Trigonometry 8.02 *[106 marks]*

1. Solve the equation $2\cos^2 x + 5\sin x = 4, 0 \leq x \leq 2\pi$. *[7 marks]*

2. Let $a = \sin b,\ 0 < b < \frac{\pi}{2}$. *[5 marks]*

Find, in terms of *b*, the solutions of $\sin 2x = -a,\ 0 \leqslant x \leqslant \pi$.

Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9, 0 \leqslant x < \frac{\pi}{2}$.

3a. Determine an expression for $f'(x)$ in terms of x. *[2 marks]*

3b. Sketch a graph of $y = f'(x)$ for $0 \leqslant x < \frac{\pi}{2}$. *[4 marks]*

3c. Find the x-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. *[2 marks]*

Let $u = \tan x$.

3d. Express $\sin x$ in terms of u. *[2 marks]*

3e. Express $\sin 2x$ in terms of u. *[3 marks]*

3f. Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. *[2 marks]*

3g. Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. *[3 marks]*

4a. Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. *[1 mark]*

4b. Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)\,x - 1 = 0$. *[7 marks]*

4c. Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. *[5 marks]*

4d. Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$. *[6 marks]*

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$.

5. It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$. *[4 marks]*

6. A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$. *[7 marks]*

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$.

7a. Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$. *[2 marks]*

7b. Show that $\sec 2x + \tan 2x = \frac{\cos x + \sin x}{\cos x - \sin x}$. *[4 marks]*

7c. Hence or otherwise find $\int_0^{\frac{\pi}{6}} (\sec 2x + \tan 2x)\,\mathrm{d}x$ in the form $\ln\left(a + \sqrt{b}\right)$ where $a, b \in \mathbb{Z}$. *[9 marks]*

Consider the function $g(x) = 4\cos x + 1$, $a \leqslant x \leqslant \frac{\pi}{2}$ where $a < \frac{\pi}{2}$.

8a. For $a = -\frac{\pi}{2}$, sketch the graph of $y = g(x)$. Indicate clearly the maximum and minimum values of the function. *[3 marks]*

8b. Write down the least value of a such that g has an inverse. *[1 mark]*

8c. For the value of a found in part (b), write down the domain of g^{-1}. *[1 mark]*

8d. For the value of a found in part (b), find an expression for $g^{-1}(x)$. *[2 marks]*

9. Solve the equation $\sec^2 x + 2\tan x = 0,\ 0 \leqslant x \leqslant 2\pi$. *[5 marks]*

10a. Find the value of $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4}$. *[2 marks]*

10b. Show that $\frac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in \mathbb{Z}$. *[2 marks]*

10c. Use the principle of mathematical induction to prove that *[9 marks]*

$$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x},\ n \in \mathbb{Z}^+,\ x \neq k\pi \text{ where } k \in \mathbb{Z}.$$

10d. Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. *[6 marks]*