We have $AC = AE + CE = AB + BD$.

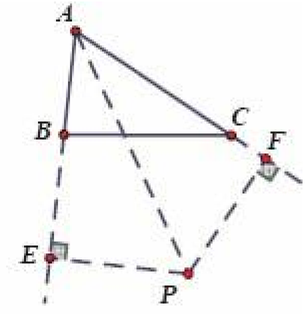


1.3 It is easy to see that $\Delta ACE \cong \Delta AGB$ (S.A.S.). Hence, we have $BG = CE$ and $\angle ACE = \angle AGB$. Let BG and CE intersect at P. Notice that $\angle CPG = \angle CAG = 90°$ (Example 1.1.6) and hence, $BG \perp CE$.



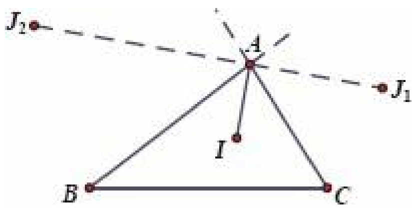
1.4 Refer to the left diagram below. Let BP, CP bisect the exterior angles of $\angle B$, $\angle C$ respectively. We are to show AP bisects $\angle A$. Draw $PD \perp BC$ at D, $PE \perp AB$ at E and $PF \perp AC$ at F. It is easy to see that $\Delta BPE \cong \Delta BPD$ (A.A.S.) and hence, $PD = PE$. Similarly, $PD = PF$.





Now we have $PE = PF$. Refer to the right diagram above. One sees that $\Delta APE \cong \Delta APF$ (H.L.) and hence, AP bisects $\angle A$.

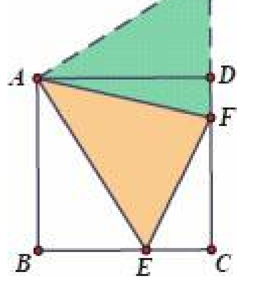
1.5 Connect AJ_1. Since AI and AJ_1 are the angle bisectors of neighboring supplementary angles, we have $AI \perp AJ_1$ (Example 1.1.9, or one may simply see that



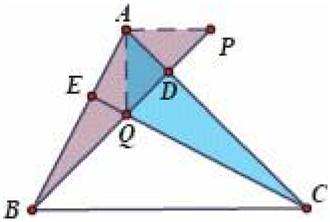
$$\angle IAJ_1 = \angle CAI + \angle CAJ_1 = \frac{1}{2}\angle BAC + \frac{1}{2}(180° - \angle BAC) = 90°\text{ .)}$$

Similarly, $AI \perp AJ_2$. Now J_1AJ_2 = 90° + 90° = 180° which implies A, J_1, J_2 are collinear and hence, $AI \perp J_1J_2$.

1.6 Choose E' on CD extended such that $DE' = BE$. Connect AE' It is easy to see that $\Delta ABE \cong \Delta ADE'$ (S.A.S.). Hence, $AE = AE'$ and $\angle BAE = \angle DAE.'$ Now we see that $\angle EAF = \angle E'AF = 45°$ and $\Delta AEF \cong \Delta AE'F$ (S.A.S.). Hence, $EF = E'F = DF + BE$.

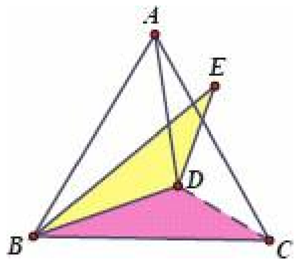


1.7 We have $\angle ABD = \angle ACE = 90° - \angle BAC$. Hence, $\Delta ABP \cong \Delta QCA$ (S.A.S.). It follows that $AQ = AP$ and $\angle QAD = \angle APD = 90° - \angle PAC$, i.e., $\angle QAD + \angle PAC = \angle PAQ = 90°$. Thus, $\angle AQP = 45°$.



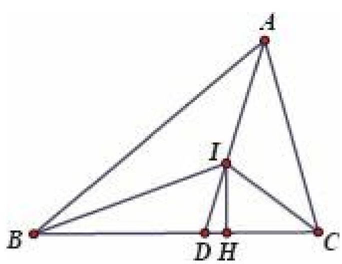
1.8 Connect CD. Since $BE = AB = BC$ and BD bisects $\angle CBE$, we have $\Delta BCD \cong \Delta BED$ (S.A.S.). Hence, $\angle BED = \angle BCD$.

Since $AD = BD$, D (and similarly C) lie on the perpendicular bisector of AB, which is indeed the line CD. It follows that CD bisects $\angle ACB$.

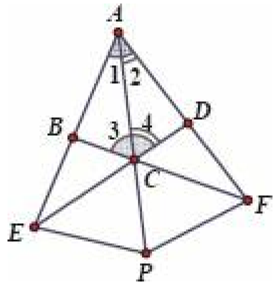


Now $\angle BED = \angle BCD = \frac{1}{2}\angle ACB = 30°$.

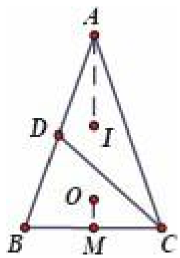
1.9 Since I is the incenter, CI bisects $\angle C$. Theorem 1.3.3 gives $\angle AIB = 90° + \frac{1}{2}\angle C$. Hence, $\angle BID$ = 180°' $\angle AIB = 90° + \frac{1}{2}\angle C$. = 90° – $\angle BCI = \angle CIH$.



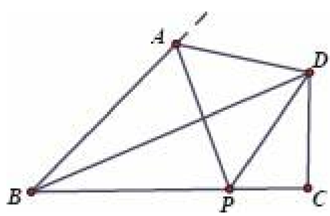
1.10 Since $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, we have $\Delta ABC \cong \Delta ADC$ (A.A.S.). Hence, $AB = AD$ and $\angle ABF = \angle ADE$. Now $\Delta ABF \cong \Delta ADE$ (A.A.S.), which implies $AE = AF$. It follows that $\Delta AEP \cong \Delta AFP$ (S.A.S.) and $PE = PF$. Note that the proof holds regardless of the position of P.



1.11 Let M be the midpoint of BC. Since O is the circumcenter of ΔBCD, OM is the perpendicular bisector of BC. On the other hand, since I is the incenter of ΔACD, AI is the angle bisector $\angle A$, which passes through M since $AB = AC$ Thus, A, I, O lie on the perpendicular bisector of BC. The conclusion follows.

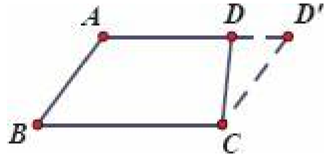


1.12 Let $\angle ABC = 2a$ and $\angle APC = 2\beta$. We have $\angle BAP = \angle APC - \angle ABC = 2(\alpha - \beta)$. Since BD, PD are angle bisectors, we have $\angle CBD = a$ and $\angle CPD = \beta$. It follows that $\angle BDP = \angle CPD - \angle CBD = \alpha - \beta$
Notice that D is the ex-center of ΔABP opposite B (Exercise 1.4), which implies that AD bisects the exterior angle of $\angle BAP$.



N o w $\angle PAD = \frac{1}{2}(180^\circ - \angle BAP) = 90^\circ - \frac{1}{2} \cdot 2(\alpha - \beta) = 90^\circ - \angle BDP$.
This completes the proof.

1.13 Suppose otherwise. Draw CD' // AB, intersecting the line AD at D' Now $ABCD'$ is a parallelogram and $AB = CD'$ $BC = AD'$ We have $AD' - CD' = BC - AB = AD - CD$.

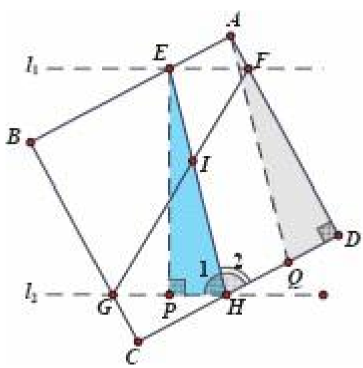


Case I: $AD < AD'$
Refer to the diagram below.
We have $DD' = AD' - AD = CD' - CD$, i.e., $DD' + CD = CD'$ This contradicts triangle inequality.

Case II: $AD > AD'$
Similarly, we have $DD' = AD - AD' = CD - CD'$, i.e., $DD' + CD' = CD$. This contradicts triangle inequality.

It follows that D and D' coincide, i.e., $ABCD$ is a parallelogram.

1.14 Draw $EP \perp \ell_2$ at P and $AQ \,//\, EH$, intersecting CD at Q. It is easy to see that $AEHQ$ is a parallelogram and hence, $EH = AQ$. Given that $EP = AD$, we must have $\Delta EPH \cong \Delta ADQ$ (H.L.). It follows that $\angle 1 = \angle AQD = \angle 2$.
Similarly, we have $\angle BGF = \angle HGF$.



Now $\angle GIH = 180° - \angle HGF - \angle 1$, where $\angle 1 = \frac{1}{2}(180° - \angle CHG)$

$= 90° - \frac{1}{2}\angle CHG$ and similarly, $\angle HGF = 90° - \frac{1}{2}\angle CGH$.

Hence, $\angle GIH = 180° - \left(90° - \frac{1}{2}\angle CHG\right) - \left(90° - \frac{1}{2}\angle CGH\right)$

$= \frac{1}{2}(\angle CGH + \angle CHG) = 45°$, because ΔCGH is a right angled triangle where $\angle C = 90°$.

Note: One may observe that I is the ex-center of ΔCGH opposite C (Exercise