Differential Calculus revision

[82 marks]

Let $f\left(x\right) = x - 8$, $g\left(x\right) = x^4 - 3$ and $h\left(x\right) = f\left(g\left(x\right)\right)$.

1a. Find $h\left(x\right)$. *[2 marks]*

Markscheme

attempt to form composite (in any order) ***(M1)***

eg $f\left(x^4 - 3\right)$, $\left(x - 8\right)^4 - 3$

$h\left(x\right) = x^4 - 11$ ***A1 N2***

[2 marks]

1b. Let C be a point on the graph of h. The tangent to the graph of h at C is parallel to the graph of f. *[5 marks]*

Find the x-coordinate of C.

Markscheme

recognizing that the gradient of the tangent is the derivative ***(M1)***

eg h'

correct derivative (seen anywhere) ***(A1)***

$h'(x) = 4x^3$

correct value for gradient of f (seen anywhere) ***(A1)***

$f'(x) = 1,\ m = 1$

setting **their** derivative equal to 1 ***(M1)***

$4x^3 = 1$

0.629960

$x = \sqrt[3]{\frac{1}{4}}$ (exact), 0.630 ***A1 N3***

[5 marks]

Let θ be an **obtuse** angle such that $\sin\theta = \frac{3}{5}$.

2a. Find the value of $\tan\theta$. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

evidence of valid approach ***(M1)***

eg sketch of triangle with sides 3 and 5, $\cos^2\theta = 1 - \sin^2\theta$

correct working ***(A1)***

eg missing side is 4 (may be seen in sketch), $\cos\theta = \frac{4}{5},\ \cos\theta = -\frac{4}{5}$

$\tan\theta = -\frac{3}{4}$ ***A2 N4***

[4 marks]

2b. Line L passes through the origin and has a gradient of $\tan\theta$. Find the equation of L. *[2 marks]*

Markscheme

correct substitution of either gradient **or** origin into equation of line ***(A1)***

(do not accept $y = mx + b$)

eg $y = x\tan\theta$, $y - 0 = m(x - 0)$, $y = mx$

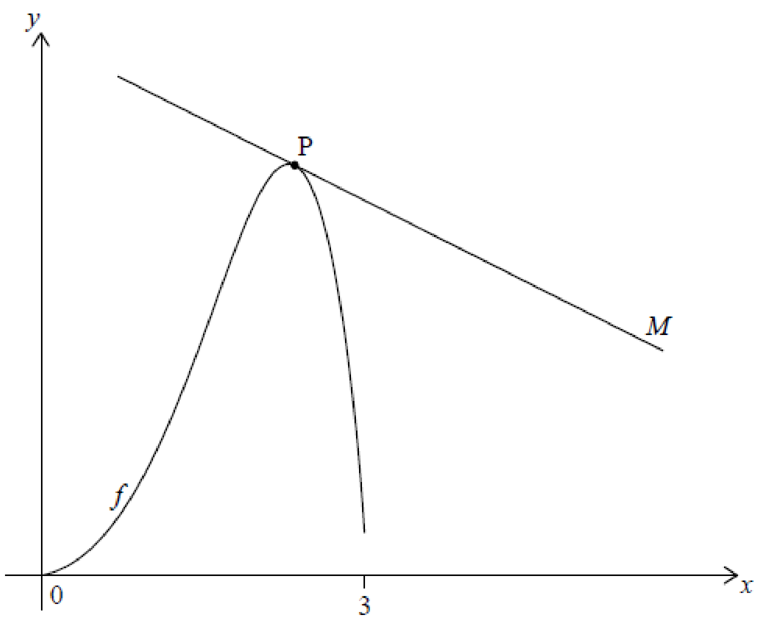
$y = -\frac{3}{4}x$ ***A2 N4***

Note: Award ***A1A0*** for $L = -\frac{3}{4}x$.

[2 marks]

Let $f(x) = \mathrm{e}^x \sin x - \frac{3x}{4}$.

2c. The following diagram shows the graph of f for 0 ≤ x ≤ 3. Line M is a tangent to the graph of f at point P. *[4 marks]*



Given that M is parallel to L, find the x-coordinate of P.

Markscheme

valid approach to equate **their** gradients ***(M1)***

eg $f' = \tan\theta$, $f' = -\frac{3}{4}$, $\mathrm{e}^x \cos x + \mathrm{e}^x \sin x - \frac{3}{4} = -\frac{3}{4}$, $\mathrm{e}^x\left(\cos x + \sin x\right) - \frac{3}{4} = -\frac{3}{4}$

correct equation without e^x ***(A1)***

eg $\sin x = -\cos x$, $\cos x + \sin x = 0$, $\frac{-\sin x}{\cos x} = 1$

correct working ***(A1)***

eg $\tan\theta = -1$, $x = 135°$

$x = \frac{3\pi}{4}$ (do not accept $135°$) ***A1 N1***

Note: Do not award the final ***A1*** if additional answers are given.

[4 marks]

3. Find the coordinates of the points on the curve $y^3 + 3xy^2 - x^3 = 27$ at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[9 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt at implicit differentiation ***M1***

$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 = 0$ ***A1A1***

Note: Award ***A1*** for the second & third terms, ***A1*** for the first term, fourth term & RHS equal to zero.

substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$3y^2 - 3x^2 = 0$

$\Rightarrow y = \pm x$ ***A1***

substitute either variable into original equation ***M1***

$y = x \Rightarrow x^3 = 9 \Rightarrow x = \sqrt[3]{9}$ (or $y^3 = 9 \Rightarrow y = \sqrt[3]{9}$) ***A1***

$y = -x \Rightarrow x^3 = 27 \Rightarrow x = 3$ (or $y^3 = -27 \Rightarrow y = -3$) ***A1***

$\left(\sqrt[3]{9}\ ,\ \sqrt[3]{9}\ \right)$, (3, −3) ***A1***

[9 marks]

The curve C is given by the equation $y = x\tan\left(\frac{\pi xy}{4}\right)$.

4a. At the point (1, 1) , show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+\pi}{2-\pi}$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to differentiate implicitly ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sec^2\left(\frac{\pi xy}{4}\right)\left[\left(\frac{\pi}{4}x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\pi}{4}y\right)\right] + \tan\left(\frac{\pi xy}{4}\right)$ ***A1A1***

Note: Award ***A1*** for each term.

attempt to substitute $x = 1$, $y = 1$ into their equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\pi}{2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\pi}{2} + 1$

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1 - \frac{\pi}{2}\right) = \frac{\pi}{2} + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+\pi}{2-\pi}$ ***AG***

[5 marks]

4b. Hence find the equation of the normal to C at the point (1, 1). *[2 marks]*

Markscheme

attempt to use gradient of normal $= \frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ ***(M1)***

$= \frac{\pi-2}{\pi+2}$

so equation of normal is $y - 1 = \frac{\pi-2}{\pi+2}(x-1)$ or $y = \frac{\pi-2}{\pi+2}x + \frac{4}{\pi+2}$ ***A1***

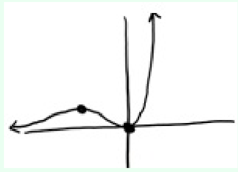
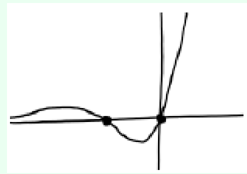
[2 marks]

Consider the function $f(x) = x^2\mathrm{e}^{3x}$, $x \in \mathbb{R}$.

5. The graph of f has a horizontal tangent line at $x = 0$ and at $x = a$. Find a. *[2 marks]*

Markscheme

valid method ***(M1)***

eg $f'(x) = 0$,  , 

$a = -0.667 \left(= -\frac{2}{3}\right)$ (accept $x = -0.667$) ***A1 N2***

[2 marks]

Consider the function $f(x) = \frac{1}{3}x^3 + \frac{3}{4}x^2 - x - 1$.

6a. Write down the y-intercept of the graph of $y = f(x)$. *[1 mark]*

Markscheme

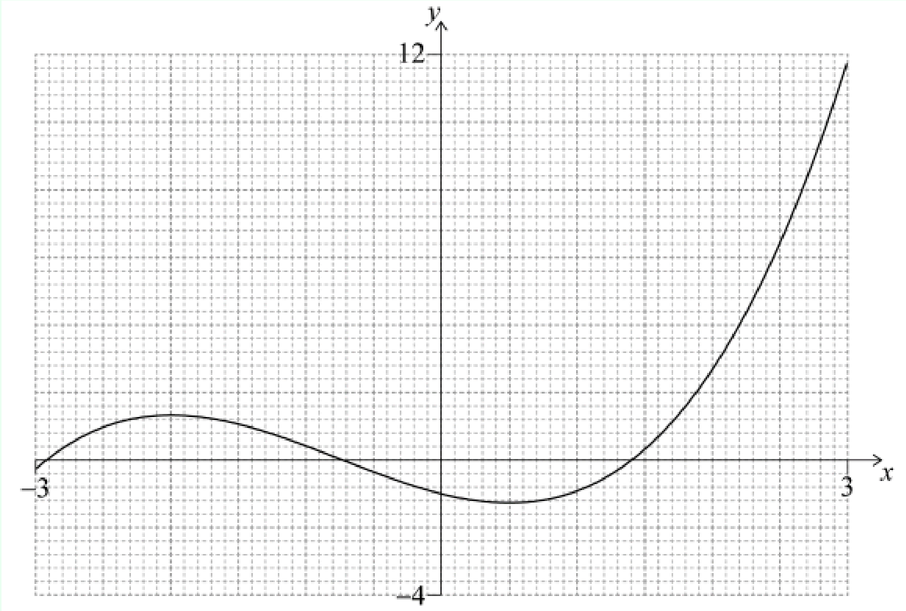
-1 ***(A1)***

Note: Accept (0, −1).

[1 mark]

6b. Sketch the graph of $y = f(x)$ for $-3 \le x \le 3$ and $-4 \le y \le 12$. *[4 marks]*

Markscheme



(A1)(A1)(A1)(A1)

Note: Award ***(A1)*** for correct window and axes labels, −3 to 3 should be indicated on the x-axis and −4 to 12 on the y-axis.

(A1)) for smooth curve with correct cubic shape;

(A1) for x-intercepts: one close to −3, the second between −1 and 0, and third between 1 and 2; and y-intercept at approximately −1;

(A1) for local minimum in the 4th quadrant and maximum in the 2nd quadrant, in approximately correct positions.

Graph paper does not need to be used. If window not given award at most ***(A0)(A1)(A0)(A1)***.

[4 marks]

The function has one local maximum at $x = p$ and one local minimum at $x = q$.

6c. Determine the range of $f\left(x\right)$ for $p \leq x \leq q$. *[3 marks]*

Markscheme

$-1.27 \leqslant f(x) \leqslant 1.33 \quad \left(-1.27083\ldots \leqslant f(x) \leqslant 1.33333\ldots,\ -\frac{61}{48} \leqslant f(x) \leqslant \frac{4}{3}\right.$

(A1)*(ft)*(A1)*(ft)*(A1)

Note: Award ***(A1)*** for −1.27 seen, ***(A1)*** for 1.33 seen, and ***(A1)*** for correct weak inequalities with **their** endpoints in the correct order. For example, award ***(A0)(A0)(A0)*** for answers like $5 \leqslant f(x) \leqslant 2$. Accept y in place of $f(x)$. Accept alternative correct notation such as [−1.27, 1.33].

Follow through from their p and q values from part (g) only if their $f(p)$ and $f(q)$ values are between −4 and 12. Award ***(A0)(A0)(A0)*** if their values from (g) are given as the endpoints.

[3 marks]

7. Let l be the tangent to the curve $y = x\mathrm{e}^{2x}$ at the point $(1, \mathrm{e}^2)$. *[4 marks]*

Find the coordinates of the point where l meets the x-axis.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

equation of tangent is $y = 22.167\ldots x - 14.778\ldots$ **OR** $y = -7.389\ldots = 22.167\ldots(x-1)$ ***(M1)(A1)***

meets the x-axis when $y = 0$

$x = 0.667$

meets x-axis at (0.667, 0)$\left(= \left(\frac{2}{3},\ 0\right)\right)$ ***A1A1***

Note: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates (x, 0) given.

METHOD 1

Attempt to differentiate ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$

when $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^2$ ***(M1)***

equation of the tangent is $y - \mathrm{e}^2 = 3\mathrm{e}^2\left(x - 1\right)$

$y = 3\mathrm{e}^2x - 2\mathrm{e}^2$

meets x-axis at $x = \frac{2}{3}$

$\left(\frac{2}{3},\ 0\right)$ ***A1A1***

Note: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates (x, 0) given.

[4 marks]

8. Using L'Hôpital's rule, find $\lim\limits_{x\to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)$. *[9 marks]*

Markscheme

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$$\lim_{x\to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)$$

$$\lim_{x\to 0}\left(\frac{3\sec^2 3x - 3\sec^2 x}{3\cos 3x - 3\cos x}\right) \quad \left(= \lim_{x\to 0}\left(\frac{\sec^2 3x - \sec^2 x}{\cos 3x - \cos x}\right)\right)$$ ***M1A1A1***

Note: Award ***M1*** for attempt at differentiation using l'Hopital's rule, ***A1*** for numerator, ***A1*** for denominator.

METHOD 1

using l'Hopital's rule again

$$= \lim_{x\to 0}\left(\frac{18\sec^2 3x\tan 3x - 6\sec^2 x\tan x}{-9\sin 3x + 3\sin x}\right) \ \left(= \lim_{x\to 0}\left(\frac{6\sec^2 3x\tan 3x - 2\sec^2 x\tan x}{-3\sin 3x + \sin x}\right)\right)$$ ***A1A1***

EITHER

$$= \lim_{x\to 0}\left(\frac{108\sec^2 3x\tan^2 3x + 54\sec^4 3x - 12\sec^2 x\tan^2 x - 6\sec^4 x}{\text{-}\ 27\cos 3x + 3\cos x}\right)$$ ***A1A1***

Note: Not all terms in numerator need to be written in final fraction. Award ***A1*** for $54\sec^4 3x + \ldots - 6\sec^4 x\ldots -$. However, if the terms are written, they
must be correct to award A1.

attempt to substitute $x = 0$ ***M1***

$= \frac{48}{-24}$

OR

$\frac{\mathrm{d}}{\mathrm{d}x}\left(18\sec^2 3x\tan 3x - 6\sec^2 x\tan x\right)|_{x=0} = 48$ ***(M1)A1***

$\frac{\mathrm{d}}{\mathrm{d}x}(-9\sin 3x + 3\sin x)|_{x=0} = -24$ ***A1***

THEN

$\left(\lim_{x\to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)\right) = -2$ ***A1***

METHOD 2

$$= \lim_{x\to 0}\left(\frac{\frac{3}{\cos^2 3x} - \frac{3}{\cos^2 x}}{3\cos 3x - 3\cos x}\right)$$ ***M1***

$$= \lim_{x\to 0}\left(\frac{\cos^2 x - \cos^2 3x}{\cos^2 3x\cos^2 x(\cos 3x - \cos x)}\right)$$ ***A1***

$$= \lim_{x\to 0}\left(\frac{\cos x + \cos 3x}{-\cos^2 3x\cos^2 x}\right)$$ ***M1A1***

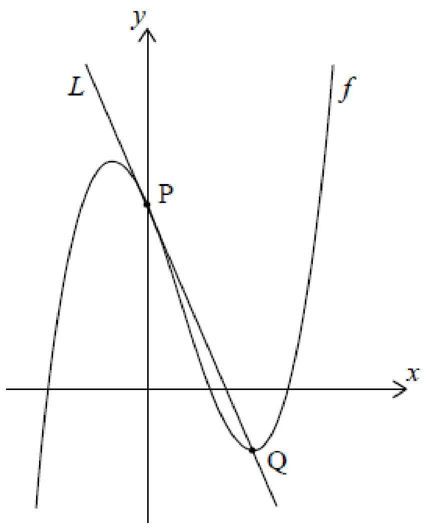
attempt to substitute $x = 0$ ***M1***

$= \frac{2}{-1}$

$= -2$ ***A1***

[9 marks]

Let $f(x) = x^3 - 2x^2 + ax + 6$. Part of the graph of f is shown in the following diagram.



The graph of f crosses the y-axis at the point P. The line *L* is tangent to the graph of f at P.

9a. Find $f'(x)$. *[2 marks]*

Markscheme

$f' = 3x^2 - 4x + a$ ***A2 N2***

[2 marks]

9b. Hence, find the equation of *L* in terms of a. *[4 marks]*

Markscheme

valid approach ***(M1)***

eg $f'(0)$

correct working ***(A1)***

eg $3(0)^2 - 4(0) + a$, slope = a, $f'(0) = a$

attempt to substitute gradient and coordinates into linear equation ***(M1)***

eg $y - 6 = a(x - 0)$, $y - 0 = a(x - 6)$, $6 = a(0) + c$, *L* $= ax + 6$

correct equation ***A1 N3***

eg $y = ax + 6$, $y - 6 = ax$, $y - 6 = a(x - 0)$

[4 marks]

9c. The graph of f has a local minimum at the point Q. The line *L* passes through Q. *[8 marks]*

Find the value of a.

Markscheme

valid approach to find intersection ***(M1)***

eg $f(x) = L$

correct equation ***(A1)***

eg $x^3 - 2x^2 + ax + 6 = ax + 6$

correct working ***(A1)***

eg $x^3 - 2x^2 = 0,\ x^2(x - 2) = 0$

$x = 2$ at Q ***(A1)***

valid approach to find minimum ***(M1)***

eg $f'(x) = 0$

correct equation ***(A1)***

eg $3x^2 - 4x + a = 0$

substitution of **their** value of x at Q into **their** $f'(x) = 0$ equation ***(M1)***

eg $3(2)^2 - 4(2) + a = 0,\ 12 - 8 + a = 0$

$a = -4$ ***A1 N0***

[8 marks]

10. Differentiate from first principles the function $f(x) = 3x^3 - x$. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ ***(A1)***

$= \frac{9x^2h+9xh^2+3h^3-h}{h}$ ***A1*** cancelling h ***M1***

$= 9x^2 + 9xh + 3h^2 - 1$

then $\lim_{h \to 0}\left(9x^2 + 9xh + 3h^2 - 1\right)$

$= 9x^2 - 1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

METHOD 2

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left((x+h)^3-x^3\right)+(x-(x+h))}{h}$ ***(A1)***

$= \frac{3h\left((x+h)^2+x(x+h)+x^2\right)-h}{h}$ ***A1***

cancelling h ***M1***

$= 3\left((x+h)^2 + x(x+h) + x^2\right) - 1$

then $\lim_{h \to 0}\left(3\left((x+h)^2 + x(x+h) + x^2\right) - 1\right)$

$= 9x^2 - 1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

[5 marks]

11. Consider $f(x)$, $g(x)$ and $h(x)$, for $x \in \mathbb{R}$ where $h(x) = \left(f \circ g\right)(x)$. *[7 marks]*

Given that $g(3) = 7$, $g'(3) = 4$ and $f'(7) = -5$, find the gradient of the normal to the curve of h at $x = 3$.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognizing the need to find *h*´ ***(M1)***

recognizing the need to find *h*´(3) (seen anywhere) ***(M1)***

evidence of choosing chain rule ***(M1)***

eg $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},\ f'\left(g\left(3\right)\right) \times g'\left(3\right),\ f'\left(g\right) \times g'$

correct working ***(A1)***

eg $f'\left(7\right) \times 4,\ -5 \times 4$

$h'\left(3\right) = -20$ ***(A1)***

evidence of taking **their** negative reciprocal for normal ***(M1)***

eg $-\frac{1}{h'(3)},\ m_1 m_2 = -1$

gradient of normal is $\frac{1}{20}$ ***A1 N4***

[7 marks]