3 Trigonometry

ESSENTIAL UNDERSTANDINGS

- Trigonometry allows us to quantify the physical world.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

- about three new trigonometric functions: sec, cosec and cot
- about inverse trigonometric functions
- how to expand expressions such as $\sin(A + B)$ (compound angle identities)
- the double angle identity for tan.

CONCEPTS

The following concepts will be addressed in this chapter:

- Different **representations** of the values of trigonometric relationships, such as exact or **approximate**, may not be **equivalent** to one another.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Write down the exact value of $\sin\left(\frac{\pi}{3}\right)$.

2 Given that $\sin\theta = \frac{1}{3}$, find the value of $\cos 2\theta$.

3 Solve the equation $3x^2 + 4x - 1 = 0$.

4 Solve the following equations for $0 \leqslant x \leqslant 2\pi$.

a $\sin x = -\cos x$ b $\cos x - \sin^2 x = 1$ c $\cos 3x = \frac{1}{2}$

■ **Figure 3.1** How do we model multifaceted situations using trigonometry?

Trigonometric functions are used to model real life situations where a quantity varies periodically in space or time. In some cases, several different trigonometric functions need to be combined to create a good model. In this chapter, you will learn several new identities which enable you to manipulate such expressions. You will also meet some new trigonometric functions and apply your knowledge of inverse functions to trigonometry.

Starter Activity

Look at the images in Figure 3.1. In small groups, discuss why these situations need to be modelled using a combination of different trigonometric functions. What other practical situations can be modelled in a similar way? What is the effect of combining different trigonometric functions in those situations?

Now look at this problem:

1 Use technology to draw the following graphs.

a $y = 3\sin x + 4\cos x$

b $y = 5\sin x - 2\cos x$

c $y = \cos x - 4\sin x$

Use your knowledge of transformations of graphs to write each expression as a single trigonometric function.

2 Use technology to draw these graphs.

a $y = \sin x + \sin 2x$

b $y = \sin x + \sin 5x$

c $y = \sin 6x + \sin 7x$

Describe how the frequencies of the two sin functions affect the shape of the graph.

LEARNER PROFILE – Knowledgeable

What are the links between mathematics and other subjects? Most people see mathematics used in science, but did you know that in Ancient Greece music was considered a branch of mathematics, just like geometry or statistics are now. At the highest levels, philosophy and mathematics are increasingly intertwined. See if you can find any surprising applications of mathematics in some of your other subjects.

3A Further trigonometric functions

Definition of the reciprocal trigonometric ratios sec θ, cosec θ and cot θ

You already know that $\tan x \equiv \frac{\sin x}{\cos x}$, so you could do all the relevant calculations using just sine and cosine. However, having the notation for $\tan x$ can simplify many expressions. Expressions of the form $\frac{\cos x}{\sin x}\left(\equiv \frac{1}{\tan x}\right)$, $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ also occur frequently, so it can be useful to have notation for these too.

Tip

You may also see cosecant abbreviated to csc instead of cosec.

KEY POINT 3.1

- secant : $\sec x \equiv \frac{1}{\cos x}$
- cosecant : $\operatorname{cosec} x \equiv \frac{1}{\sin x}$
- cotangent : $\cot x \equiv \frac{1}{\tan x} \equiv \frac{\cos x}{\sin x}$

In a right-angled triangle, $\sin\theta$ was defined as $\frac{\text{opposite}}{\text{hypotenuse}}$. There was nothing special about having the ratio this way round. You could have been taught all about right-angled triangles using $\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}$.

TOK Links

Does having names for reciprocals of other functions add to our body of knowledge? Why might it be useful? Why do you think the mathematical community decided to make sine the fundamental function rather than secant?

WORKED EXAMPLE 3.1

Evaluate $\sec\left(\frac{\pi}{3}\right)$.

We need to link this expression to one of the functions we are more familiar with

$$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$$

Remember that by default we are working in radians

$\cos\left(\frac{\pi}{3}\right)$ is one of the exact values you should know

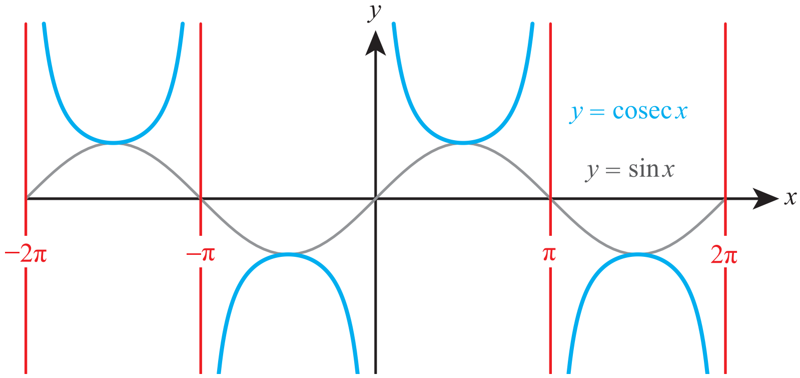
$$= \frac{1}{(1/2)}$$

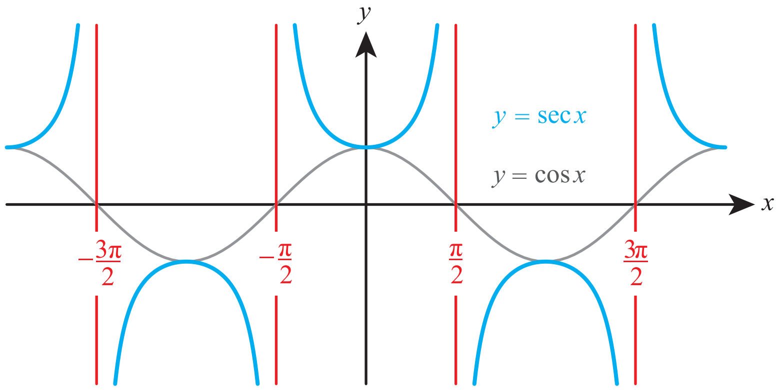
$$= 2$$

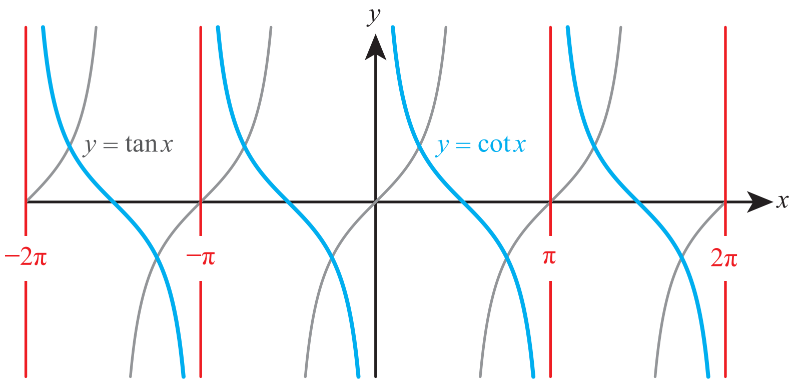
What is the origin of the word 'secant'?

You can use technology to graph these reciprocal functions.

You will see how to relate graphs of $y = \mathrm{f}(x)$ to $y = \dfrac{1}{\mathrm{f}(x)}$ in Chapter 7.







From these graphs you can deduce the range and domain of each function:

Function	Domain	Range
$\mathrm{f}(x) = \operatorname{cosec} x$	$x \neq \left(n + \frac{1}{2}\right)\pi,\ \ n \in \mathbb{Z}$	$\mathrm{f}(x) \geqslant 1$ or $\mathrm{f}(x) \leqslant -1$
$\mathrm{f}(x) = \sec x$	$x \neq n\pi,\ \ n \in \mathbb{Z}$	$\mathrm{f}(x) \geqslant 1$ or $\mathrm{f}(x) \leqslant -1$
$\mathrm{f}(x) = \cot x$	$x \neq n\pi,\ \ n \in \mathbb{Z}$	$\mathrm{f}(x) \in \mathbb{R}$

Tip

$\cot \dfrac{\pi}{2}$ is zero. Why?

WORKED EXAMPLE 3.2

Solve $\cot x = \sqrt{3}$, $0 \leqslant x \leqslant 2\pi$.

Virtually every problem with the reciprocal trigonometric functions involves, at some stage, rewriting the expression in terms of the standard trigonometric functions

$$\frac{1}{\tan x} = \sqrt{3}$$

Rearrange to get the equation with $\tan x$ as the subject

$$\tan x = \frac{1}{\sqrt{3}}$$

This is one of the standard exact values you should know. Do not forget that there can be further solutions too. When solving this type of tan equation, we can always add on π to get more answers

$$x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$

Pythagorean identities

Perhaps the most common usage of these functions is in the following identities, which can be deduced from the familiar $\cos^2 x + \sin^2 x \equiv 1$ by dividing through by $\cos^2 x$ and $\sin^2 x$ respectively.

KEY POINT 3.2

- $1 + \tan^2 x \equiv \sec^2 x$
- $\cot^2 x + 1 \equiv \operatorname{cosec}^2 x$

WORKED EXAMPLE 3.3

If θ is an acute angle and $\tan\theta = \frac{1}{2}$, find the exact value of $\cos\theta$.

We need a link between tan and cos. The identity from Key Point 3.2 involving tan and sec provides this

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\frac{1}{4} + 1 = \sec^2\theta$$

We can take the reciprocal of both sides to get $\cos^2 x$

$$\frac{5}{4} = \sec^2\theta$$

$$\frac{4}{5} = \cos^2\theta$$

Square root both sides

$$\cos\theta = \pm\frac{2}{\sqrt{5}}$$

For all acute angles, cosine is positive

$$\text{Since } \theta \text{ is acute, } \cos\theta = \frac{2}{\sqrt{5}}$$

Tip

The reciprocal trigonometric functions follow the same conventions as the normal trigonometric functions, so $\sec^2 x = (\sec x)^2$.

WORKED EXAMPLE 3.4

Solve the equation $\tan^2 x - 4\sec x + 5 = 0$ for $0 < x < 2\pi$.

We would like to have only one trigonometric function, so use the identity $\tan^2 x + 1 \equiv \sec^2 x$ from Key Point 3.2

$$(\sec^2 x - 1) - 4\sec x + 5 = 0$$

This is a disguised quadratic in $\sec x$. First write in standard quadratic form

$$\sec^2 x - 4\sec x + 4 = 0$$

We could use a substitution $u = \sec x$ to help make it look more familiar

If $u = \sec x$,

$$u^2 - 4u + 4 = 0$$

Factorize (or you could use the quadratic formula)

$$(u - 2)^2 = 0$$

We do not know how to find inverse sec, so express $\sec x$ in terms of $\cos x$

$$u = 2 \text{ so } \sec x = 2 \text{ so } \frac{1}{\cos x} = 2$$

Therefore, $\cos x = \dfrac{1}{2}$

Finally, we solve this trigonometric equation, remembering to give all the possible solutions

$$x = \frac{\pi}{3} \text{ or } 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Inverse trigonometric functions

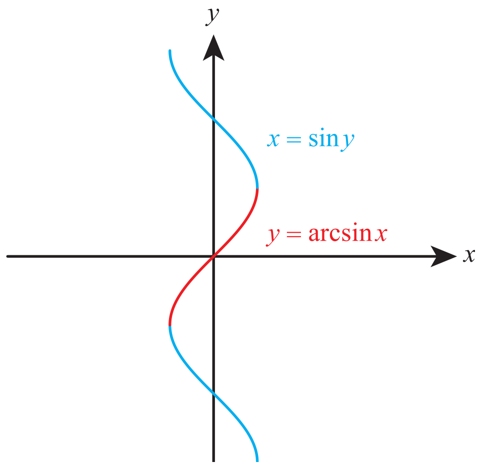
You have already met the inverse sin, cos and tan operations when you used them to solve trigonometric equations. However, in this section we shall now look at them as functions.

The best way to study the inverse sine function is to consider the relationship between functions and their inverses graphically – they are reflections in the line $y = x$.

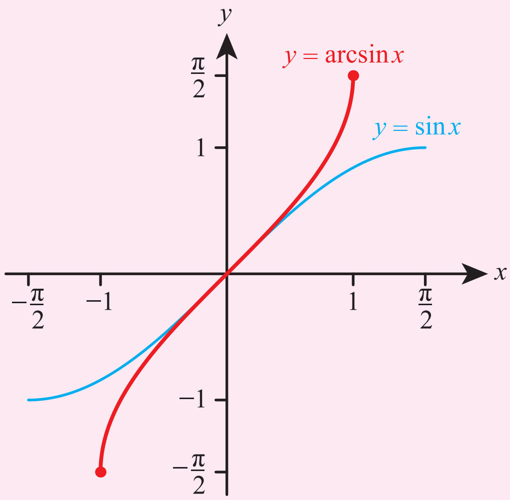
You met this relationship in Chapter 14 of the Mathematics: analysis and approaches SL book.

If the graph of $y = \sin x$ is reflected in the line $y = x$, the result is not a function because each x value corresponds to more than one y value.

However, if the original graph of $y = \sin x$ is restricted to $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$ then when it is reflected in $y = x$, the result (shown in red on the graph) is a function, called $\arcsin x$.



KEY POINT 3.3



The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$ and the range is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

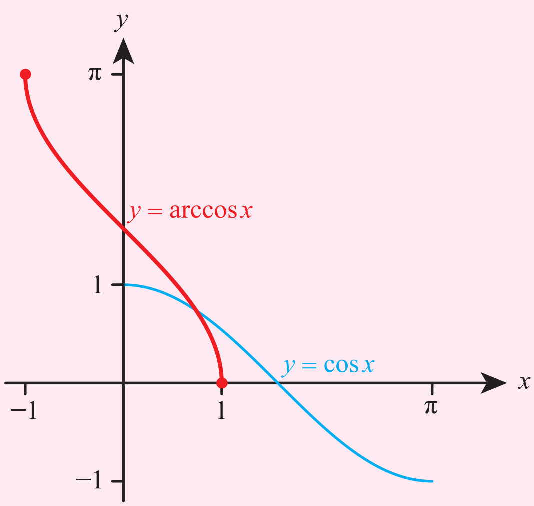
Tip

$\arcsin x$ is also known as $\sin^{-1}x$, especially on calculators. This is not the same as $\frac{1}{\sin x}$.

Remember that the domain is the set of all numbers allowed into a function and the range is the corresponding set of outputs. This was covered in Chapter 3 of the Mathematics: analysis and approaches SL book.

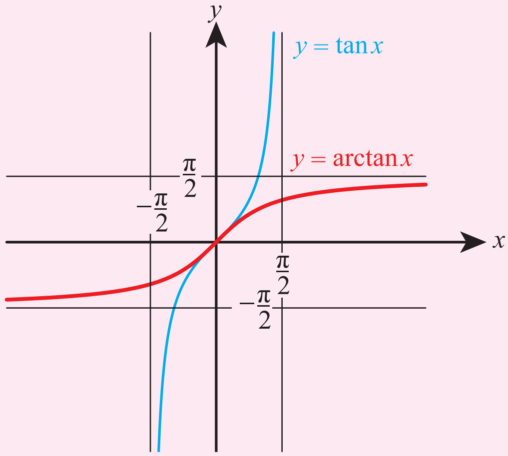
A similar argument leads to inverse functions of the cosine and tangent functions over restricted domains.

KEY POINT 3.4



The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$ and the range is $0 \leqslant y \leqslant \pi$.

KEY POINT 3.5



The domain of $y = \arctan x$ is $x \in \mathbb{R}$ and the range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

WORKED EXAMPLE 3.5

Evaluate $\arccos 0.5$.

$\arccos 0.5$ is equivalent to asking for the solution to $\cos x = 0.5$ with $0 \leqslant x \leqslant \pi$. This is one of the exact values you should know

$$\arccos 0.5 = \frac{\pi}{3}$$

Exact values you should know are covered in Key Point 18.9 of the Mathematics: analysis and approaches SL book.

Once you are familiar with the inverse trigonometric functions, you can apply them in algebraic expressions. People often find this quite tricky – the example below is towards the top end of examination difficulty.

WORKED EXAMPLE 3.6

Simplify $\sin(\arccos x)$ if $-1 \leqslant x \leqslant 1$.

We cannot directly apply sine to an arccosine, so we need to use an identity to apply cosine rather than a sine to $\arccos x$. The appropriate one is $\sin^2\theta + \cos^2\theta \equiv 1$

$$\sin^2(\arccos x) + \cos^2(\arccos x) = 1$$

Since cosine and arccosine are inverse functions, we can use $\cos(\arccos x) \equiv x$

$$\sin^2(\arccos x) + x^2 = 1$$

$$\sin^2(\arccos x) = 1 - x^2$$

So,

$$\sin(\arccos x) = \pm\sqrt{1 - x^2}$$

We then need to decide if there is a reason why we should choose the positive or the negative root. To do this, we need to consider the range of $\arccos x$

The range of $\arccos(x)$ is between 0 and π inclusive. Sine of these values is never negative, so we can exclude the negative root. Therefore,

$$\sin(\arccos x) = \sqrt{1 - x^2}$$

TOOLKIT: Problem Solving

Find the values of x for which it is true that:

a $\sin(\arcsin x) = x$

b $\arcsin(\sin x) = x$

Be the Examiner 3.1

Evaluate $\cos^{-1} 0$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\cos^{-1} 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$	$\cos^{-1} 0 = \frac{\pi}{2}$	$\cos^{-1} 0 = \pi$

Exercise 3A

1 Use the method demonstrated in Worked Example 3.1 to evaluate the following expressions on your calculator. Give your answers to three significant figures.

a i $\sec 2.4$ **b i** $\cot(-1)$ **c i** $\operatorname{cosec}\left(\frac{3\pi}{5}\right)$

ii $\operatorname{cosec} 3$ **ii** $\sec(-2)$ **ii** $\cot\left(\frac{7\pi}{3}\right)$

2 Use the method demonstrated in Worked Example 3.1 to find the exact value of the following expressions.

a i $\operatorname{cosec}\left(\frac{\pi}{2}\right)$ **b i** $\sec(0)$ **c i** $\cot\left(\frac{\pi}{4}\right)$ **d i** $\operatorname{cosec}\left(\frac{3\pi}{2}\right)$

ii $\operatorname{cosec}\left(\frac{\pi}{4}\right)$ **ii** $\sec\left(\frac{\pi}{6}\right)$ **ii** $\cot\left(\frac{\pi}{3}\right)$ **ii** $\cot\left(\frac{\pi}{2}\right)$

3 Use the method demonstrated in Worked Example 3.2 to solve the following equations, giving all answers in the region $0 \leqslant x \leqslant 2\pi$. Give your answer to three significant figures.

a i $\sec x = 2.5$ **b i** $\operatorname{cosec} x = -3$ **c i** $\cot x = 2$ **d i** $\sec 2x = 1.5$

ii $\sec x = 4$ **ii** $\operatorname{cosec} x = 3$ **ii** $\cot x = 0.4$ **ii** $\operatorname{cosec} 2x = 5$

4 Use the method demonstrated in Worked Example 3.2 to solve the following equations, giving all answers in the region $0 \leqslant x \leqslant 2\pi$. Give your answer in an exact form.

a i $\operatorname{cosec}\theta = 2$ **b i** $\cot\theta = \frac{1}{\sqrt{3}}$ **c i** $\sec\theta = 1$ **d i** $\cot\theta = 0$

ii $\operatorname{cosec}\theta = 1$ **ii** $\cot\theta = 1$ **ii** $\sec\theta = \frac{2}{\sqrt{3}}$ **ii** $\cot\theta = -1$

5 Use the method demonstrated in Worked Example 3.3 to find the exact value of the required trigonometric ratio.

a i Given that $\tan\theta = \frac{3}{4}$ and $0 < \theta < \frac{\pi}{2}$, find $\sec\theta$.

ii Given that $\tan\theta = \frac{1}{5}$ and $0 < \theta < \frac{\pi}{2}$, find $\sec\theta$.

b **i** Given that $\operatorname{cosec}\theta = 4$ and $\frac{\pi}{2} < \theta < \pi$, find $\cot\theta$.

ii Given that $\operatorname{cosec}\theta = 7$ and $\frac{\pi}{2} < \theta < \pi$, find $\cot\theta$.

c **i** Given that $\cot\theta = 2$ and $0 < \theta < \pi$, find the exact value of $\sin\theta$.

ii Given that $\cot\theta = 3$ and $\pi < \theta < 2\pi$, find the exact value of $\sin\theta$.

d **i** Given that $\tan\theta = \sqrt{2}$, find the possible values of $\cos\theta$.

ii Given that $\cot\theta = -4$, find the possible values of $\sin\theta$.

6 Use technology to evaluate each of the following, giving your answers in radians to three significant figures.

a **i** $\arcsin 0.8$ **b** **i** $\cos^{-1}(-0.75)$ **c** **i** $\arctan(\pi)$

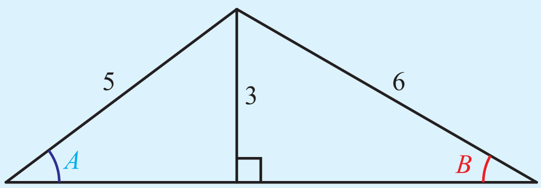
ii $\arcsin(-0.6)$ **ii** $\cos^{-1}(0.01)$ **ii** $\arctan(10)$

7 Use the method demonstrated in Worked Example 3.5 to find the exact value of the given expression. Give your answer in radians.

a **i** $\arccos\frac{\sqrt{3}}{2}$ **b** **i** $\arcsin\frac{1}{2}$ **c** **i** $\arctan\sqrt{3}$ **d** **i** $\arccos(-0.5)$

ii $\arccos 1$ **ii** $\arcsin 1$ **ii** $\arctan 1$ **ii** $\arcsin(-0.5)$

8 Find the values of $\operatorname{cosec} A$ and $\sec B$.



9 Prove that $\sin^2\theta + \cot^2\theta\sin^2\theta = 1$.

10 Solve the equation $\tan x + \sec x = 2$ for $0 \leqslant x \leqslant 2\pi$.

11 A function is defined by $f(x) = \tan x + \operatorname{cosec} x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

a Find the coordinates of the minimum and maximum points on the graph of $y = f(x)$.

b Hence write down the range of f.

12 Show that $\sin A \cot A = \cos A$.

13 Show that $\tan B \operatorname{cosec} B = \sec B$.

14 Evaluate $\arcsin(\sin\pi)$.

15 Sketch $y = \sec 2x$ for $0 \leqslant x \leqslant 2\pi$, labelling all maximum and minimum points.

16 Sketch $y = 3\cot 2x$ for $0 \leqslant x \leqslant \pi$, labelling all asymptotes.

17 Sketch $y = \operatorname{cosec}(x - \pi)$ for $0 \leqslant x \leqslant 2\pi$.

18 Show that $\tan x + \cot x = \sec x \operatorname{cosec} x$.

19 Prove that $\sec x - \cos x \equiv \sin x \tan x$.

20 Prove that $\frac{\sin\theta}{1-\cos\theta} - \frac{\sin\theta}{1+\cos\theta} = 2\cot\theta$.

21 Solve the equation $2\tan^2 x + \frac{3}{\cos x} = 0$ for $-\pi < x < \pi$.

22 How many solutions are there to the equation $\arccos x = 2x$? Justify your answer.

23 Write $\arccos(-x)$ in terms of $\arccos x$.

24 **a** Given that $\sec^2 x - 3\tan x + 1 = 0$, show that $\tan^2 x - 3\tan x + 2 = 0$.

b Find the possible values of $\tan x$.

c Hence solve the equation $\sec^2 x - 3\tan x + 1 = 0$ for $x \in [0, 2\pi]$.

25 Prove that $\operatorname{cosec}(2x) = \dfrac{\sec x \operatorname{cosec} x}{2}$.

26 Show that $\sec(2\theta) = \dfrac{\sec^2\theta}{2-\sec^2\theta}$.

27 Find the inverse function of $\sec x$ in terms of the arccosine function.

28 A straight line has equation $(4\cos\theta)x + (5\sin\theta)y = 20$, where θ is a constant. The line intersects the x-axis at P and the y-axis at Q. The midpoint of PQ is M.

a Show that the coordinates of M are $\left(\frac{5}{2}\sec\theta,\ 2\operatorname{cosec}\theta\right)$.

b Hence show that M lies on the curve with equation $\dfrac{25}{x^2} + \dfrac{16}{y^2} = 4$.

29 a State the largest possible domain and range of the function $f(x) = \arccos(\cos x)$.

b Simplify $\arccos(\cos x)$ for:

i $2\pi \leqslant x \leqslant 3\pi$ ii $\pi \leqslant x \leqslant 2\pi$ iii $-\pi \leqslant x \leqslant 0$.

30 The function $\operatorname{isin} x$ is defined as the inverse function of $f(x) = \sin x$ for $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Write $\operatorname{invsin} x$ in terms of $\arcsin x$.

3B Compound angle identities

Compound angle identities are used to expand expressions such as $\sin(A + B)$ or $\tan(A - B)$. They are derived in a similar way to the double angle identities.

You met the identities for $\sin 2\theta$ and $\cos 2\theta$ in Chapter 18 of the Mathematics: analysis and approaches SL book.

Tip

Notice the signs in the cosine identities: in the identity for the sum you use the minus sign, and in the identity for the difference the plus sign.

Compound angle identities for sin and cos

KEY POINT 3.6

- $\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$

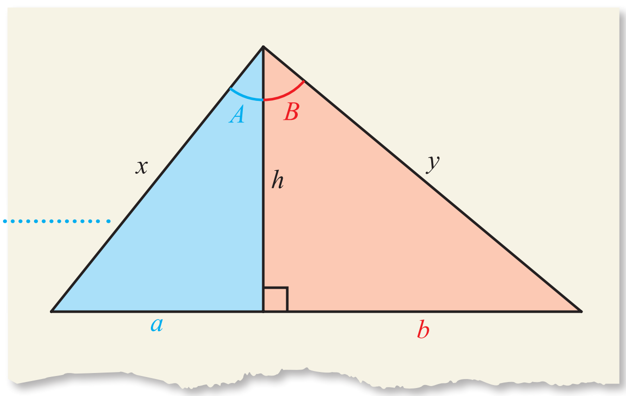
Proof 3.1

Prove the identity $\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$ for $0 < A,\ B < \frac{\pi}{2}$.

To work with trigonometric ratios, it is useful to try to form a diagram containing some right-angled triangles including the given angles

Consider the triangle in the diagram. The angle on top is $A + B$ and the height divides it into two angles of size A and B

Label the sides of the triangle x and y



$\sin(A+B)$ appears in the formula for the area of the triangle

For the whole triangle:

$$\text{Area} = \frac{1}{2}xy\sin(A+B) \quad (1)$$

But you can also find the area of the two right-angled triangles and add them together

You first need to express the base and the height of each triangle in terms of x, y, A and B

From the diagram:

$$a = x\sin A, \quad b = y\sin B$$

$$h = x\cos A = y\cos B$$

You have two different expressions for the height. In the identity you want to prove, both A and B appear in each term. So, use the expression containing B for the first triangle and the expression containing A for the second

For the first triangle:

$$\text{Area} = \frac{1}{2}ah$$

$$= \frac{1}{2}(x\sin A)(y\cos B) \quad (2)$$

For the second triangle:

$$\text{Area} = \frac{1}{2}bh$$

$$= \frac{1}{2}(y\sin B)(x\cos A) \quad (3)$$

$(2)+(3)=(1)$:

Add the two small triangles to get the large triangle

$$\frac{1}{2}xy\sin A\cos B + \frac{1}{2}xy\cos A\sin B = \frac{1}{2}xy\sin(A+B)$$

Divide both sides by $\frac{1}{2}xy$

$$\therefore \sin A\cos B + \cos A\sin B = \sin(A+B)$$

The proof above works whenever A and B are between 0 and $\frac{\pi}{2}$, so they can be angles in the two right-angled triangles. It can be shown, by using symmetries of trigonometric graphs, that the identity in fact holds for all values of A and B.

You can use compound angle identities to find certain exact values of trigonometric functions.

The other compound angle formulae from Key Point 3.6 can be proved by starting from this formula and applying the symmetries of the trigonometric functions, such as $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$.

WORKED EXAMPLE 3.7

Find the exact value of: **a** $\sin 75°$ **b** $\cos\frac{\pi}{12}$.

Notice that $30° + 45° = 75°$ – you know the exact values of sin and cos for 30° and 45°

a $\sin 75° = \sin(30° + 45°)$

$$= \sin 30°\cos 45° + \cos 30°\sin 45°$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

This time use the fact that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

b $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

Make sure to use the correct sign in the cos identity

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Tip

If you use more than one $\pm$ in a single expression or equation, the interpretation is that they pair up exactly (all the upper options produce one equation and all the lower options make a second option; there is no suggestion that every combination should be used). Here, the + on the left side is paired specifically with a + in the numerator and a – in the denominator.

Compound angle identities for tan

You can use the compound angle identities for sin and cos to derive the identities for tan.

KEY POINT 3.7

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Proof 3.2

Write $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.

Express tan in terms of sin and cos

$$\tan(A+B) \equiv \frac{\sin(A+B)}{\cos(A+B)}$$

Use the identities for $\sin(A+B)$ and $\cos(A+B)$

$$\equiv \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

You want to write this in terms of tan. Looking at the top of the fraction, if you divide by $\cos A$ you will get $\tan A$ in the first term, and if you divide by $\cos B$ you will get $\tan B$ in the second term. So, divide top and bottom by $\cos A\cos B$

$$\equiv \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

$$\equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

The identity for $\tan(A-B)$ is proved similarly.

WORKED EXAMPLE 3.8

Show that $\tan 105° = -2 - \sqrt{3}$.

Find two special angles whose sum or difference is 105. For example, you could use 60 + 45, or 135 – 30

$$\tan 105° = \tan(60° + 45°)$$

Use the compound angle formula for $\tan(A+B)$

$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= \frac{(\sqrt{3}+1)(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

To get the expression into the required form, rationalise the denominator

$$= \frac{1+3+2\sqrt{3}}{1-3}$$

$$= -2-\sqrt{3}$$

Be the Examiner 3.2

Find the exact value of $\tan 15°$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\tan(15°) = \tan(45° - 30°)$ $= \tan(45°) - \tan(30°)$ $= 1 - \frac{1}{\sqrt{3}}$ $= \frac{3-\sqrt{3}}{3}$	$\tan(15°) = \tan(45° - 30°)$ $= \frac{\tan(45°) - \tan(30°)}{1 + \tan(45°)\tan(30°)}$ $= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$ $= \frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\tan(15°) = \tan(45° - 30°)$ $= \frac{\tan(45°) + \tan(30°)}{1 - \tan(45°)\tan(30°)}$ $= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$ $= 2 + \sqrt{3}$

Link with double angle identities

The compound angle identities can be used to derive the double angle identities which you already know. For example, setting $A = B = \theta$ in the identity for $\cos(A + B)$ gives $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$.

You can similarly prove a new double angle identity for tan.

KEY POINT 3.8

$$\tan 2\theta \equiv \frac{2\tan\theta}{1 - \tan^2\theta}$$

You can use these new identities to solve more complicated trigonometric equations.

WORKED EXAMPLE 3.9

Solve the equation $\tan 2x = 3\tan x$ for $0 \leqslant x \leqslant 2\pi$.

Use the double angle identity to express the left-hand side in terms of $\tan x$	$\frac{2\tan x}{1 - \tan^2 x} = 3\tan x$
Multiply through by the denominator	$2\tan x = 3\tan x - 3\tan^3 x$
Get everything on one side and factorize	$3\tan^3 x - \tan x = 0$ $\tan x(3\tan^2 x - 1) = 0$
Don't forget $\pm$ when taking the square root	$\tan x = 0$ or $\tan x = \pm\frac{1}{\sqrt{3}}$
Solve each equation separately	$\tan x = 0 \Rightarrow x = 0, \pi, 2\pi$
Remember to add π to get to the next solution	$\tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}, \frac{7\pi}{6}$ $\tan x = -\frac{1}{\sqrt{3}} \Rightarrow x = \frac{5\pi}{6}, \frac{11\pi}{6}$

You can also derive further identities involving multiple angles.

WORKED EXAMPLE 3.10

Starting from the identity for $\cos(A+B)$, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Write 3θ as $2\theta + \theta$ and use the compound angle identity

$$\cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta \sin\theta$$

Use the double angle identities for sin and cos. Since the required answer only contains cos, use the version of the cos 2θ identity which only contains cos

$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

Write $\sin^2\theta$ in terms of $\cos^2\theta$

$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

In Chapter 4 you will learn how to use complex numbers to derive multiple angle identities.

Symmetries of trigonometric graphs

In Section 18B of the Mathematics: analysis and approaches SL book, you learnt about various symmetries of trigonometric functions, such as $\sin(\pi + x) = -\sin x$, which you can illustrate using either a graph or the unit circle. You can now also derive them using compound angle identities.

WORKED EXAMPLE 3.11

Use a compound angle identity to simplify $\sin(\pi + x)$.

Use $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

$$\sin(\pi + x) \equiv \sin\pi\cos x + \cos\pi\sin x$$

$\sin\pi \equiv 0$ and $\cos\pi \equiv -1$

$$\equiv 0\cos x - 1\sin x$$
$$\equiv -\sin x$$

Exercise 3B

For questions 1 to 4, use the technique demonstrated in Worked Example 3.7 to find the exact values of the trigonometric functions.

1 a $\sin 105°$ b $\sin 165°$

2 a $\cos 15°$ b $\cos 75°$

3 a $\cos \frac{5\pi}{12}$ b $\cos \frac{7\pi}{12}$

4 a $\sin \frac{\pi}{12}$ b $\sin \frac{5\pi}{12}$

For questions 5 to 7, use the technique demonstrated in Worked Example 3.8 to find the exact values of the trigonometric functions.

5 a $\tan 15°$ b $\tan 165°$

6 a $\tan 330°$ b $\tan 225°$

7 a $\tan \frac{7\pi}{12}$ b $\tan \frac{5\pi}{12}$

For questions 8 to 10, use the technique demonstrated in Worked Example 3.11 to simplify the given expressions.

8 a $\sin(x-\pi)$ b $\sin\left(x+\frac{\pi}{2}\right)$

9 a $\cos(\pi+x)$ b $\cos\left(\frac{\pi}{2}-x\right)$

10 a $\tan(\pi-x)$ b $\tan(x+\pi)$

11 a Use a compound angle formula to show that $\sin(\pi+x) = -\sin x$ for all values of x.

b Sketch the graph of $y = \sin x$ and illustrate the result from part a in the case when $x = \frac{\pi}{6}$.

12 Show that $\sin\left(\theta+\frac{\pi}{3}\right)+\sin\left(\theta-\frac{\pi}{3}\right) \equiv \sin\theta$.

13 Prove that, for all values of x, $\cos x + \cos\left(x+\frac{2\pi}{3}\right)+\cos\left(x+\frac{4\pi}{3}\right) = 0$.

14 a Simplify $\sin(x+45°)+\cos(x+45°)$.

b Hence solve the equation $\sin(x+45°)+\cos(x+45°) = \frac{\sqrt{2}}{2}$ for $0° < x < 360°$.

15 Prove the identity $\cot x - \tan x \equiv 2\cot 2x$.

16 Show that $\tan\left(\theta+\frac{\pi}{4}\right)-\tan\left(\theta-\frac{\pi}{4}\right) \equiv \frac{4\tan\theta}{1-\tan^2\theta}$.

17 Given that $\tan x = \frac{1}{3}$, find the exact value of:

a $\tan 2x$

b $\tan 3x$.

18 Given that $\tan(\theta-45°) = \frac{1}{2}$, find the exact value of $\tan\theta$.

19 Given that $\sin A = \frac{1}{3}$ and $\sin B = \frac{4}{5}$, and that $0 < A,\ B < \frac{\pi}{2}$, find the exact value of

a $\cos A$

b $\cos(A+B)$.

20 Given that x and y are acute angles such that $\sin x = \frac{3}{5}$ and $\sin y = \frac{5}{13}$, find the exact value of $\sin(x-y)$.

21 Given that $\sin(\theta + 60°) = 2\cos\theta$, find the exact value of $\tan\theta$.

22 Given that $\tan(x - y) = 2$, express $\tan y$ in terms of $\tan x$.

23 Given that $\tan 2A = -\frac{3}{4}$, find the possible values of $\tan A$.

24 Find the exact value of $\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{24}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{24}\right)$.

25 Find the exact value of $\cos 15° + \sqrt{3}\sin 15°$.

26 Show that $\cot(A + B) \equiv \frac{\cot A \cot B - 1}{\cot A + \cot B}$.

27 a Use the identity for $\sin(A + B)$ to express $\sin 2x$ in terms of $\sin x$ and $\cos x$.

b Hence solve the equation $\sin 2x = \tan x$ for $0 \leqslant x \leqslant 2\pi$.

28 Use a double angle identity to find the exact value of $\tan 22.5°$.

29 a Show that $\tan\left(x - \frac{\pi}{4}\right) \equiv \frac{\tan x - 1}{\tan x + 1}$.

b Given that $\tan\left(\theta - \frac{\pi}{4}\right) = 6\tan\theta$, find the two possible values of $\tan\theta$.

30 a Use the identity for $\sin(A + B)$ to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

b Hence solve the equation $\sin 3x = 2\sin x$ for $0 \leqslant x \leqslant \pi$.

31 a Use a compound angle formula to show that $\sin(\pi - x) = \sin x$.

b An isosceles triangle has angles A, A and C. Show that $\frac{\sin C}{\sin A} = 2\cos A$.

32 a Find the value of R such that $3\sin x - 7\cos x = R\sin(x - \theta)$ for some $\theta \in \left(0, \frac{\pi}{2}\right)$.

b Hence find the maximum value of the expression $\frac{3}{10 + 3\sin x - 7\cos x}$.

33 Let $f(x) = \frac{1}{3\sin 2x + \sqrt{3}\cos 2x + 6}$.

a Express $3\sin\theta + \sqrt{3}\cos\theta$ in the form $R\sin(\theta + \alpha)$.

b Hence find the minimum value of $f(x)$ and the smallest positive value of x for which it occurs.

34 Find the exact value of $\tan\left(\arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{5}\right)\right)$.

35 a Use the identity for $\sin(A + B)$ to show that $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

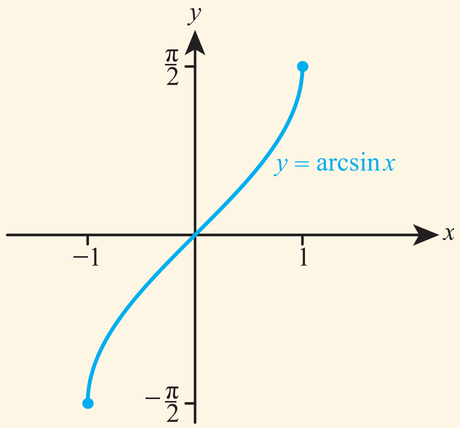
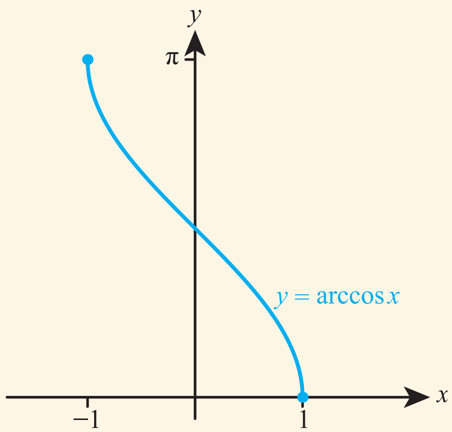
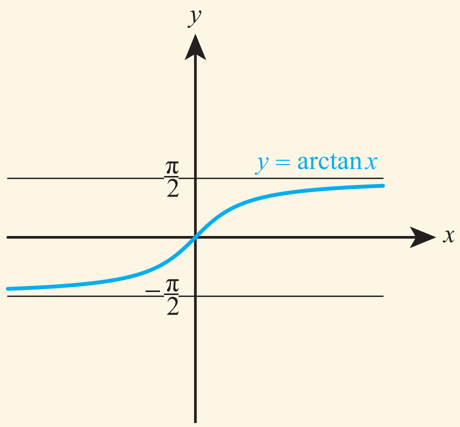
b Starting from the identity for $\sin(A + B)$, prove the identity $\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$.

36 a Use the formula for $\tan(A + B)$ to show that $\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

b Hence show that $\tan 10°$ is a root of the equation $t^3 - \sqrt{3}t^2 - 3t + \frac{\sqrt{3}}{3} = 0$.

Checklist

- You should know the reciprocal trigonometric functions:
  - secant: $\sec x = \dfrac{1}{\cos x}$
  - cosecant: $\operatorname{cosec} x = \dfrac{1}{\sin x}$
  - cotangent: $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$
- You should be able to work with the Pythagorean identities:
  - $\tan^2 x + 1 \equiv \sec^2 x$
  - $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$
- You should know the inverse trigonometric functions:

Function	Graph	Domain	Range
$\arcsin x$		$-1 \leqslant x \leqslant 1$	$-\dfrac{\pi}{2} < y \leqslant \dfrac{\pi}{2}$
$\arccos x$		$-1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$\arctan x$		$x \in \mathbb{R}$	$-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

- You should be able to work with the compound angle identities for sin and cos:
  - $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$
  - $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$
- You should be able to work with the compound and double angle identities for tan:
  - $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
  - $\tan 2\theta \equiv \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Mixed Practice

1 Given that $\sec\theta = 3$, find the exact value of $\cos 2\theta$.

2 **a** Show that $\cos\left(x+\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right)\equiv\sqrt{2}\cos x$.

b Hence solve the equation $\cos\left(x+\frac{\pi}{4}\right)+\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin x$ for $0\leqslant x<\pi$.

3 **a** Show that

$$\sin\left(x+\frac{\pi}{3}\right)+\cos\left(x+\frac{\pi}{6}\right)\equiv\sqrt{3}\cos x.$$

b Hence solve the equation

$$\sin\left(x+\frac{\pi}{3}\right)+\cos\left(x+\frac{\pi}{6}\right)=\sin x$$

for $0<x<2\pi$.

4 An acute angle θ has $\tan\theta = 3$. Find the exact value of

a $\tan 2\theta$

b $\sec\theta$.

5 Given that $\cos A=\frac{1}{2}$ and $\cos B=\frac{1}{3}$, find the exact value of $\cos(A+B)$.

6 Solve the equation $(\arcsin x)^2=\frac{\pi^2}{9}$.

7 Find the exact value of $\tan 105°$.

8 Solve the equation

$$\sin\left(x+\frac{\pi}{3}\right)=\sin\left(x-\frac{\pi}{3}\right)$$

for $0<x<2\pi$.

9 Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0°\leqslant x<360°$.

Mathematics HL May 2015 Paper 1 TZ2 Q3

10 **a** Given that $\arctan\frac{1}{2}-\arctan\frac{1}{3}=\arctan a$, $a\in\mathbb{Q}^+$, find the value of a.

b Hence, or otherwise, solve the equation $\arcsin x = \arctan a$.

Mathematics HL November 2011 Paper 2 Q4

11 **a** Use the formulae for $\sin 2\theta$ and $\cos 2\theta$ to derive a formula for $\tan 2\theta$ in terms of $\tan\theta$.

b Hence find the exact value of $\tan 112.5°$.

12 Solve the equation $\sin\left(x+\frac{\pi}{6}\right)+\sin\left(x-\frac{\pi}{6}\right)=3\cos x$ for $0\leqslant x\leqslant\pi$.

13 Given that $\cos y=\sin(x+y)$, show that $\tan y=\sec x-\tan x$.

14 Prove that $\cot 2x=\dfrac{\cot^2 x-1}{2\cot x}$.

15 **a** Prove that $\operatorname{cosec} 2x - \cot 2x \equiv \tan x$.

b Hence find the exact value of $\tan\left(\frac{3}{8}\pi\right)$.

16 **a** Prove the following identities:

i $\cos^4\theta - \sin^4\theta = \cos 2\theta$

ii $\sin^2 2\theta(\cot^2\theta - \tan^2\theta) \equiv 4(\cos^4\theta - \sin^4\theta)$.

b Hence solve the equation $\sin^2 2\theta(\cot^2\theta - \tan^2\theta) = 2$ for $0 \leqslant \theta \leqslant 2\pi$.

17 **a** Write $3\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\theta)$ where $R > 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$.

b Hence solve the equation $3\sin x + \sqrt{3}\cos x = 3$ for $-\pi < x < \pi$.

18 If x satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$, show that $11\tan x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^+$.

Mathematics HL May 2010 Paper 1 TZ2 Q6

19 Let $f(x) = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

a For what values of x does $f(x)$ not exist?

b Simplify the expression $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$.

Mathematics HL May 2012 Paper 1 TZ1 Q5

20 **a** Show that $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$.

b Hence solve the equation $\cos 3x + \cos x = 3\cos 2x$ for $0 \leqslant \theta < 2\pi$.

21 Find the exact value of:

a $\tan(\arctan 3 - \arctan 2)$

b $\tan\left(2\arctan\left(\frac{1}{2}\right)\right)$.

22 Let θ be the acute angle between the lines with equations $y = x$ and $y = 2x$.
Find the exact value of $\tan\theta$.

23 **a** Use the identity for $\tan(A+B)$ to express $\tan 3x$ in terms of $\tan x$.

b Hence solve the equation $\tan x + \tan 3x = 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

24 **a** Show that $\sin\left(2x + \frac{\pi}{2}\right) \equiv \cos 2x$.

b Hence solve the equation $\sin 3x = \cos 2x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

c Show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$ and hence express $\cos 2x - \sin 3x$ in terms of $\sin x$.

Let $f(s) = 4s^3 - 2s^2 - 3s + 1$.

d Show that $(s-1)$ is a factor of $f(s)$ and factorize $f(s)$ completely.

e Hence find the exact value of $\sin\left(\frac{\pi}{10}\right)$.

25 Compactness is a measure of how compact an enclosed region is. The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$ where A is the area of the region and d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of diameter x units.

a If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$.

If $n > 1$ and odd, it can be shown that $C = \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

b Find the regular polygon with the least number of sides for which the compactness is more than 0.99.

c Comment briefly on whether C is a good measure of compactness.

Mathematics HL November 2014 Paper 2 Q9

26 **a** Given that $\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) = \arctan\left(\frac{1}{p}\right)$, where $p \in \mathbb{Z}^+$, find p.

b Hence find the value of $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$.

Mathematics HL May 2013 Paper 1 TZ2 Q10

14 a $\begin{cases} -a+b-c=7 \\ 8a+4b+2c=4 \\ 27a+9b+3c=3 \end{cases}$

b $a=-1,\ b=4,\ c=-2$

15 a $k=-3$

b $x=2,\ y=-1,\ z=4$

16 a Proof

b $x=2.1,\ y=-1.7,\ z=1.8$

17 a $a=-2$

b $x=\lambda+0.4,\ y=\lambda,\ z=-0.8$

18 $k=2$ or -1

19 5

20 $k=2$ or 7

21 a $k\neq 1$ b $k=1$ and $c=\frac{4}{7}$

c $k=1$ and $c\neq\frac{4}{7}$

22 754

Chapter 2 Mixed Practice

1 $1-\frac{3}{2}x-\frac{9}{8}x^2-\frac{27}{16}x^3+\ldots$

2 a $\frac{1}{3}-\frac{2}{27}x+\frac{2}{81}x^2+\ldots$

b $|x|<\frac{9}{4}$

3 a $a=3,\ n=-2$

b $1-6x+27x^2-108x^3+\ldots$

4 $A=-1,\ B=4$

5 $\frac{3}{2(3x-4)}-\frac{1}{2(x+2)}$

6 $\frac{2}{x-5}-\frac{3}{x+6}$

7 $x=-3,\ y=2,\ z=4$

8 $x=2\lambda-8,\ y=\lambda-4,\ z=\lambda$

9 $x=-1-4\lambda,\ y=1,\ z=\lambda$

10 a $\begin{cases} 4a-2b+c=12 \\ a-b+c=1 \\ a+b+c=-3 \end{cases}$

b $a=3,\ b=-2,\ c=-4$

11 a $1-x^2+2x^3+\ldots$

b $|x|<1$

12 $a=4$

13 a $4+2x-\frac{1}{4}x^2+\ldots$ b $|x|<\frac{4}{3}$

c 4.639

14 a $\frac{3}{1+3x}+\frac{4}{2-5x}$

b $5-4x+\frac{79}{2}x^2+\ldots$

c $|x|<\frac{1}{3}$

15 a $k=9$

b $x=6-\lambda,\ y=1+4\lambda,\ z=7\lambda$

16 $a=-3,\ b=-4$

17 $1-x+x^3+\ldots$

18 $-1+\frac{4}{3}x+\frac{34}{9}x^2+\ldots$

19 a $1-\frac{7}{2}x+\frac{287}{8}x^2+\ldots$ b $|x|<\frac{1}{12}$

c 3.87

20 $b=-\frac{35}{2}$

21 -270

22 a i $\alpha=2,\ \beta\neq 0$

ii $\alpha\neq 1$

iii $\alpha=2,\ \beta=0$

b $\frac{x+2}{-2}=\frac{y-4}{-2}=z$

Chapter 3 Prior Knowledge

1 $\frac{\sqrt{3}}{2}$

2 $\frac{7}{9}$

3 $\frac{-2\pm\sqrt{7}}{3}$

4 a $\frac{3\pi}{4},\ \frac{7\pi}{4}$ b $0,\ 2\pi$

c $\frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \frac{11\pi}{9},\ \frac{13\pi}{9},\ \frac{17\pi}{9}$

Exercise 3A

1 a i -1.36 ii 7.09

b i -0.642 ii -2.40

c i 1.05 ii 0.577

2 a i 1 ii $\sqrt{2}$

b i 1 ii $\frac{2\sqrt{3}}{3}$

c i 1 ii $\frac{\sqrt{3}}{3}$

d i -1 ii 0

3 a i $1.16, 5.12$ b i $3.48, 5.94$
ii $1.32, 4.97$ ii $0.340, 2.80$
c i $0.464, 3.61$ ii $1.19, 4.33$
d i $0.421, 2.72, 3.56, 5.86$
ii $0.101, 1.47, 3.24, 4.61$

4 a i $\frac{\pi}{6}, \frac{5\pi}{6}$ ii $\frac{\pi}{2}$
b i $\frac{\pi}{3}, \frac{4\pi}{3}$ ii $\frac{\pi}{4}, \frac{5\pi}{4}$
c i $0, 2\pi$ ii $\frac{\pi}{6}, \frac{11\pi}{6}$
d i $\frac{\pi}{2}, \frac{3}{2}\pi$ ii $\frac{3\pi}{4}, \frac{7\pi}{4}$

5 a i $\frac{5}{4}$ ii $\frac{\sqrt{26}}{5}$
b i $-\sqrt{15}$ ii $-\sqrt{48}$
c i $\frac{1}{\sqrt{5}}$ ii $\frac{-1}{\sqrt{10}}$
d i $\pm\frac{1}{\sqrt{3}}$ ii $\pm\frac{1}{\sqrt{17}}$

6 a i 0.927 ii -0.644
b i 2.42 ii 1.56
c i 1.26 ii 1.47

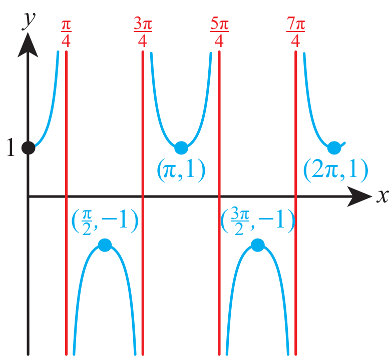
7 a i $\frac{\pi}{6}$ ii 0
b i $\frac{\pi}{6}$ ii $\frac{\pi}{2}$
c i $\frac{\pi}{3}$ ii $\frac{\pi}{4}$
d i $\frac{2\pi}{3}$ ii $-\frac{\pi}{6}$

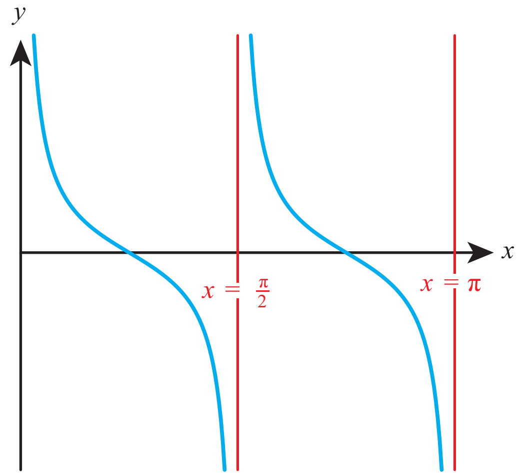
8 $\operatorname{cosec} A = \frac{5}{3}$, $\sec B = \frac{2}{\sqrt{3}}$

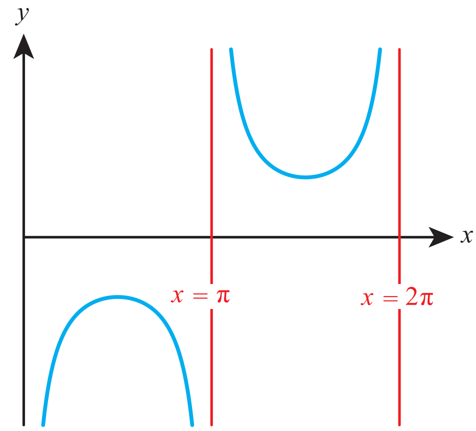
10 0.644

11 $(0.715, 2.39)$
$f(x) \geqslant 2.39$

14 0

15


16


17


21 $\pm\frac{2\pi}{3}$

22 1

23 $\pi - \arccos x$

24 b $1, 2$ c $\frac{\pi}{4}, \frac{5\pi}{4}, 1.11, 4.25$

27 $\arccos\left(\frac{1}{x}\right)$

28 a $x \in \mathbb{R}, -1 \leqslant f(x) \leqslant 1$
b i $x - 2\pi$ ii $2\pi - x$
iii $-x$

29 $\operatorname{isin} x = \pi - \arcsin x$

Exercise 3B

1 a $\frac{\sqrt{6}+\sqrt{2}}{4}$ b $\frac{\sqrt{6}-\sqrt{2}}{4}$

2 a $\frac{\sqrt{6}+\sqrt{2}}{4}$ b $\frac{\sqrt{6}-\sqrt{2}}{4}$

3 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ b $\frac{-\sqrt{6}+\sqrt{2}}{4}$

4 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ b $\frac{\sqrt{6}+\sqrt{2}}{4}$

5 a $2-\sqrt{3}$ b $-2+\sqrt{3}$

6 a $-\frac{1}{\sqrt{3}}$ b 1

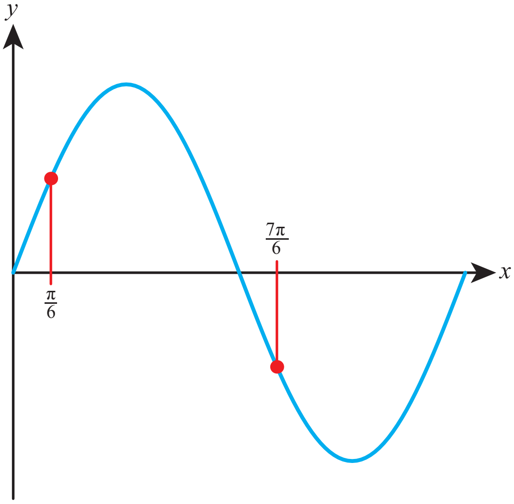
7 a $-2-\sqrt{3}$ b $2+\sqrt{3}$

8 a $-\sin x$ b $\cos x$

9 a $\cos x$ b $\sin x$

10 a $-\tan x$ b $\tan x$

11 b



14 a $\sqrt{2}\cos x$ b 60°, 300°

17 a $\frac{3}{4}$ b $\frac{13}{9}$

18 3

19 a $\frac{2\sqrt{2}}{3}$ b $\frac{6\sqrt{2}-4}{15}$

20 $\frac{16}{65}$

21 $4-\sqrt{3}$

22 $\tan y = \frac{\tan x - 2}{1 + 2\tan x}$

23 $3, -\frac{1}{3}$

24 $\frac{\sqrt{3}}{2}$

25 $\sqrt{2}$

27 a $2\sin x\cos x$

b $0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

28 $\sqrt{2}-1$

29 b $-\frac{1}{2}, -\frac{1}{3}$

30 b $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

32 a $\sqrt{58}$ b $\frac{3}{10-\sqrt{58}}$

33 a $2\sqrt{3}\sin\left(\theta+\frac{\pi}{6}\right)$ b $\frac{3-\sqrt{3}}{12}$ for $x=\frac{\pi}{6}$

34 $\frac{3}{11}$

Chapter 3 Mixed Practice

1 $-\frac{7}{9}$

2 b $\frac{\pi}{4}$

3 b $\frac{\pi}{3}, \frac{4\pi}{3}$

4 a $-\frac{3}{4}$ b $\sqrt{10}$

5 $\frac{1\pm 2\sqrt{6}}{6}$

6 $\pm\frac{\sqrt{3}}{2}$

7 $-2-\sqrt{3}$

8 $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

9 0°, 60°, 120°, 180°, 240°, 300°

10 a $\frac{1}{7}$ b $\frac{1}{\sqrt{50}}$

11 b $-\sqrt{2}-1$

12 $\frac{\pi}{3}$

15 b $1+\sqrt{2}$

16 b $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

17 a $2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)$ b $\frac{\pi}{6}, \frac{\pi}{2}$

18 $a=6, b=1$

19 a $\frac{n\pi}{2}, n \in \mathbb{Z}$ b 2

20 b $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

21 a $\frac{1}{7}$ b $\frac{4}{3}$

22 $\frac{1}{3}$

23 a $\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ b $-\frac{\pi}{4}, 0, \frac{\pi}{4}$

24 b $\frac{\pi}{10}, \frac{\pi}{2}$

$1 - 3\sin x - 2\sin^2 2x + 4\sin^3 x$

d $(s-1)\left(2s + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(2s + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

e $\frac{-1+\sqrt{5}}{4}$

25 b 21

c e.g. As n increases, the compactness of a polygon with n sides gets closer to that of a circle ($c = 1$).

26 a $p = 3$ b $\frac{\pi}{4}$

Chapter 4 Prior Knowledge

1 $\frac{3 \pm \sqrt{3}}{3}$

2 $5 - 2\sqrt{5}$

3 a $\frac{\sqrt{2}}{2}$ b $-\frac{1}{2}$

4 a $\sin\frac{3\pi}{10}$ b $\frac{5\pi}{24}$

5 $e^{(\ln 4)x}$

6 $a(x-3)(x+5)$

7 $\frac{1}{1-x}$

Exercise 4A

1 a i b −1

2 a 4i b 5

3 a −9 b −8i

4 a 1 b −1

5 a $x = \pm 3i$ b $x = \pm 6i$

6 a $x = \pm 2\sqrt{2}i$ b $x = \pm 5\sqrt{3}i$

7 a $x = 1 \pm 2i$ b $x = 2 \pm 3i$

8 a $x = -1 \pm \frac{\sqrt{2}}{2}i$ b $x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}i$

9 a $11 + 4i$ b $-4 + 4i$

10 a $1 - 2i$ b $-6 + 10i$

11 a $8 - i$ b $16 + 2i$

12 a $8 + 6i$ b $7 - 24i$

13 a $4 - 2i$ b $-3 + 3i$

14 a $4 + 3i$ b $1 - i$

15 a $\frac{1}{2} + \frac{1}{2}i$ b $-\frac{2}{13} - \frac{23}{13}i$

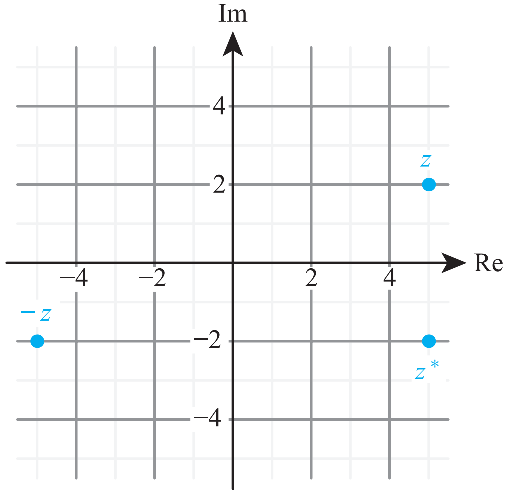
16 a $a = 5, b = 7$ b $a = -3, b = 9$

17 a $a = -\frac{3}{2}, b = \frac{31}{2}$ b $a = \frac{1}{2}, b = -8$

18 a $z = -4 - i$ b $z = 1 + 2i$

19 a $z = \frac{2}{3} + 7i$ b $z = -\frac{5}{3} + \frac{1}{3}i$

20 a



b