Vectors 10.02 *[60 marks]*

A line L_1 passes through the points $\mathrm{A}(0, 1, 8)$ and $\mathrm{B}(3, 5, 2)$.

1a. Find $\overrightarrow{AB}$. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $A - B,\ -\begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$ ***A1 N2***

[2 marks]

1b. Hence, write down a vector equation for L_1. *[2 marks]*

Markscheme

any correct equation in the form ***r*** = ***a*** + t***b*** (any parameter for t) ***A2*** ***N2***

where ***a*** is $\begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$, and ***b*** is a scalar multiple of $\begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$

eg ***r*** = $\begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$, ***r*** = $\begin{pmatrix} 3+3t \\ 5+4t \\ 2-6t \end{pmatrix}$, ***r*** = ***j*** + 8***k*** + t(3***i*** + 4***j*** – 6***k***)

Note: Award ***A1*** for the form ***a*** + t***b***, ***A1*** for the form ***L*** = ***a*** + t***b***, ***A0*** for the form ***r*** = ***b*** + t***a***.

[2 marks]

1c. *[3 marks]*

A second line L_2, has equation ***r*** = $\begin{pmatrix} 1 \\ 13 \\ -14 \end{pmatrix} + s\begin{pmatrix} p \\ 0 \\ 1 \end{pmatrix}$.

Given that L_1 and L_2 are perpendicular, show that $p = 2$.

Markscheme

valid approach ***(M1)***

eg $a \bullet b = 0$

choosing correct direction vectors (may be seen in scalar product) ***A1***

eg $\begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} p \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} p \\ 0 \\ 1 \end{pmatrix} = 0$

correct working/equation ***A1***

eg $3p - 6 = 0$

$p = 2$ ***AG*** ***N0***

[3 marks]

Given that L_1 and L_2 are perpendicular, show that $p = 2$.

1d. The lines L_1 and L_1 intersect at $C(9,\ 13,\ z)$. Find z. *[5 marks]*

Markscheme

valid approach ***(M1)***

$egL_1 = \begin{pmatrix} 9 \\ 13 \\ z \end{pmatrix},\ L_1 = L_2$

one correct equation (must be different parameters if both lines used) ***(A1)***

$eg3t = 9,\ 1 + 2s = 9,\ 5 + 4t = 13,\ 3t = 1 + 2s$

one correct value ***A1***

$egt = 3,\ s = 4,\ t = 2$

valid approach to substitute their t or s value ***(M1)***

$eg8 + 3(-6),\ -14 + 4(1)$

$z = -10$ ***A1*** ***N3***

[5 marks]

1e. Find a unit vector in the direction of L_2. *[2 marks]*

Markscheme

$\left|\vec{d}\right| = \sqrt{2^2 + 1}\ \left(= \sqrt{5}\right)$ ***(A1)***

$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\ \left(\text{accept} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{0}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\right)$ ***A1*** ***N2***

[2 marks]

1f. Hence or otherwise, find one point on L_2 which is $\sqrt{5}$ units from C. *[3 marks]*

Markscheme

METHOD 1 (using unit vector)

valid approach ***(M1)***

$eg \begin{pmatrix} 9 \\ 13 \\ -10 \end{pmatrix} \pm \sqrt{5}\hat{d}$

correct working ***(A1)***

$eg \begin{pmatrix} 9 \\ 13 \\ -10 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 13 \\ -10 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

one correct point ***A1*** ***N2***

$eg (11,\ 13,\ -9),\ (7,\ 13,\ -11)$

METHOD 2 (distance between points)

attempt to use distance between $(1+2s,\ 13,\ -14+s)$ and $(9,\ 13,\ -10)$ ***(M1)***

$eg (2s-8)^2 + 0^2 + (s-4)^2 = 5$

solving $5s^2 - 40s + 75 = 0$ leading to $s = 5$ or $s = 3$ ***(A1)***

one correct point ***A1*** ***N2***

$eg (11,\ 13,\ -9),\ (7,\ 13,\ -11)$

[3 marks]

2. Find the Cartesian equation of plane *Π* containing the points $\mathrm{A}\,(6, 2, 1)$ and $\mathrm{B}\,(3, -1, 1)$ and perpendicular to the plane $x + 2y - z - 6 = 0$. *[6 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix}$ ***(A1)***

$\begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***M1A1***

$= \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ ***A1***

$x - y - z = k$ ***M1***

$k = 3$ equation of plane *Π* is $x - y - z = 3$ or equivalent ***A1***

METHOD 2

let plane *Π* be $ax + by + cz = d$

attempt to form one or more simultaneous equations: ***M1***

$a + 2b - c = 0$ (1) ***A1***

$6a + 2b + c = d$ (2)

$3a - b + c = d$ (3) ***A1***

Note: Award second ***A1*** for equations (2) and (3).

attempt to solve ***M1***

EITHER

using GDC gives $a = \frac{d}{3}, b = -\frac{d}{3}, c = -\frac{d}{3}$ ***(A1)***

equation of plane *Π* is $x - y - z = 3$ or equivalent ***A1***

OR

row reduction ***M1***

equation of plane *Π* is $x - y - z = 3$ or equivalent ***A1***

[6 marks]

3. Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \neq \boldsymbol{0}$ prove that $\boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ where s is a scalar. *[4 marks]*

Markscheme

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METHOD 1

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c}$

$(\boldsymbol{a} \times \boldsymbol{b}) - (\boldsymbol{b} \times \boldsymbol{c}) = 0$

$(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \boldsymbol{b}) = 0$ ***M1A1***

$(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b} = 0$ ***A1***

$(\boldsymbol{a} + \boldsymbol{c})$ is parallel to $\boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***R1AG***

Note: Condone absence of arrows, underlining, or other otherwise "correct" vector notation throughout this question.

Note: Allow "is in the same direction to", for the final ***R*** mark.

METHOD 2

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} \Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$ ***M1A1***

$a_2b_3 - a_3b_2 = b_2c_3 - b_3c_2 \Rightarrow b_3(a_2 + c_2) = b_2(a_3 + c_3)$

$a_3b_1 - a_1b_3 = b_3c_1 - b_1c_3 \Rightarrow b_1(a_3 + c_3) = b_3(a_1 + c_1)$

$a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 \Rightarrow b_2(a_1 + c_1) = b_1(a_2 + c_2)$

$\frac{(a_1+c_1)}{b_1} = \frac{(a_2+c_2)}{b_2} = \frac{(a_3+c_3)}{b_3} = s$ ***A1***

$\Rightarrow a_1 + c_1 = sb_1$

$\Rightarrow a_2 + c_2 = sb_2$

$\Rightarrow a_3 + c_3 = sb_3$

$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = s\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ ***A1***

$\Rightarrow \boldsymbol{a} + \boldsymbol{c} = s\boldsymbol{b}$ ***AG***

[4 marks]

The points A, B and C have the following position vectors with respect to an origin O.

$\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{OC} = \boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$

4a. Find the vector equation of the line (BC). *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\overrightarrow{BC} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) - (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = -\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ ***(A1)***

$\boldsymbol{r} = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$

(or $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$) ***(M1)A1***

Note: Do not award ***A1*** unless ***r*** = or equivalent correct notation seen.

[3 marks]

4b. Determine whether or not the lines (OA) and (BC) intersect. *[6 marks]*

Markscheme

attempt to write in parametric form using two different parameters **AND** equate ***M1***

$2\mu = 2 - \lambda$

$\mu = -1 + 4\lambda$

$-2\mu = 2 + \lambda$ ***A1***

attempt to solve first pair of simultaneous equations for two parameters ***M1***

solving first two equations gives $\lambda = \frac{4}{9},\ \mu = \frac{7}{9}$ ***(A1)***

substitution of these two values in third equation ***(M1)***

since the values do not fit, the lines do not intersect ***R1***

Note: Candidates may note that adding the first and third equations immediately leads to a contradiction and hence they can immediately deduce that the lines do not intersect.

[6 marks]

4c. Find the Cartesian equation of the plane Π_1, which passes through C and is perpendicular to $\overrightarrow{\mathrm{OA}}$. *[3 marks]*

Markscheme

METHOD 1

plane is of the form $\boldsymbol{r} \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = d$ ***(A1)***

$d = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -1$ ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

METHOD 2

plane is of the form $2x + y - 2z = d$ ***(A1)***

substituting $(1,\ 3,\ 3)$ (to find gives $2 + 3 - 6 = -1$) ***(M1)***

hence Cartesian form of plane is $2x + y - 2z = -1$ ***A1***

[3 marks]

4d. Show that the line (BC) lies in the plane Π_1. *[2 marks]*

Markscheme

METHOD 1

attempt scalar product of direction vector BC with normal to plane ***M1***

$(-\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = -2 + 4 - 2$

$= 0$ ***A1***

hence BC lies in Π_1 ***AG***

METHOD 2

substitute eqn of line into plane ***M1***

$$\text{line } r = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}. \text{ Plane } \pi_1 : 2x + y - 2z = -1$$

$2(2 - \lambda) + (-1 + 4\lambda) - 2(2 + \lambda)$

$= -1$ ***A1***

hence BC lies in Π_1 ***AG***

Note: Candidates may also just substitute $2i - j + 2k$ into the plane since they are told C lies on π_1.

Note: Do not award ***A1FT***.

[2 marks]

The plane Π_2 contains the points O, A and B and the plane Π_3 contains the points O, A and C.

4e. Verify that $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2. *[3 marks]*

Markscheme

METHOD 1

applying scalar product to $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ ***M1***

$(2\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 0$ ***A1***

$(2\boldsymbol{j} + \boldsymbol{k}) \bullet (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) = 0$ ***A1***

METHOD 2

attempt to find cross product of $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ ***M1***

plane Π_2 has normal $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OB}} = -8\boldsymbol{j} - 4\boldsymbol{k}$ ***A1***

since $-8\boldsymbol{j} - 4\boldsymbol{k} = -4(2\boldsymbol{j} + \boldsymbol{k})$, $2\boldsymbol{j} + \boldsymbol{k}$ is perpendicular to the plane Π_2 ***R1***

[3 marks]

4f. Find a vector perpendicular to the plane Π_3. *[1 mark]*

Markscheme

plane Π_3 has normal $\overrightarrow{\mathrm{OA}} \times \overrightarrow{\mathrm{OC}} = 9\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$ ***A1***

[1 mark]

4g. Find the acute angle between the planes Π_2 and Π_3. *[4 marks]*

Markscheme

attempt to use dot product of normal vectors ***(M1)***

$\cos\theta = \frac{(2j+k)\bullet(9i-8j+5k)}{|2j+k||9i-8j+5k|}$ ***(M1)***

$= \frac{-11}{\sqrt{5}\sqrt{170}}$ $(= -0.377\ldots)$ ***(A1)***

Note: Accept $\frac{11}{\sqrt{5}\sqrt{170}}$. acute angle between planes $= 67.8^\circ$ $(= 1.18^\circ)$ ***A1***

[4 marks]

5. Find the coordinates of the point of intersection of the planes defined by the equations $x + y + z = 3$, $x - y + z = 5$ and $x + y + 2z = 6$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

for eliminating one variable from two equations ***(M1)***

eg, $\begin{cases}(x+y+z=3)\\ 2x+2z=8\\ 2x+3z=11\end{cases}$ ***A1A1***

for finding correctly one coordinate

eg, $\begin{cases}(x+y+z=3)\\ (2x+2z=8)\\ z=3\end{cases}$ ***A1***

for finding correctly the other two coordinates ***A1***

$\Rightarrow \begin{cases}x=1\\ y=-1\\ z=3\end{cases}$

the intersection point has coordinates $(1, -1, 3)$

METHOD 2

for eliminating two variables from two equations or using row reduction ***(M1)***

eg, $\begin{cases} (x+y+z=3) \\ -2=2 \\ z=3 \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ***A1A1***

for finding correctly the other coordinates ***A1A1***

$\Rightarrow \begin{cases} x=1 \\ y=-1 \\ (z=3) \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

the intersection point has coordinates $(1, -1, 3)$

METHOD 3

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2$ ***(A1)***

attempt to use Cramer's rule ***M1***

$x = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 5 & -1 & 1 \\ 6 & 1 & 2 \end{vmatrix}}{-2} = \frac{-2}{-2} = 1$ ***A1***

$y = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 5 & 1 \\ 1 & 6 & 2 \end{vmatrix}}{-2} = \frac{2}{-2} = -1$ ***A1***

$z = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 1 & 6 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3$ ***A1***

Note: Award ***M1*** only if candidate attempts to determine at least one of the variables using this method.

[5 marks]

Consider the lines l_1 and l_2 defined by

$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ and $l_2 : \frac{6-x}{3} = \frac{y-2}{4} = 1-z$ where a is a constant.

Given that the lines l_1 and l_2 intersect at a point P,

6a. find the value of a; *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$$l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = -3 + \beta \\ y = -2 + 4\beta \\ z = a + 2\beta \end{cases}$$ ***M1***

$\frac{6-(-3+\beta)}{3} = \frac{(-2+4\beta)-2}{4} \Rightarrow 4 = \frac{4\beta}{3} \Rightarrow \beta = 3$ ***M1A1***

$\frac{6-(-3+\beta)}{3} = 1 - (a + 2\beta) \Rightarrow 2 = -5 - a \Rightarrow a = -7$ ***A1***

METHOD 2

$$\begin{cases} -3 + \beta = 6 - 3\lambda \\ -2 + 4\beta = 4\lambda + 2 \\ a + 2\beta = 1 - \lambda \end{cases}$$ ***M1***

attempt to solve ***M1***

$\lambda = 2,\ \beta = 3$ ***A1***

$a = 1 - \lambda - 2\beta = -7$ ***A1***

[4 marks]

6b. determine the coordinates of the point of intersection P. *[2 marks]*

Markscheme

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} + 3 \bullet \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} 0 \\ 10 \\ -1 \end{pmatrix}$$ ***A1***

$\therefore \mathrm{P}(0,\ 10,\ -1)$

[2 marks]