Kinematics *[147 marks]*

A particle moves in a straight line such that its velocity, $v\ \mathrm{ms}^{-1}$, at time t seconds is given by

$v = 4t^2 - 6t + 9 - 2\sin(4t), 0 \le t \le 1.$

The particle's acceleration is zero at $t = T$.

1a. Find the value of T. *[2 marks]*

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts either graphical or symbolic means to find the value of t when $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ **(M1)**

$T = 0.465\,(\mathrm{s})$ **A1**

[2 marks]

1b. Let s_1 be the distance travelled by the particle from $t = 0$ to $t = T$ and let s_2 be the distance travelled by the particle from $t = T$ to $t = 1$. *[3 marks]*

Show that $s_2 > s_1$.

Markscheme

attempts to find the value of either $s_1 = \int_0^{0.46494\ldots} v \, \mathrm{d}t$ or $s_2 = \int_{0.46494\ldots}^{1} v \, \mathrm{d}t$ **(M1)**

$s_1 = 3.02758\ldots$ and $s_2 = 3.47892\ldots$ **A1A1**

Note: Award as above for obtaining, for example, $s_2 - s_1 = 0.45133\ldots$ or $\frac{s_2}{s_1} = 1.14907\ldots$

Note: Award a maximum of **M1A1A0FT** for use of an incorrect value of T from part (a).

so $s_2 > s_1$ **AG**

[3 marks]

A particle moves along a straight line so that its velocity, $v \mathrm{~m\,s^{-1}}$, after t seconds is given by $v(t) = \mathrm{e}^{\sin t} + 4 \sin t$ for $0 \leq t \leq 6$.

2a. Find the value of t when the particle is at rest. *[2 marks]*

Markscheme

recognizing at rest $v = 0$ ***(M1)***

$t = 3.34692\ldots$

$t = 3.35$ (seconds) ***A1***

Note: Award ***(M1)A0*** for additional solutions to $v = 0$ eg $t = -0.205$ or $t = 6.08$.

[2 marks]

2b. Find the acceleration of the particle when it changes direction. *[3 marks]*

Markscheme

recognizing particle changes direction when $v = 0$ OR when $t = 3.34692\ldots$ ***(M1)***

$a = -4.71439\ldots$

$a = -4.71\left(\mathrm{ms}^{-2}\right)$ ***A2***

[3 marks]

2c. Find the total distance travelled by the particle. *[2 marks]*

Markscheme

distance travelled $= \int_0^6 |v| \mathrm{d}\,t$ OR

$\int_0^{3.34\ldots}\left(\mathrm{e}^{\sin(t)} + 4\sin(t)\right)\mathrm{d}\,t - \int_{3.34\ldots}^{6}\left(\mathrm{e}^{\sin(t)} + 4\sin(t)\right)\mathrm{d}\,t \ \ (= 14.3104\ldots + 6.4$ ***(A1)***

$= 20.7534\ldots$

$= 20.8$ (metres) ***A1***

[2 marks]

A particle moves in a straight line such that its velocity, $v\ \mathrm{m\,s}^{-1}$, at time t seconds is given by $v = \frac{\left(t^2+1\right)\cos t}{4},\ 0 \le t \le 3$.

3a. Determine when the particle changes its direction of motion. *[2 marks]*

Markscheme

recognises the need to find the value of t when $v = 0$ ***(M1)***

$t = 1.57079\ldots \left(= \frac{\pi}{2}\right)$

$t = 1.57 \left(= \frac{\pi}{2}\right)$ (s) ***A1***

[2 marks]

3b. Find the times when the particle's acceleration is $-1.9\,\mathrm{m\,s^{-2}}$. *[3 marks]*

Markscheme

recognises that $a(t) = v'(t)$ ***(M1)***

$t_1 = 2.26277\ldots,\ t_2 = 2.95736\ldots$

$t_1 = 2.26,\ t_2 = 2.96$ (s) ***A1A1***

Note: Award ***M1A1A0*** if the two correct answers are given with additional values outside $0 \le t \le 3$.

[3 marks]

3c. Find the particle's acceleration when its speed is at its greatest. *[2 marks]*

Markscheme

speed is greatest at $t = 3$ ***(A1)***

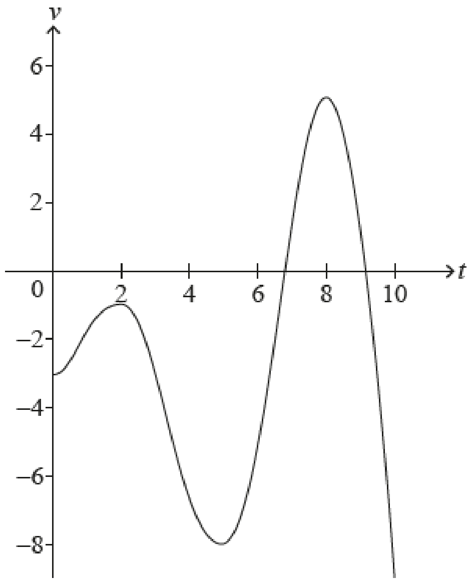
$a = -1.83778\ldots$

$a = -1.84\ \left(\mathrm{m\,s^{-2}}\right)$ ***A1***

[2 marks]

A particle moves in a straight line. The velocity, $v\ \mathrm{ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t \sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of v.



4a. Find the smallest value of t for which the particle is at rest. *[2 marks]*

Markscheme

recognising $v = 0$ ***(M1)***

$t = 6.74416\ldots$

$= 6.74$ (sec) ***A1***

Note: Do not award ***A1*** if additional values are given.

[2 marks]

4b. Find the total distance travelled by the particle. *[2 marks]*

Markscheme

$\int_0^{10} |v(t)|\,\mathrm{d}\,t$ OR $-\int_0^{6.74416\ldots} v(t)\ \mathrm{d}\,t + \int_{6.74416\ldots}^{9.08837\ldots} v(t)\ \mathrm{d}\,t - \int_{9.08837\ldots}^{10} v(t)\ \mathrm{d}\,t$ ***(A1)***

$= 37.0968\ldots$

$= 37.1(\mathrm{m})$ ***A1***

[2 marks]

4c. Find the acceleration of the particle when $t = 7$. *[2 marks]*

Markscheme

recognizing acceleration at $t = 7$ is given by $v'(7)$ ***(M1)***

acceleration $= 5.93430\ldots$

$= 5.93\left(\text{ms}^{-2}\right)$ ***A1***

[2 marks]

A particle P moves in a straight line such that after time t seconds, its velocity, v in $\text{m}\,\text{s}^{-1}$, is given by $v = \text{e}^{-3t}\sin 6t$, where $0 < t < \frac{\pi}{2}$.

5a. Find the times when P comes to instantaneous rest. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\frac{\pi}{6}(= 0.524)$ ***A1***

$\frac{\pi}{3}(= 1.05)$ ***A1***

[2 marks]

At time t, P has displacement $s(t)$; at time $t = 0$, $s(0) = 0$.

5b. Find an expression for s in terms of t. *[7 marks]*

Markscheme

attempt to use integration by parts ***M1***

$s = \int \mathrm{e}^{-3t} \sin 6t \, \mathrm{d}t$

EITHER

$= -\frac{\mathrm{e}^{-3t} \sin 6t}{3} - \int -2\mathrm{e}^{-3t} \cos 6t \, \mathrm{d}t$ ***A1***

$= -\frac{\mathrm{e}^{-3t} \sin 6t}{3} - \left(\frac{2\mathrm{e}^{-3t} \cos 6t}{3} - \int -4\mathrm{e}^{-3t} \sin 6t \, \mathrm{d}t \right)$ ***A1***

$= -\frac{\mathrm{e}^{-3t} \sin 6t}{3} - \left(\frac{2\mathrm{e}^{-3t} \cos 6t}{3} + 4s \right)$

$5s = \frac{-3\mathrm{e}^{-3t} \sin 6t - 6\mathrm{e}^{-3t} \cos 6t}{9}$ ***M1***

OR

$= -\frac{\mathrm{e}^{-3t} \cos 6t}{6} - \int \frac{1}{2}\mathrm{e}^{-3t} \cos 6t \, \mathrm{d}t$ ***A1***

$= -\frac{\mathrm{e}^{-3t} \cos 6t}{6} - \left(\frac{\mathrm{e}^{-3t} \sin 6t}{12} + \int \frac{1}{4}\mathrm{e}^{-3t} \sin 6t \, \mathrm{d}t \right)$ ***A1***

$= -\frac{\mathrm{e}^{-3t} \cos 6t}{6} - \left(\frac{\mathrm{e}^{-3t} \sin 6t}{12} + \frac{1}{4}s \right)$

$\frac{5}{4}s = \frac{-2\mathrm{e}^{-3t} \cos 6t - \mathrm{e}^{-3t} \sin 6t}{12}$ ***M1***

THEN

$s = -\frac{\mathrm{e}^{-3t} (\sin 6t + 2\cos 6t)}{15} (+c)$ ***A1***

at $t = 0, \; s = 0 \Rightarrow 0 = -\frac{2}{15} + c$ ***M1***

$c = \frac{2}{15}$ ***A1***

$s = \frac{2}{15} - \frac{\mathrm{e}^{-3t} (\sin 6t + 2\cos 6t)}{15}$

[7 marks]

5c. Find the maximum displacement of P, in metres, from its initial position. *[2 marks]*

Markscheme

EITHER

substituting $t = \frac{\pi}{6}$ into their equation for s ***(M1)***

$\left(s = \frac{2}{15} - \frac{e^{-\frac{\pi}{2}}(\sin\pi + 2\cos\pi)}{15}\right)$

OR

using GDC to find maximum value ***(M1)***

OR

evaluating $\int_0^{\frac{\pi}{6}} v\mathrm{d}t$ ***(M1)***

THEN

$= 0.161\left(= \frac{2}{15}\left(1 + e^{-\frac{\pi}{2}}\right)\right)$ ***A1***

[2 marks]

5d. Find the total distance travelled by P in the first 1.5 seconds of its motion. *[2 marks]*

Markscheme

METHOD 1

EITHER

distance required $= \int_0^{1.5} \left|e^{-3t} \sin 6t\right| \, dt$ ***(M1)***

OR

distance required $= \int_0^{\frac{\pi}{6}} e^{-3t} \sin 6t \, dt + \left|\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} e^{-3t} \sin 6t \, dt\right| + \int_{\frac{\pi}{3}}^{1.5} e^{-3t} \sin 6t \, dt$

(M1)

$(= 0.16105\ldots + 0.033479\ldots + 0.006806\ldots)$

THEN

$= 0.201 \text{ (m)}$ ***A1***

METHOD 2

using successive minimum and maximum values on the displacement graph ***(M1)***

$0.16105\ldots + (0.16105\ldots - 0.12757\ldots) + (0.13453\ldots - 0.12757\ldots)$

$= 0.201 \text{ (m)}$ ***A1***

[2 marks]

At successive times when the acceleration of P is 0 m s^{-2}, the velocities of P form a geometric sequence. The acceleration of P is zero at times t_1, t_2, t_3 where $t_1 < t_2 < t_3$ and the respective velocities are v_1, v_2, v_3.

5e. Show that, at these times, $\tan 6t = 2$. *[2 marks]*

Markscheme

valid attempt to find $\frac{\mathrm{d}v}{\mathrm{d}t}$ using product rule and set $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{e}^{-3t}6\cos 6t - 3\mathrm{e}^{-3t}\sin 6t$ ***A1***

$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \Rightarrow \tan 6t = 2$ ***AG***

[2 marks]

5f. Hence show that $\frac{v_2}{v_1} = \frac{v_3}{v_2} = -\mathrm{e}^{-\frac{\pi}{2}}$. *[5 marks]*

Markscheme

attempt to evaluate $t_1,\ t_2,\ t_3$ in exact form ***M1***

$6t_1 = \arctan 2\left(\Rightarrow t_1 = \frac{1}{6}\arctan 2\right)$

$6t_2 = \pi + \arctan 2\left(\Rightarrow t_2 = \frac{\pi}{6} + \frac{1}{6}\arctan 2\right)$

$6t_3 = 2\pi + \arctan 2\left(\Rightarrow t_3 = \frac{\pi}{3} + \frac{1}{6}\arctan 2\right)$ ***A1***

Note: The ***A1*** is for any two consecutive correct, or showing that $6t_2 = \pi + 6t_1$ or $6t_3 = \pi + 6t_2$.

showing that $\sin 6t_{n+1} = -\sin 6t_n$

eg $\tan 6t = 2 \Rightarrow \sin 6t = \pm\frac{2}{\sqrt{5}}$ ***M1A1***

showing that $\frac{\mathrm{e}^{-3t_{n+1}}}{\mathrm{e}^{-3t_n}} = \mathrm{e}^{-\frac{\pi}{2}}$ ***M1***

eg $\mathrm{e}^{-3\left(\frac{\pi}{6}+k\right)} \div \mathrm{e}^{-3k} = \mathrm{e}^{-\frac{\pi}{2}}$

Note: Award the ***A1*** for any two consecutive terms.

$\frac{v_3}{v_2} = \frac{v_2}{v_1} = -\mathrm{e}^{-\frac{\pi}{2}}$ ***AG***

[5 marks]

A rocket is travelling in a straight line, with an initial velocity of $140\,\mathrm{m\,s^{-1}}$. It accelerates to a new velocity of $500\,\mathrm{m\,s^{-1}}$ in two stages.

During the first stage its acceleration, $a\,\mathrm{m\,s^{-2}}$, after t seconds is given by $a(t) = 240\sin(2t)$, where $0 \leqslant t \leqslant k$.

6a. Find an expression for the velocity, $v\,\mathrm{m\,s^{-1}}$, of the rocket during the first stage. *[4 marks]*

Markscheme

recognizing that $v = \int a$ ***(M1)***

correct integration ***A1***

eg $-120\cos(2t) + c$

attempt to find c using their $v(t)$ ***(M1)***

eg $-120\cos(0) + c = 140$

$v(t) = -120\cos(2t) + 260$ ***A1 N3***

[4 marks]

The first stage continues for k seconds until the velocity of the rocket reaches 375 $\mathrm{m\,s^{-1}}$.

6b. Find the distance that the rocket travels during the first stage. *[4 marks]*

Markscheme

evidence of valid approach to find time taken in first stage ***(M1)***

eg graph, $-120\cos(2t) + 260 = 375$

$k = 1.42595$ ***A1***

attempt to substitute **their** v and/or **their** limits into distance formula ***(M1)***

eg $\int_0^{1.42595} |v|$, $\int 260 - 120\cos(2t)$, $\int_0^k (260 - 120\cos(2t))\,\mathrm{d}t$

353.608

distance is 354 (m) ***A1 N3***

[4 marks]

6c. During the second stage, the rocket accelerates at a constant rate. The distance which the rocket travels during the second stage is the same as the distance it travels during the first stage. *[6 marks]*

Find the total time taken for the two stages.

Markscheme

recognizing velocity of second stage is linear (seen anywhere) ***R1***

eg graph, $s = \frac{1}{2}h(a+b)$, $v = mt + c$

valid approach ***(M1)***

eg $\int v = 353.608$

correct equation ***(A1)***

eg $\frac{1}{2}h(375+500) = 353.608$

time for stage two $= 0.808248$ (0.809142 from 3 sf) ***A2***

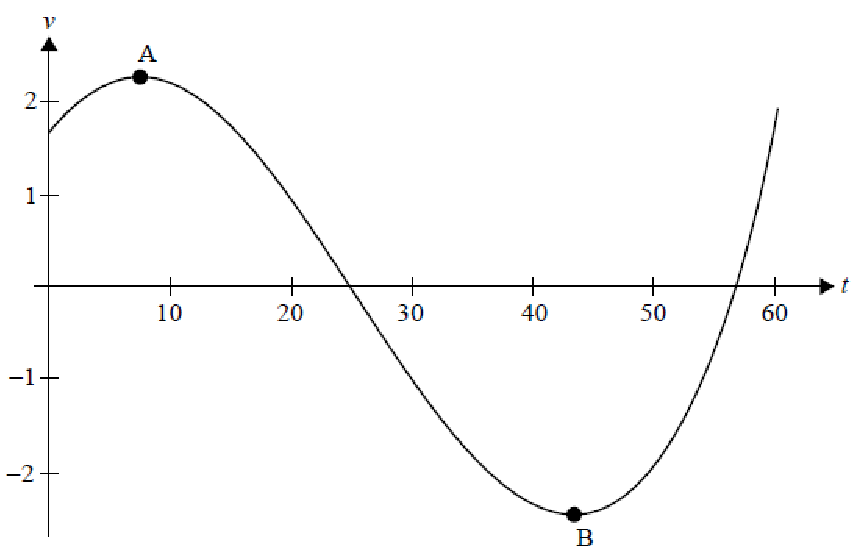
2.23420 (2.23914 from 3 sf)

2.23 seconds (2.24 from 3 sf) ***A1 N3***

[6 marks]

A body moves in a straight line such that its velocity, $v\,\text{ms}^{-1}$, after t seconds is given by $v = 2\sin\left(\frac{t}{10} + \frac{\pi}{5}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right)$ for $0 \leqslant t \leqslant 60$.

The following diagram shows the graph of v against t. Point A is a local maximum and point B is a local minimum.



7a. Determine the coordinates of point A and the coordinates of point B. *[4 marks]*

Markscheme

$\mathrm{A}\,(7.47,\,2.28)$ and $\mathrm{B}\,(43.4,\,-\,2.45)$ ***A1A1A1A1***

[4 marks]

7b. Hence, write down the maximum speed of the body. *[1 mark]*

Markscheme

maximum speed is $2.45\ \left(\mathrm{ms}^{-1}\right)$ ***A1***

[1 mark]

The body first comes to rest at time $t = t_1$. Find

7c. the value of t_1. *[2 marks]*

Markscheme

$v = 0 \Rightarrow t_1 = 25.1\ (\mathrm{s})$ ***(M1)A1***

[2 marks]

7d. the distance travelled between $t = 0$ and $t = t_1$. *[2 marks]*

Markscheme

$\int_0^{t_1} v\,\mathrm{d}t$ ***(M1)***

$= 41.0\ (\mathrm{m})$ ***A1***

[2 marks]

7e. the acceleration when $t = t_1$. *[2 marks]*

Markscheme

$a = \frac{dv}{dt}$ at $t = t_1 = 25.1$ ***(M1)***

$a = -0.200\ \left(\text{ms}^{-2}\right)$ ***A1***

Note: Accept $a = -0.2$.

[2 marks]

7f. Find the distance travelled in the first 30 seconds. *[3 marks]*

Markscheme

attempt to integrate between 0 and 30 ***(M1)***

Note: An unsupported answer of 38.6 can imply integrating from 0 to 30.

EITHER

$\int_0^{30} |v|\ dt$ ***(A1)***

OR

$41.0 - \int_{t_1}^{30} v\, dt$ ***(A1)***

THEN

$= 43.3\ (\text{m})$ ***A1***

[3 marks]

8. A particle moves along a horizontal line such that at time t seconds, $t \geq 0$, its acceleration a is given by $a = 2t - 1$. When $t = 6$, its displacement s from a fixed origin O is 18.25 m. When $t = 15$, its displacement from O is 922.75 m. Find an expression for s in terms of t. *[6 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to integrate a to find v ***M1***

$v = \int a \, \mathrm{d}t = \int (2t - 1) \, \mathrm{d}t$

$= t^2 - t + c$ ***A1***

$s = \int v \, \mathrm{d}t = \int \left(t^2 - t + c\right) \, \mathrm{d}t$

$= \frac{t^3}{3} - \frac{t^2}{2} + ct + d$ ***A1***

attempt at substitution of given values ***(M1)***

at $t = 6,\ 18.25 = 72 - 18 + 6c + d$

at $t = 15,\ 922.75 = 1125 - 112.5 + 15c + d$

solve simultaneously: ***(M1)***

$c = -6,\ d = 0.25$ ***A1***

$\Rightarrow s = \frac{t^3}{3} - \frac{t^2}{2} + -6t + \frac{1}{4}$

[6 marks]

A particle moves along a straight line so that its velocity, v m s^{-1}, after t seconds is given by $v\left(t\right) = 1.4^t - 2.7$, for 0 ≤ t ≤ 5.

9a. Find when the particle is at rest. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach ***(M1)***

eg $v\left(t\right) = 0$, sketch of graph

2.95195

$t = \log_{1.4} 2.7$ (exact), $t = 2.95$ (s) ***A1 N2***

[2 marks]

9b. Find the acceleration of the particle when $t = 2$. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $a(t) = v'(t)$, $v'(2)$

0.659485

$a(2)$ = 1.96 ln 1.4 (exact), $a(2)$ = 0.659 (m s^{-2}) ***A1 N2***

[2 marks]

9c. Find the total distance travelled by the particle. *[3 marks]*

Markscheme

correct approach ***(A1)***

eg $\int_0^5 |v(t)| \, \mathrm{d}t$, $\int_0^{2.95} (-v(t)) \, \mathrm{d}t + \int_{295}^{5} v(t) \, \mathrm{d}t$

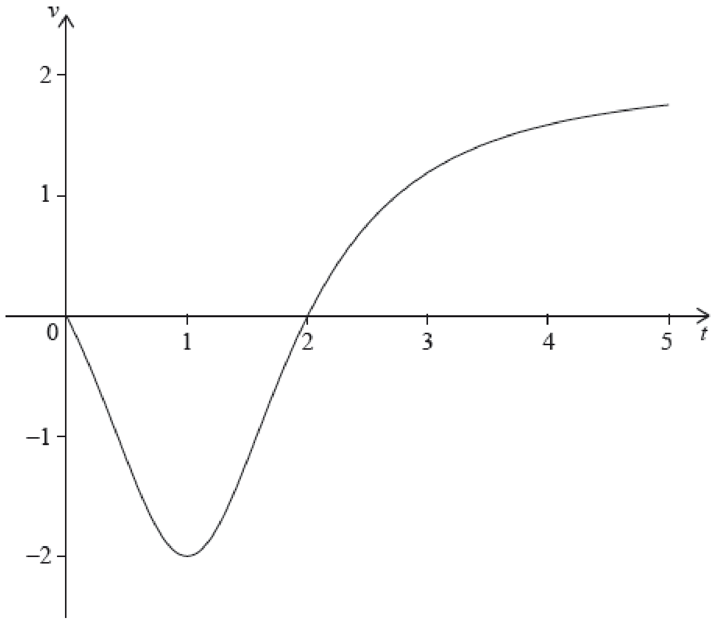
5.3479

distance = 5.35 (m) ***A2 N3***

[3 marks]

10. **Note: In this question, distance is in metres and time is in seconds.** *[6 marks]*

A particle moves along a horizontal line starting at a fixed point A. The velocity v of the particle, at time t, is given by $v(t) = \frac{2t^2-4t}{t^2-2t+2}$, for $0 \leqslant t \leqslant 5$. The following diagram shows the graph of v



There are t-intercepts at $(0,\ 0)$ and $(2,\ 0)$.

Find the maximum distance of the particle from A during the time $0 \leqslant t \leqslant 5$ and justify your answer.

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1 (displacement)

recognizing $s = \int v\mathrm{d}t$ ***(M1)***

consideration of displacement at $t = 2$ **and** $t = 5$ (seen anywhere) ***M1***

eg $\int_0^2 v$ and $\int_0^5 v$

Note: Must have both for any further marks.

correct displacement at $t = 2$ and $t = 5$ (seen anywhere) ***A1A1***

-2.28318 (accept 2.28318), 1.55513

valid reasoning comparing correct displacements ***R1***

eg $|-2.28| > |1.56|$, more left than right

2.28 (m) ***A1*** ***N1***

Note: Do not award the final ***A1*** without the ***R1****.*

METHOD 2 (distance travelled)

recognizing distance $= \int |v| \, \mathrm{d}t$ ***(M1)***

consideration of distance travelled from $t = 0$ to 2 **and** $t = 2$ to 5 (seen anywhere) ***M1***

eg $\int_0^2 v$ and $\int_2^5 v$

Note: Must have both for any further marks

correct distances travelled (seen anywhere) ***A1A1***

2.28318, (accept -2.28318), 3.83832

valid reasoning comparing correct distance values ***R1***

eg $3.84 - 2.28 < 2.28, \; 3.84 < 2 \times 2.28$

2.28 (m) ***A1*** ***N1***

Note: Do not award the final ***A1*** without the ***R1***.

[6 marks]

A particle P moves along the x-axis. The velocity of P is $v \text{ m s}^{-1}$ at time t seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0, \; P$ is at the origin O.

11a. Find the value of t when P reaches its maximum velocity. *[2 marks]*

Markscheme

valid approach to find turning point ($v' = 0,\ -\frac{b}{2a}$, average of roots)
(M1)

$4 - 6t = 0$ OR $-\frac{4}{2(-3)}$ OR $\frac{-\frac{2}{3}+2}{2}$

$t = \frac{2}{3}$ (s) ***A1***

[2 marks]

11b. Show that the distance of P from O at this time is $\frac{88}{27}$ metres. *[5 marks]*

Markscheme

attempt to integrate v ***(M1)***

$\int v \, \mathrm{d}t = \int \left(4 + 4t - 3t^2\right) \, \mathrm{d}t = 4t + 2t^2 - t^3 (+c)$ ***A1A1***

Note: Award ***A1*** for $4t + 2t^2$, ***A1*** for $-t^3$.

attempt to substitute their t into their solution for the integral ***(M1)***

distance $= 4\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$

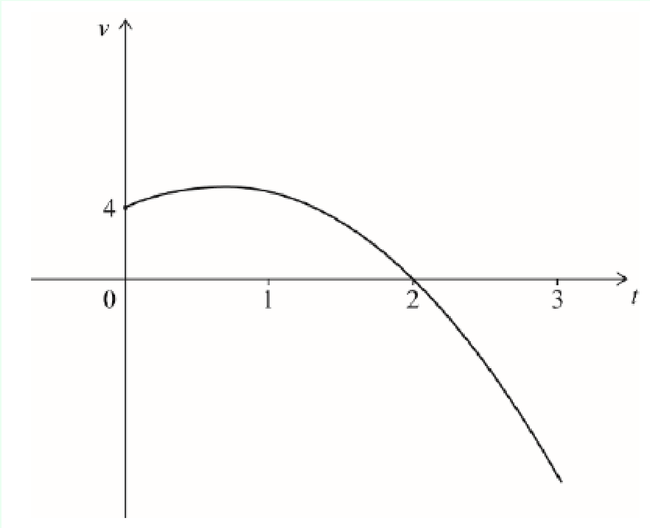
$= \frac{8}{3} + \frac{8}{9} - \frac{8}{27}$ (or equivalent) ***A1***

$= \frac{88}{27}$ (m) ***AG***

[5 marks]

11c. Sketch a graph of v against t, clearly showing any points of intersection with the axes. *[4 marks]*

Markscheme



valid approach to solve $4 + 4t - 3t^2 = 0$ (may be seen in part (a))
(M1)

$(2 - t)(2 + 3t)$ OR $\frac{-4 \pm \sqrt{16+48}}{-6}$

correct x- intercept on the graph at $t = 2$ ***A1***

Note: The following two ***A*** marks may only be awarded if the shape is a concave down parabola. These two marks are independent of each other and the ***(M1)***.

correct domain from 0 to 3 starting at $(0, 4)$ ***A1***

Note: The 3 must be clearly indicated.

vertex in approximately correct place for $t = \frac{2}{3}$ and $v > 4$ ***A1***

[4 marks]

11d. Find the total distance travelled by P.   *[5 marks]*

Markscheme

recognising to integrate between 0 and 2, or 2 and 3 OR $\int_0^3 \left|4+4t-3t^2\right| \mathrm{d}t$ ***(M1)***

$\int_0^2 \left(4+4t-3t^2\right) \mathrm{d}t$

$=8$ ***A1***

$\int_2^3 \left(4+4t-3t^2\right) \mathrm{d}t$

$=-5$ ***A1***

valid approach to sum the two areas (seen anywhere) ***(M1)***

$\int_0^2 v \, \mathrm{d}t - \int_2^3 v \, \mathrm{d}t$ OR $\int_0^2 v \, \mathrm{d}t + \left|\int_2^3 v \, \mathrm{d}t\right|$

total distance travelled $=13$ (m) ***A1***

[5 marks]

The acceleration, $a \text{ ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v \text{ ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t=0$, the particle is at a fixed origin O and has initial velocity $v_0 \text{ ms}^{-1}$.

12a. By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1+v_0)\mathrm{e}^{-t} - 1$. *[6 marks]*

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}t} = -(1+v)$ ***(A1)***

$\int 1 \, \mathrm{d}t = \int -\frac{1}{1+v} \, \mathrm{d}v$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$C = \ln(1 + v_0)$

$t = \ln(1 + v_0) - \ln(1 + v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow \mathrm{e}^t = \frac{1+v_0}{1+v}$ ***A1***

$\mathrm{e}^t(1 + v) = 1 + v_0$

$1 + v = (1 + v_0)\mathrm{e}^{-t}$ ***A1***

$v(t) = (1 + v_0)\mathrm{e}^{-t} - 1$ ***AG***

OR

$t - C = -\ln(1 + v) \Rightarrow \mathrm{e}^{t-C} = \frac{1}{(1+v)}$

Attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$\mathrm{e}^{-C} = \frac{1}{(1+v_0)} \Rightarrow C = \ln(1 + v_0)$

$t - \ln(1 + v_0) = -\ln(1 + v) \Rightarrow t = \ln(1 + v_0) - \ln(1 + v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow \mathrm{e}^t = \frac{1+v_0}{1+v}$ ***A1***

$\mathrm{e}^t(1 + v) = 1 + v_0$

$1 + v = (1 + v_0)\mathrm{e}^{-t}$ ***A1***

$v(t) = (1 + v_0)\mathrm{e}^{-t} - 1$ ***AG***

OR

$t - C = -\ln(1 + v) \Rightarrow \mathrm{e}^{-t+C} = 1 + v$ ***A1***

$k\mathrm{e}^{-t} - 1 = v$

Attempt to find k with initial conditions $t = 0, v = v_0$ ***M1***

$k = 1 + v_0$

$\mathrm{e}^{-t}(1 + v_0) = 1 + v$ ***A1***

$v(t) = (1 + v_0)\mathrm{e}^{-t} - 1$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

12b. Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\mathrm{e}^T = 1 + v_0$. *[2 marks]*

Markscheme

recognition that when $t = T, v = 0$ ***M1***

$(1 + v_0)\mathrm{e}^{-T} - 1 = 0 \Rightarrow \mathrm{e}^{-T} = \frac{1}{1+v_0}$ ***A1***

$\mathrm{e}^T = 1 + v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = \mathrm{e}^T - 1$ into v and showing that $v = 0$.

[6 marks]

12c. By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. *[5 marks]*

Markscheme

$s(t) = \int v(t)\mathrm{d}\,t\left(= \int\left((1 + v_0)\mathrm{e}^{-t} - 1\right)\mathrm{d}\,t\right)$ ***(M1)***

$= -(1 + v_0)\mathrm{e}^{-t} - t(+D)$ ***A1***

($t = 0, s = 0$ so) $D = 1 + v_0$ ***A1***

$s(t) = -(1 + v_0)\mathrm{e}^{-t} - t + 1 + v_0$

at $s_{\max}$, $\mathrm{e}^T = 1 + v_0 \Rightarrow T = \ln(1 + v_0)$

Substituting into $s(t)\left(= -(1 + v_0)\mathrm{e}^{-t} - t + 1 + v_0\right)$ ***M1***

$s_{\max} = -(1 + v_0)\left(\frac{1}{1+v_0}\right) - \ln(1 + v_0) + v_0 + 1$ ***A1***

$\left(s_{\max} = v_0 - \ln(1 + v_0)\right)$

[5 marks]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v\left(T-k\right) = (1+v_0)\mathrm{e}^{-(T-k)} - 1.$$

12d. By using the result to part (b) (i), show that $v\left(T-k\right) = \mathrm{e}^k - 1$. *[2 marks]*

Markscheme

METHOD 1

$v(T-k) = (1+v_0)\mathrm{e}^{-T}\mathrm{e}^k - 1$ ***(M1)***

$= (1+v_0)\left(\frac{1}{1+v_0}\right)\mathrm{e}^k - 1$ ***A1***

$= \mathrm{e}^k - 1$ ***AG***

METHOD 2

$v(T-k) = (1+v_0)\mathrm{e}^{-\left(T-k\right)} - 1$

$= e^T\mathrm{e}^{-\left(T-k\right)} - 1$ ***M1***

$= e^{T-T+k} - 1$ ***A1***

$= \mathrm{e}^k - 1$ ***AG***

[2 marks]

Similarly, let $v\left(T+k\right)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

12e. Deduce a similar expression for $v\left(T+k\right)$ in terms of k. *[2 marks]*

Markscheme

METHOD 1

$v(T+k) = (1+v_0)\mathrm{e}^{-T}\mathrm{e}^{-k} - 1$ ***(A1)***

$= \mathrm{e}^{-k} - 1$ ***A1***

METHOD 2

$v(T+k) = (1+v_0)\mathrm{e}^{-(T+k)} - 1$ ***(A1)***

$= \mathrm{e}^{T}\mathrm{e}^{-(T+k)} - 1$

$= \mathrm{e}^{T-T-k} - 1$

$= \mathrm{e}^{-k} - 1$ ***A1***

[2 marks]

12f. Hence, show that $v(T-k)+v(T+k) \geq 0$. *[3 marks]*

Markscheme

METHOD 1

$v(T-k)+v(T+k) = \mathrm{e}^{k} + \mathrm{e}^{-k} - 2$ ***A1***

attempt to express as a square ***M1***

$= \left(\mathrm{e}^{\frac{k}{2}} - \mathrm{e}^{-\frac{k}{2}}\right)^2 (\geq 0)$ ***A1***

so $v(T-k)+v(T+k) \geq 0$ ***AG***

METHOD 2

$v(T-k)+v(T+k) = \mathrm{e}^{k} + \mathrm{e}^{-k} - 2$ ***A1***

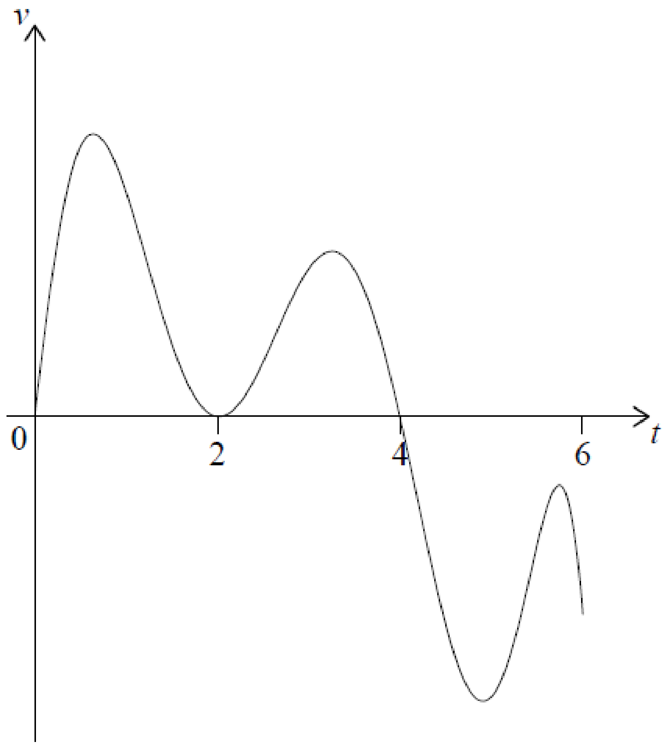
Attempt to solve $\frac{\mathrm{d}}{\mathrm{d}k}\left(\mathrm{e}^{k} + \mathrm{e}^{-k}\right) = 0 (\Rightarrow k = 0)$ ***M1***

minimum value of 2, (when $k=0$), hence $\mathrm{e}^{k} + \mathrm{e}^{-k} \geq 2$ ***R1***

so $v(T-k)+v(T+k) \geq 0$ ***AG***

[3 marks]

A particle P starts from point O and moves along a straight line. The graph of its velocity, v ms^{-1} after t seconds, for $0 \le t \le 6$, is shown in the following diagram.



The graph of v has t-intercepts when t = 0, 2 and 4.

The function $s(t)$ represents the displacement of P from O after t seconds.

It is known that P travels a distance of 15 metres in the first 2 seconds. It is also known that $s(2) = s(5)$ and $\int_2^4 v\,\mathrm{d}t = 9$.

13a. Find the value of $s(4) - s(2)$. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognizing relationship between v and s ***(M1)***

eg $\int v = s,\ s' = v$

$s(4) - s(2) = 9$ ***A1 N2***

[2 marks]

13b. Find the total distance travelled in the first 5 seconds. *[5 marks]*

Markscheme

correctly interpreting distance travelled in first 2 seconds (seen anywhere, including part (a) or the area of 15 indicated on diagram) ***(A1)***

eg $\int_0^2 v = 15,\ s(2) = 15$

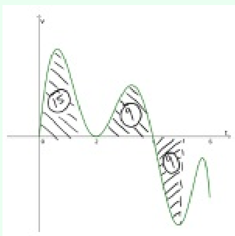
valid approach to find total distance travelled ***(M1)***

eg sum of 3 areas, $\int_0^4 v + \int_4^5 v$, shaded areas in diagram between 0 and 5

Note: Award ***M0*** if only $\int_0^5 |v|$ is seen.

correct working towards finding distance travelled between 2 and 5 (seen anywhere including within total area expression or on diagram) ***(A1)***

eg $\int_2^4 v - \int_4^5 v,\ \int_2^4 v = \int_4^5 |v|,\ \int_4^5 v\,\mathrm{d}t = -9,\ s(4) - s(2) - [s(5) - s(4)]$,

equal areas 

correct working using $s(5) = s(2)$ ***(A1)***

eg $15 + 9 - (-9),\ 15 + 2[s(4) - s(2)],\ 15 + 2(9),\ 2 \times s(4) - s(2)$, $48 - 15$

total distance travelled = 33 (m) ***A1 N2***

[5 marks]

14. A particle moves in a straight line such that at time t seconds $(t \geqslant 0)$, its velocity v, in ms^{-1}, is given by $v = 10t\mathrm{e}^{-2t}$. Find the exact distance travelled by the particle in the first half-second. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$s = \int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\mathrm{d}t$

attempt at integration by parts ***M1***

$= \left[-5t\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -5\mathrm{e}^{-2t}\mathrm{d}t$ ***A1***

$= \left[-5t\mathrm{e}^{-2t} - \frac{5}{2}\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}}$ ***(A1)***

Note: Condone absence of limits (or incorrect limits) and missing factor of 10 up to this point.

$s = \int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\mathrm{d}t$ ***(M1)***

$= -5\mathrm{e}^{-1} + \frac{5}{2} \left(= \frac{-5}{\mathrm{e}} + \frac{5}{2}\right) \left(= \frac{5\mathrm{e}-10}{2\mathrm{e}}\right)$ ***A1***

[5 marks]

A particle moves along a straight line. Its displacement, s metres, at time t seconds is given by $s = t + \cos 2t, t \geqslant 0$. The first two times when the particle is at rest are denoted by t_1 and t_2, where $t_1 < t_2$.

15a. Find t_1 and t_2. *[5 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$s = t + \cos 2t$

$\frac{\mathrm{d}s}{\mathrm{d}t} = 1 - 2\sin 2t$ ***M1A1***

$= 0$ ***M1***

$\Rightarrow \sin 2t = \frac{1}{2}$

$t_1 = \frac{\pi}{12}(s), t_2 = \frac{5\pi}{12}(s)$ ***A1A1***

Note: Award ***A0A0*** if answers are given in degrees.

[5 marks]

15b. Find the displacement of the particle when $t = t_1$ *[2 marks]*

Markscheme

$s = \frac{\pi}{12} + \cos\frac{\pi}{6}\ \left(s = \frac{\pi}{12} + \frac{\sqrt{3}}{2}(m)\right)$ ***A1A1***

[2 marks]