Normal distribution 03.02 *[106 marks]*

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

1a. Find the probability that a bag selected at random is rejected. *[2 marks]*

Markscheme

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let $X =$ mass of a bag of sugar

evidence of identifying the correct area ***(M1)***

$\mathrm{P}(X < 995) = 0.0765637\ldots$

$= 0.0766$ ***A1***

[2 marks]

1b. Estimate the number of bags which will be rejected from a random sample of 100 bags. *[1 mark]*

Markscheme

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let $X =$ mass of a bag of sugar

0.0766×100

≈ 8 ***A1***

Note: Accept 7.66.

[1 mark]

1c. Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. *[3 marks]*

Markscheme

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let $X =$ mass of a bag of sugar

recognition that $\mathrm{P}\left(X > 1005 \middle| X \geq 995\right)$ is required ***(M1)***

$\frac{\mathrm{P}\left(X \geq 995 \cap X > 1005\right)}{\mathrm{P}\left(X \geq 995\right)}$

$\frac{\mathrm{P}\left(X > 1005\right)}{\mathrm{P}\left(X \geq 995\right)}$ ***(A1)***

$\frac{0.0765637...}{1 - 0.0765637...}\left(= \frac{0.0765637...}{0.923436...}\right)$

$= 0.0829$ ***A1***

[3 marks]

The weights, in grams, of individual packets of coffee can be modelled by a normal distribution, with mean 102 g and standard deviation 8 g.

2a. Find the probability that a randomly selected packet has a weight less than 100 g. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$X \sim \mathrm{N}\left(102,\ 8^2\right)$

$\mathrm{P}(X < 100) = 0.401$ ***(M1)A1***

[2 marks]

2b. The probability that a randomly selected packet has a weight greater than w grams is 0.444. Find the value of w. *[2 marks]*

Markscheme

$\mathrm{P}(X > w) = 0.444$ ***(M1)***

$\Rightarrow w = 103\ (\mathrm{g})$ ***A1***

[2 marks]

2c. A packet is randomly selected. Given that the packet has a weight greater than 105 g, find the probability that it has a weight greater than 110 g. *[3 marks]*

Markscheme

$\mathrm{P}\left(X > 100 \mid X > 105\right) = \frac{\mathrm{P}(X>100 \cap X>105)}{\mathrm{P}(X>105)}$ ***(M1)***

$= \frac{\mathrm{P}(X>100)}{\mathrm{P}(X>105)}$ ***(A1)***

$= \frac{0.15865\ldots}{0.35383\ldots}$

$= 0.448$ ***A1***

[3 marks]

2d. From a random sample of 500 packets, determine the number of packets that would be expected to have a weight lying within 1.5 standard deviations of the mean. *[3 marks]*

Markscheme

EITHER

$\mathrm{P}(90 < X < 114) = 0.866\ldots$ ***(A1)***

OR

$\mathrm{P}(-1.5 < Z < 1.5) = 0.866\ldots$ ***(A1)***

THEN

$0.866\ldots \times 500$ ***(M1)***
$= 433$ ***A1***

[3 marks]

2e. Packets are delivered to supermarkets in batches of 80. Determine the probability that at least 20 packets from a randomly selected batch have a weight less than 95 g. *[4 marks]*

Markscheme

$p = \mathrm{P}(X < 95) = 0.19078\ldots$ ***(A1)***

recognising $Y \sim \mathrm{B}(80,\ p)$ ***(M1)***

now using $Y \sim \mathrm{B}(80,\ 0.19078\ldots)$ ***(M1)***

$\mathrm{P}(Y \geq 20) = 0.116$ ***A1***

[4 marks]

3. Rachel and Sophia are competing in a javelin-throwing competition. *[7 marks]*

The distances, R metres, thrown by Rachel can be modelled by a normal distribution with mean 56.5 and standard deviation 3.

The distances, S metres, thrown by Sophia can be modelled by a normal distribution with mean 57.5 and standard deviation 1.8.

In the first round of competition, each competitor must have five throws. To qualify for the next round of competition, a competitor must record at least one throw of 60 metres or greater in the first round.

Find the probability that only one of Rachel or Sophia qualifies for the next round of competition.

Markscheme

Rachel: $R \sim \mathrm{N}\left(56.5, 3^2\right)$

$\mathrm{P}(R \geq 60) = 0.1216\ldots$ ***(A1)***

Sophia: $S \sim \mathrm{N}\left(57.5, 1.8^2\right)$

$\mathrm{P}(S \geq 60) = 0.0824\ldots$ ***(A1)***

recognises binomial distribution with $n = 5$ ***(M1)***

let N_R represent the number of Rachel's throws that are longer than 60 metres

$N_R \sim \mathrm{B}(5, 0.1216\ldots)$

either $\mathrm{P}(N_R \geq 1) = 0.4772\ldots$ or $\mathrm{P}(N_R = 0) = 0.5227\ldots$ ***(A1)***

let N_S represent the number of Sophia's throws that are longer than 60 metres

$N_S \sim \mathrm{B}(5, 0.0824\ldots)$

either $\mathrm{P}(N_S \geq 1) = 0.3495\ldots$ or $\mathrm{P}(N_S = 0) = 0.6504\ldots$ ***(A1)***

EITHER

uses $\mathrm{P}(N_R \geq 1)\mathrm{P}(N_S = 0) + \mathrm{P}(N_S \geq 1)\mathrm{P}(N_R = 0)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = (0.4772\ldots \times 0.6504) + (0.3495\ldots \times 0.52$

OR

uses $\mathrm{P}(N_R \geq 1) + \mathrm{P}(N_S \geq 1) - 2 \times \mathrm{P}(N_R \geq 1) \times \mathrm{P}(N_S \geq 1)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = 0.4772\ldots + 0.3495\ldots - 2 \times 0.4772\ldots \times$

THEN

$= 0.4931\ldots$

$= 0.493$ ***A1***

Note: ***M*** marks are not dependent on the previous ***A*** marks.

[7 marks]

The length, X mm, of a certain species of seashell is normally distributed with mean 25 and variance, σ^2.

The probability that X is less than 24.15 is 0.1446.

4a. Find P(24.15 < X < 25). *[2 marks]*

Markscheme

attempt to use the symmetry of the normal curve ***(M1)***

eg diagram, 0.5 − 0.1446

P(24.15 < X < 25) = 0.3554 ***A1***

[2 marks]

4b. Find σ, the standard deviation of X. *[3 marks]*

Markscheme

use of inverse normal to find z score ***(M1)***

$z = -1.0598$

correct substitution $\frac{24.15-25}{\sigma} = -1.0598$ ***(A1)***

σ = 0.802 ***A1***

[3 marks]

4c. Hence, find the probability that a seashell selected at random has a length greater than 26 mm. *[2 marks]*

Markscheme

P(X > 26) = 0.106 ***(M1)A1***

[2 marks]

A random sample of 10 seashells is collected on a beach. Let Y represent the number of seashells with lengths greater than 26 mm.

4d. Find E(Y). *[3 marks]*

Markscheme

recognizing binomial probability ***(M1)***

$E(Y) = 10 \times 0.10621$ ***(A1)***

$= 1.06$ ***A1***

[3 marks]

4e. Find the probability that exactly three of these seashells have a length greater than 26 mm. *[2 marks]*

Markscheme

$P(Y = 3)$ ***(M1)***

$= 0.0655$ ***A1***

[2 marks]

4f. A seashell selected at random has a length less than 26 mm. *[3 marks]*

Find the probability that its length is between 24.15 mm and 25 mm.

Markscheme

recognizing conditional probability ***(M1)***

correct substitution ***A1***

$\frac{0.3554}{1-0.10621}$

$= 0.398$ ***A1***

[3 marks]

The time, T minutes, taken to complete a jigsaw puzzle can be modelled by a normal distribution with mean μ and standard deviation 8.6.

It is found that 30% of times taken to complete the jigsaw puzzle are longer than 36.8 minutes.

5a. By stating and solving an appropriate equation, show, correct to two decimal places, that $\mu = 32.29$. *[4 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$T \sim \mathrm{N}\left(\mu,\ 8.6^2\right)$

$\mathrm{P}(T \leq 36.8) = 0.7$ **(A1)**

states a correct equation, for example, $\frac{36.8-\mu}{8.6} = 0.5244\ldots$ **A1**

attempts to solve their equation **(M1)**

$\mu = 36.8 - (0.5244\ldots)(8.6)(= 32.2902\ldots)$ **A1**

the solution to the equation is $\mu = 32.29$, correct to two decimal places **AG**

[4 marks]

Use $\mu = 32.29$ in the remainder of the question.

5b. Find the 86th percentile time to complete the jigsaw puzzle. *[2 marks]*

Markscheme

let $t_{0.86}$ be the 86th percentile

attempts to use the inverse normal feature of a GDC to find $t_{0.86}$ **(M1)**

$t_{0.86} = 41.6$ (mins) **A1**

[2 marks]

5c. Find the probability that a randomly chosen person will take more than 30 minutes to complete the jigsaw puzzle. *[2 marks]*

Markscheme

evidence of identifying the correct area under the normal curve **(M1)**

Note: Award **M1** for a clearly labelled sketch.

$\mathrm{P}(T > 30) = 0.605$ **A1**

[2 marks]

Six randomly chosen people complete the jigsaw puzzle.

5d. Find the probability that at least five of them will take more than 30 minutes to complete the jigsaw puzzle. *[3 marks]*

Markscheme

let X represent the number of people out of the six who take more than 30 minutes to complete the jigsaw puzzle

$X \sim \mathrm{B}(6, 0.6049\ldots)$ **(M1)**

for example, $\mathrm{P}(X = 5) + \mathrm{P}(X = 6)$ or $1 - \mathrm{P}(X \leq 4)$ **(A1)**

$\mathrm{P}(X \geq 5) = 0.241$ **A1**

[3 marks]

5e. Having spent 25 minutes attempting the jigsaw puzzle, a randomly chosen person had not yet completed the puzzle. *[4 marks]*

Find the probability that this person will take more than 30 minutes to complete the jigsaw puzzle.

Markscheme

recognizes that $\mathrm{P}\left(T > 30 \middle| T \geq 25\right)$ is required **(M1)**

Note: Award **M1** for recognizing conditional probability.

$= \frac{\mathrm{P}\left(T>30\cap T\geq 25\right)}{\mathrm{P}\left(T\geq 25\right)}$ **(A1)**

$= \frac{\mathrm{P}\left(T>30\right)}{\mathrm{P}\left(T\geq 25\right)} = \frac{0.6049\ldots}{0.8016\ldots}$ **M1**

$= 0.755$ **A1**

[4 marks]

The time it takes Suzi to drive from home to work each morning is normally distributed with a mean of 35 minutes and a standard deviation of σ minutes.

On 25% of days, it takes Suzi longer than 40 minutes to drive to work.

6a. Find the value of σ. *[4 marks]*

Markscheme

METHOD 1

$T \sim \mathrm{N}\left(35, \sigma^2\right)$

$\mathrm{P}(T > 40) = 0.25$ or $\mathrm{P}(T < 40) = 0.75$ ***(M1)***

attempt to solve for σ graphically or numerically using the GDC ***(M1)***

graph of normal curve $T \sim \mathrm{N}\left(35, \sigma^2\right)$ for $\mathrm{P}(T > 40)$ and $y = 0.25$ OR
$\mathrm{P}(T < 40)$ and $y = 0.75$
OR table of values for $\mathrm{P}(T < 40)$ or $\mathrm{P}(T > 40)$

$\sigma = 7.413011\ldots$

$\sigma = 7.41$ (min) ***A2***

METHOD 2

$T \sim \mathrm{N}\left(35, \sigma^2\right)$

$\mathrm{P}(T > 40) = 0.25$ or $\mathrm{P}(T < 40) = 0.75$ ***(M1)***

$z = 0.674489\ldots$ ***(A1)***

valid equation using their z-score (clearly identified as z-score and not a probability) ***(M1)***

$\frac{40-35}{\sigma} = 0.674489\ldots$ OR $5 = 0.674489\ldots\sigma$

$7.413011\ldots$

$\sigma = 7.41$ (min) ***A1***

[4 marks]

6b. On a randomly selected day, find the probability that Suzi’s drive to work will take longer than 45 minutes. *[2 marks]*

Markscheme

$\mathrm{P}(T > 45)$ ***(M1)***

$= 0.0886718\ldots$

$= 0.0887$ ***A1***

[2 marks]

Suzi will be late to work if it takes her longer than 45 minutes to drive to work. The time it takes to drive to work each day is independent of any other day.

Suzi will work five days next week.

6c. Find the probability that she will be late to work at least one day next week. *[3 marks]*

Markscheme

recognizing binomial probability ***(M1)***

L~$\mathrm{B}(5,\ 0.0886718\ldots)$

$\mathrm{P}(L \geq 1) = 1 - \mathrm{P}(L = 0)$ OR

$\mathrm{P}(L \geq 1) = \mathrm{P}(L = 1) + \mathrm{P}(L = 2) + \mathrm{P}(L = 3) + \mathrm{P}(L = 4) + \mathrm{P}(L = 5)$ ***(M1)***

$0.371400\ldots$

$\mathrm{P}(L \geq 1) = 0.371$ ***A1***

[3 marks]

6d. Given that Suzi will be late to work at least one day next week, find the probability that she will be late less than three times. *[5 marks]*

Markscheme

recognizing conditional probability in context ***(M1)***

finding $\{L < 3\} \cap \{L \geq 1\} = \{L = 1,\ L = 2\}$ (may be seen in conditional probability) ***(A1)***

$\mathrm{P}(L = 1) + \mathrm{P}(L = 2) = 0.36532\ldots$ (may be seen in conditional probability) ***(A1)***

$P\left(L < 3 \,\middle|\, L \geq 1\right) = \frac{0.36532\ldots}{0.37140\ldots}$ ***(A1)***

$0.983636\ldots$

0.984 ***A1***

[5 marks]

Suzi will work 22 days this month. She will receive a bonus if she is on time at least 20 of those days.

So far this month, she has worked 16 days and been on time 15 of those days.

6e. Find the probability that Suzi will receive a bonus. *[4 marks]*

Markscheme

METHOD 1

recognizing that Suzi can be late no more than once (in the remaining six days) ***(M1)***

$\mathrm{X{\sim}B}(6,\ 0.0886718\ldots)$, where X is the number of days late ***(A1)***

$\mathrm{P}(X \leq 1) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$ ***(M1)***

$= 0.907294\ldots$

$\mathrm{P}(\text{Suzi gets a bonus}) = 0.907$ ***A1***

Note: The first two marks may be awarded independently.

METHOD 2

recognizing that Suzi must be on time at least five times (of the remaining six days) ***(M1)***

$\mathrm{X{\sim}B}(6,\ 0.911328\ldots)$, where X is the number of days on time ***(A1)***

$\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4)$ OR $1 - 0.0927052\ldots$ OR $\mathrm{P}(X = 5) + (X = 6)$ OR $0.334434\ldots + 0.572860\ldots$ ***(M1)***

$= 0.907294\ldots$

$\mathrm{P}(\text{Suzi gets a bonus}) = 0.907$ ***A1***

Note: The first two marks may be awarded independently.

[4 marks]

A bakery makes two types of muffins: chocolate muffins and banana muffins.

The weights, C grams, of the chocolate muffins are normally distributed with a mean of 62 g and standard deviation of 2.9 g.

7a. Find the probability that a randomly selected chocolate muffin weighs less than 61 g. *[2 marks]*

Markscheme

$\mathrm{P}(C < 61)$ ***(M1)***

$= 0.365112\ldots$

$= 0.365$ ***A1***

[2 marks]

7b. In a random selection of 12 chocolate muffins, find the probability that exactly 5 weigh less than 61 g. *[2 marks]*

Markscheme

recognition of binomial eg $X \sim \mathrm{B}(12,\ 0.365\ldots)$ ***(M1)***

$\mathrm{P}(X = 5) = 0.213666\ldots$

$= 0.214$ ***A1***

[2 marks]

The weights, B grams, of the banana muffins are normally distributed with a mean of 68 g and standard deviation of 3.4 g.

Each day 60% of the muffins made are chocolate.

On a particular day, a muffin is randomly selected from all those made at the bakery.

7c. Find the probability that the randomly selected muffin weighs less than 61 g. *[4 marks]*

Markscheme

Let CM represent 'chocolate muffin' and BM represent 'banana muffin'

$\mathrm{P}(B < 61) = 0.0197555\ldots$ ***(A1)***

EITHER

$\mathrm{P}(CM) \times \mathrm{P}\left(C < 61 \mid CM\right) + \mathrm{P}(BM) \times \mathrm{P}\left(B < 61 \mid BM\right)$ (or equivalent in words) ***(M1)***

OR

tree diagram showing two ways to have a muffin weigh < 61 ***(M1)***

THEN

$(0.6 \times 0.365\ldots) + (0.4 \times 0.0197\ldots)$ ***(A1)***

$= 0.226969\ldots$

$= 0.227$ ***A1***

[4 marks]

7d. Given that a randomly selected muffin weighs less than 61 g, find the probability that it is chocolate. *[3 marks]*

Markscheme

recognizing conditional probability ***(M1)***

Note: Recognition must be shown in context either in words or symbols, not just $\mathrm{P}\left(A \mid B\right)$

$\frac{0.6 \times 0.365112\ldots}{0.226969\ldots}$ ***(A1)***

$= 0.965183\ldots$

$= 0.965$ ***A1***

[3 marks]

The machine that makes the chocolate muffins is adjusted so that the mean weight of the chocolate muffins remains the same but their standard deviation changes to σ g. The machine that makes the banana muffins is not adjusted. The probability that the weight of a randomly selected muffin from these machines is less than 61 g is now 0.157.

7e. Find the value of σ. *[5 marks]*

Markscheme

METHOD 1

$\mathrm{P}(CM)\times\mathrm{P}\left(C<61 \mid CM\right)\times\mathrm{P}(BM)\times\mathrm{P}\left(B<61 \mid BM\right)=0.157$ ***(M1)***

$(0.6\times\mathrm{P}(C<61))+(0.4\times0.0197555\ldots)=0.157$

$\mathrm{P}(C<61)=0.248496\ldots$ ***(A1)***

attempt to solve for σ using GDC ***(M1)***

Note: Award ***(M1)*** for a graph or table of values to show their $\mathrm{P}(C<61)$ with a variable standard deviation.

$\sigma=1.47225\ldots$

$\sigma=1.47\,(\mathrm{g})$ ***A2***

METHOD 2

$\mathrm{P}(CM)\times\mathrm{P}\left(C<61 \mid CM\right)\times\mathrm{P}(BM)\times\mathrm{P}\left(B<61 \mid BM\right)=0.157$ ***(M1)***

$(0.6\times\mathrm{P}(C<61))+(0.4\times0.0197555\ldots)=0.157$

$\mathrm{P}(C<61)=0.248496\ldots$ ***(A1)***

use of inverse normal to find z score of their $\mathrm{P}(C<61)$ ***(M1)***

$z=-0.679229\ldots$

correct substitution ***(A1)***

$\frac{61-62}{\sigma}=-0.679229\ldots$

$\sigma=1.47225\ldots$

$\sigma=1.47\,(\mathrm{g})$ ***A1***

[5 marks]

The flight times, T minutes, between two cities can be modelled by a normal distribution with a mean of 75 minutes and a standard deviation of σ minutes.

8a. Given that 2% of the flight times are longer than 82 minutes, find the value of σ. *[3 marks]*

Markscheme

use of inverse normal to find z-score ***(M1)***

$z = 2.0537\ldots$

$2.0537\ldots = \frac{82-75}{\sigma}$ ***(A1)***

$\sigma = 3.408401\ldots$

$\sigma = 3.41$ ***A1***

[3 marks]

8b. Find the probability that a randomly selected flight will have a flight time of more than 80 minutes. *[2 marks]*

Markscheme

evidence of identifying the correct area under the normal curve ***(M1)***

$\mathrm{P}(T > 80) = 0.071193\ldots$

$\mathrm{P}(T > 80) = 0.0712$ ***A1***

[2 marks]

8c. Given that a flight between the two cities takes longer than 80 minutes, find the probability that it takes less than 82 minutes. *[4 marks]*

Markscheme

recognition that $\mathrm{P}(80 < T < 82)$ is required ***(M1)***

$\mathrm{P}\left(T < 82 \,\middle|\, T > 80\right) = \frac{\mathrm{P}(80<T<82)}{\mathrm{P}(T>80)} = \left(\frac{0.051193\ldots}{0.071193\ldots}\right)$ ***(M1)(A1)***

$= 0.719075\ldots$

$= 0.719$ ***A1***

[4 marks]

On a particular day, there are 64 flights scheduled between these two cities.

8d. Find the expected number of flights that will have a flight time of more than 80 minutes. *[3 marks]*

Markscheme

recognition of binomial probability ***(M1)***

$X \sim \mathrm{B}(64,\ 0.071193\ldots)$ or $\mathrm{E}(X) = 64 \times 0.071193\ldots$ ***(A1)***

$\mathrm{E}(X) = 4.556353\ldots$

$\mathrm{E}(X) = 4.56$ (flights) ***A1***

[3 marks]

8e. Find the probability that more than 6 of the flights on this particular day will have a flight time of more than 80 minutes. *[3 marks]*

Markscheme

$\mathrm{P}(X > 6) = \mathrm{P}(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 1 - 0.83088\ldots$ ***(A1)***

$= 0.1691196\ldots$

$= 0.169$ ***A1***

[3 marks]