Differential Calculus Intro *[93 marks]*

1. Find the equation of the tangent to the curve $y = \mathrm{e}^{2x} - 3x$ at the point where $x = 0$. *[5 marks]*

2. Consider the curve with equation $y = (2x - 1)\mathrm{e}^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$. *[5 marks]*

   The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5\mathrm{e}^{k}x$.

   Find the value of k.

The curve C has equation $\mathrm{e}^{2y} = x^3 + y$.

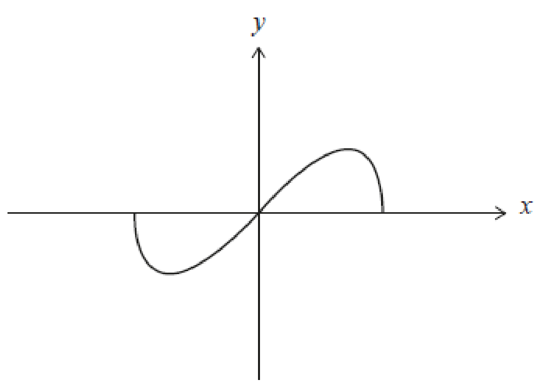
3a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y} - 1}$. *[3 marks]*

3b. The tangent to C at the point P is parallel to the y-axis. *[4 marks]*

Find the x-coordinate of P.

A function f is defined by $f(x) = x\sqrt{1 - x^2}$ where $-1 \le x \le 1$.

The graph of $y = f(x)$ is shown below.



4a. Show that f is an odd function. *[2 marks]*

4b. The range of f is $a \leq y \leq b$, where $a,\ b \in \mathbb{R}$. *[6 marks]*

Find the value of a and the value of b.

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

5a. Write down the value of $f'(4)$. *[1 mark]*

5b. Find $f(4)$. *[1 mark]*

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

5c. Find $h(4)$. *[2 marks]*

5d. Hence find the equation of the tangent to the graph of h at $x = 4$. *[3 marks]*

Consider the curve C given by $y = x - xy\ln(xy)$ where $x > 0,\ y > 0$.

6a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$. *[3 marks]*

6b. Hence find the equation of the tangent to C at the point where $x = 1$. *[5 marks]*

Consider the functions $f(x) = -(x - h)^2 + 2k$ and $g(x) = \mathrm{e}^{x-2} + k$ where $h,\ k \in \mathbb{R}$.

7a. Find $f'(x)$. *[1 mark]*

The graphs of f and g have a common tangent at $x = 3$.

7b. Show that $h = \frac{\mathrm{e}+6}{2}$. *[3 marks]*

7c. Hence, show that $k = \mathrm{e} + \frac{\mathrm{e}^2}{4}$. *[3 marks]*

Let $y = \frac{\ln x}{x^4}$ for $x > 0$.

8a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-4\ln x}{x^5}$. *[3 marks]*

Consider the function defined by $f(x)\frac{\ln x}{x^4}$ for $x > 0$ and its graph $y = f(x)$.

8b. The graph of f has a horizontal tangent at point P. Find the coordinates of P. *[5 marks]*

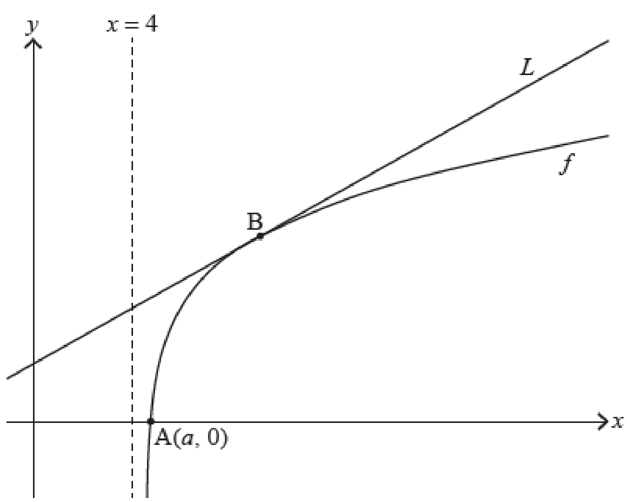
8c. Given that f ''(x)= $\frac{20\ln x-9}{x^6}$, show that P is a local maximum point. *[3 marks]*

8d. Solve $f(x)> 0$ for $x > 0$. *[2 marks]*

8e. Sketch the graph of f, showing clearly the value of the x-intercept and the approximate position of point P. *[3 marks]*

Consider the function f defined by $f\left(x\right) = \ln\left(x^2 - 16\right)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



9a. Find the exact value of a. *[3 marks]*

9b. Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. *[6 marks]*

Consider the graph of the function $f(x) = x^2 - \frac{k}{x}$.

10a. Write down $f'(x)$. *[3 marks]*

The equation of the tangent to the graph of $y = f(x)$ at $x = -2$ is $2y = 4 - 5x$.

10b. Write down the gradient of this tangent. *[1 mark]*

10c. Find the value of k. *[2 marks]*

Consider the curve C defined by $y^2 = \sin(xy), y \neq 0$.

11a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$. *[5 marks]*

11b. Prove that, when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = \pm 1$. *[5 marks]*

11c. Hence find the coordinates of all points on C, for $0 < x < 4\pi$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[5 marks]*