Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 1

26 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

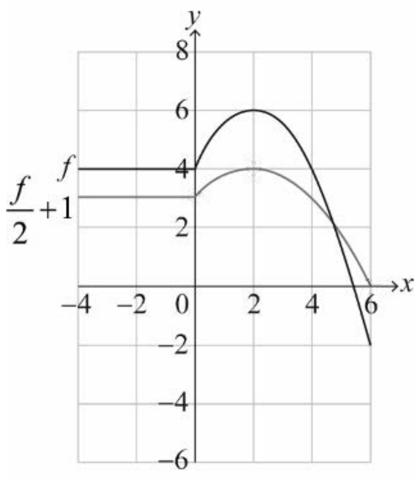
Section A

1. (a) (i) $f(2)=6$ ***A1***

(ii) $(f \circ f)(2)=-2$ ***A1***

[2 marks]

(b)



M1A1A1

> **Note:** Award ***M1*** for an attempt to apply any vertical stretch or vertical translation,
> ***A1*** for a correct horizontal line segment between –4 and 0 (located roughly at $y=3$),
> ***A1*** for a correct concave down parabola including max point at (2,4) and for correct end points at (0,3) and (6,0) (within circles). Points do not need to be labelled.

[3 marks]
Total [5 marks]

2 **METHOD 1 (finding u_1 first, from S_8)**

$4(u_1+8)=8$ ***(A1)***

$u_1=-6$ ***A1***

$u_1+7d=8$ OR $4(2u_1+7d)=8$ (may be seen with their value of u_1) ***(A1)***

attempt to substitute their u_1 ***(M1)***

$d=2$ ***A1***

METHOD 2 (solving simultaneously)

$u_1+7d=8$ ***(A1)***

$4(u_1+8)=8$ OR $4(2u_1+7d)=8$ OR $u_1=-3d$ ***(A1)***

attempt to solve linear or simultaneous equations ***(M1)***

$u_1=-6, d=2$ ***A1A1***

[5 marks]

3. (a) attempt to use definition of outlier

$1.5\times 20+Q_3$ ***(M1)***

$1.5\times 20+U\geq 75$ $\left(\Rightarrow U\geq 45\right.$, accept $\left.U>45\right)$ OR $1.5\times 20+Q_3=75$ ***A1***

minimum value of $U=45$ ***A1***

[3 marks]

(b) attempt to use interquartile range ***(M1)***

$U-L=20$ (may be seen in part (a)) OR $L\geq 25$ (accept $L>25$)

minimum value of $L=25$ ***A1***

[2 marks]

Total [5 marks]

4. (a) $f'(x) = -2(x-h)$ ***A1***

[1 mark]

(b) $g'(x) = e^{x-2}$ OR $g'(3) = e^{3-2}$ (may be seen anywhere) ***A1***

Note: The derivative of g must be explicitly seen, either in terms of x or 3.

recognizing $f'(3) = g'(3)$ ***(M1)***

$-2(3-h) = e^{3-2} \ (= e)$

$-6 + 2h = e$ OR $3 - h = -\frac{e}{2}$ ***A1***

Note: The final ***A1*** is dependent on one of the previous marks being awarded.

$h = \frac{e+6}{2}$ ***AG***

[3 marks]

(c) $f(3) = g(3)$ ***(M1)***

$-(3-h)^2 + 2k = e^{3-2} + k$

correct equation in k

EITHER

$-\left(3 - \frac{e+6}{2}\right)^2 + 2k = e^{3-2} + k$ ***A1***

$k = e + \left(\frac{6-e-6}{2}\right)^2 \left(= e + \left(\frac{-e}{2}\right)^2\right)$ ***A1***

OR

$k = e + \left(3 - \frac{e+6}{2}\right)^2$ ***A1***

$k = e + 9 - 3e - 18 + \frac{e^2 + 12e + 36}{4}$ ***A1***

THEN

$k = e + \frac{e^2}{4}$ ***AG***

[3 marks]
Total [7 marks]

5. (a)

Note: Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS= $2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$ =RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

(b) attempt to factorise ***M1***

$(\cos x - \sin x)(2\sin x + 1) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \frac{\sin x}{\cos x} = \tan x = 1$ OR $\sin x = -\frac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\frac{\pi}{4}$ OR $\frac{\pi}{6}$ (accept $-\frac{\pi}{6}, \frac{7\pi}{6}$)

Note: This ***(M1)(A1)*** is independent of the previous ***M1A1.***

$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{4}$ ***A2***

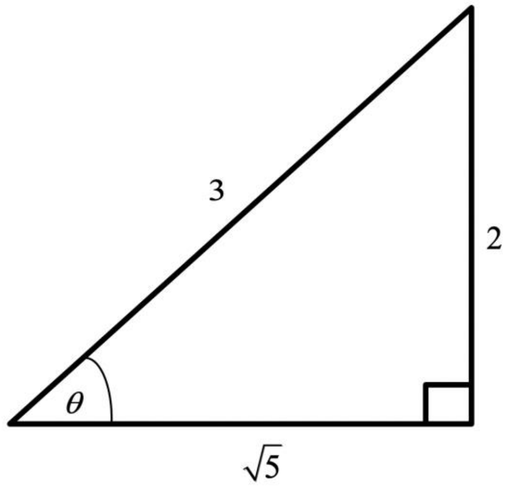
Note: Award ***A1*** for any two correct (radian) answers.
Award ***A1A0*** if additional values given with the four correct (radian) answers. Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

Total [8 marks]

6. METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

> **Note:** Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
> The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

METHOD 2

Attempt to use $1+\cot^2\theta = \operatorname{cosec}^2\theta$ ***M1***

$1+\cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

> **Note:** Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
> The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

7. METHOD 1

other two roots are $a-b\mathrm{i}$ and $b-a\mathrm{i}$ ***A1***

sum of roots $=-4$ and product of roots $=400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a+b\mathrm{i}+a-b\mathrm{i}+b+a\mathrm{i}+b-a\mathrm{i}=-4$

$2a+2b=-4\left(\Rightarrow a+b=-2\right)$ ***A1***

$\left(a+b\mathrm{i}\right)\left(a-b\mathrm{i}\right)\left(b+a\mathrm{i}\right)\left(b-a\mathrm{i}\right)=400$

$\left(a^2+b^2\right)^2=400$ ***A1***

$a^2+b^2=20$

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

METHOD 2

other two roots are $a-b\mathrm{i}$ and $b-a\mathrm{i}$ ***A1***

$\left(z-\left(a+b\mathrm{i}\right)\right)\left(z-\left(a-b\mathrm{i}\right)\right)\left(z-\left(b+a\mathrm{i}\right)\right)\left(z-\left(b-a\mathrm{i}\right)\right)(=0)$ ***A1***

$\left((z-a)^2+b^2\right)\left((z-b)^2+a^2\right)(=0)$

$\left(z^2-2az+a^2+b^2\right)\left(z^2-2bz+b^2+a^2\right)(=0)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a-2b=4$ and $\left(a^2+b^2\right)^2=400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

8. attempt to differentiate numerator and denominator ***M1***

$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$$ ***A1A1***

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x=0$ ***(M1)***

$$=\frac{2}{3}$$ ***A1***

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

9. (a) **METHOD 1**

B has one less pen to select *(M1)*

EITHER

A and B can be placed in 6×5 ways *(A1)*

C, D, E have 6 choices each *(A1)*

OR

A (or B), C, D, E have 6 choices each *(A1)*

B (or A) has only 5 choices *(A1)*

THEN

$5\times 6^4\left(=6480\right)$ ***A1***

METHOD 2

total number of ways $=6^5$ *(A1)*

number of ways with Amber and Brownie together $=6^4$ *(A1)*

attempt to subtract (may be seen in words) *(M1)*

6^5-6^4

$=5\times 6^4\left(=6480\right)$ ***A1***

[4 marks]

(b) **METHOD 1**

total number of ways $=6!\left(=720\right)$ *(A1)*

number of ways with Amber and Brownie sharing a boundary

$=2\times 7\times 4!\left(=336\right)$ *(A1)*

attempt to subtract (may be seen in words) *(M1)*

$720-336=384$ ***A1***

METHOD 2

case 1: number of ways of placing A in corner pen

$3\times 4\times 3\times 2\times 1$

Four corners total no of ways is $4\times\left(3\times 4\times 3\times 2\times 1\right)=12\times 4!\left(=288\right)$ *(A1)*

case 2: number of ways of placing A in the middle pen

$2\times 4\times 3\times 2\times 1$

two middle pens so $2\times\left(2\times 4\times 3\times 2\times 1\right)=4\times 4!\left(=96\right)$ *(A1)*

attempt to add (may be seen in words) *(M1)*

total no of ways $=288+96$

$=16\times 4!\left(=384\right)$ ***A1***

[4 marks]

Total [8 marks]

Section B

10. (a) recognising probabilities sum to 1 *(M1)*

$p + p + p + \frac{1}{2}p = 1$

$p = \frac{2}{7}$ ***A1***

[2 marks]

(b) valid attempt to find $\mathrm{E}(X)$ *(M1)*

$1 \times p + 2 \times p + 3 \times p + 4 \times \frac{1}{2}p \left(= 8p\right)$

$\mathrm{E}(X) = \frac{16}{7}$ ***A1***

[2 marks]

(c) (i) $0 \le r \le 1$ ***A1***

(ii) Attempt to find a value of q *(M1)*

$0 \le 1 - 3q \le 1$ OR $r = 0 \Rightarrow q = \frac{1}{3}$ OR $r = 1 \Rightarrow q = 0$

$0 \le q \le \frac{1}{3}$ ***A1***

[3 marks]

(d) $\mathrm{E}(Y) = 1 \times q + 2 \times q + 3 \times q + 4 \times r \left(= 2 + 2r \text{ OR } 4 - 6q\right)$ *(A1)*

one correct boundary value ***A1***

$1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} + 4 \times 0 \left(= 2\right)$ OR

$1 \times 0 + 2 \times 0 + 3 \times 0 + 4 \times 1 \left(= 4\right)$ OR

$2 + 2(0) \left(= 2\right)$ OR

$2 + 2(1) \left(= 4\right)$ OR

$4 - 6(0) \left(= 4\right)$ OR $4 - 6\left(\frac{1}{3}\right)\left(= 2\right)$

$2 \le \mathrm{E}(Y) \le 4$ ***A1***

[3 marks]

continued…

Question 10 continued

(e) **METHOD 1**

evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

solving for either q or r ***M1***

$\frac{6}{7}(q+1-3q)=\frac{1}{2}$ OR $\frac{6}{7}\left(\frac{1-r}{3}+r\right)=\frac{1}{2}$ OR $3pq+3p(1-3q)=\frac{1}{2}$

OR $3p\left(\frac{1-r}{3}\right)+3pr=\frac{1}{2}$

EITHER two correct values

$q=\frac{5}{24}$ and $r=\frac{3}{8}$ ***A1A1***

OR one correct value

$q=\frac{5}{24}$ OR $r=\frac{3}{8}$ ***A1***

substituting their value for q or r ***A1***

$4-6\left(\frac{5}{24}\right)$ OR $2+2\left(\frac{3}{8}\right)$

THEN

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]

continued…

Question 10 continued

METHOD 2 (solving for $\mathrm{E}(Y)$)
evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

rearranging to make q the subject ***M1***

$q=\frac{4-\mathrm{E}(Y)}{6}$

$3pq+3p(1-3q)=\frac{1}{2}$ ***M1***

$\frac{6}{7}\times\left(\frac{4-\mathrm{E}(Y)}{6}\right)+\frac{6}{7}\left(1-3\left(\frac{4-\mathrm{E}(Y)}{6}\right)\right)=\frac{1}{2}$ ***A1***

$\frac{2(\mathrm{E}(Y)-1)}{7}=\frac{1}{2}$

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]
Total [16 marks]

11. (a) (i) $\frac{-1+1}{2}=0=3-3$ ***A1***

the point $(-1,0,3)$ lies on L_1. ***AG***

(ii) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$\frac{x+1}{2}=y=3-z=\lambda \Rightarrow x=2\lambda-1,\ y=\lambda,\ z=3-\lambda$ OR $\frac{x+1}{2}=\frac{y-0}{1}=\frac{z-3}{-1}$

correct direction vector $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ or equivalent seen in vector form ***(A1)***

$\boldsymbol{r}=\begin{pmatrix}-1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $\boldsymbol{r}=$ is omitted.

[4 marks]

(b) attempt to use the scalar product formula ***(M1)***

$\begin{pmatrix}2\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}a\\1\\-1\end{pmatrix}=(\pm)\sqrt{6}\sqrt{a^2+2}\cos 45^\circ$ ***(A1)(A1)***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$2a+2=\frac{(\pm)\sqrt{6}\sqrt{a^2+2}\sqrt{2}}{2}\left(\Rightarrow 2a+2=(\pm)\sqrt{3}\sqrt{a^2+2}\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$4a^2+8a+4=3a^2+6$

$a^2+8a-2=0$ ***A1***

attempt to solve their quadratic ***M1***

$a=\frac{-8\pm\sqrt{64+8}}{2}=\frac{-8\pm\sqrt{72}}{2}\left(=-4\pm3\sqrt{2}\right)$ ***A1***

[8 marks]

continued…

Question 11 continued

(c) **METHOD 1**

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

Solutions exist unless $a - 2 = 0$

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

$t = \dfrac{1}{a-2}$ or $\lambda = \dfrac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\dfrac{a}{a-2}, 1 + \dfrac{1}{a-2}, 2 - \dfrac{1}{a-2}\right)\left(= \left(\dfrac{a}{a-2}, \dfrac{a-1}{a-2}, \dfrac{2a-5}{a-2}\right)\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

METHOD 2

no unique point of intersection implies direction vectors of L_1 and L_2 parallel

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

$t = \dfrac{1}{a-2}$ or $\lambda = \dfrac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\dfrac{a}{a-2}, 1 + \dfrac{1}{a-2}, 2 - \dfrac{1}{a-2}\right)\left(= \left(\dfrac{a}{a-2}, \dfrac{a-1}{a-2}, \dfrac{2a-5}{a-2}\right)\right)$ ***A2***

continued…

Question 11 continued

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

[7 marks]
Total [19 marks]

12. (a) attempt to use the chain rule ***M1***

$f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$ ***A1***

$f''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$ ***A1***

$=-\frac{1}{4\sqrt{(1+x)^3}}$ ***AG***

Note: Award ***M1A0A0*** for $f'(x)=\frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) let $n=2$

$f''(x)=\left(-\frac{1}{4\sqrt{(1+x)^3}}=\right)\left(-\frac{1}{4}\right)^1\frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$ ***R1***

Note: Award ***R0*** for not starting at $n=2$. Award subsequent marks as appropriate.

assume true for $n=k$, (so $f^{(k)}(x)=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

consider $n=k+1$

$\text{LHS}=f^{(k+1)}(x)=\frac{d\left(f^{(k)}(x)\right)}{dx}$ ***M1***

$=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

EITHER

$\text{RHS}=f^{(k+1)}(x)=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

$=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

continued…

Question 12 continued

Note: Award ***A1*** for $\frac{(2k-1)!}{(k-1)!}=\frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}$ $\left(=\frac{2(2k-1)(2k-3)!}{(k-2)!}\right)$

$$=\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

$$\left(=\left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}\right)$$

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$$=\left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

OR

Note: The following ***A*** marks can be awarded in any order.

$$=\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-k-1}$$

$$=\left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for isolating $(2k-1)$ correctly.

$$=\left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-1)!}{(2k-2)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for multiplying top and bottom by $(k-1)$ or $2(k-1)$.

$$=\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1}\frac{(2k-1)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$$=\left(-\frac{1}{4}\right)^{k}\frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$$ ***A1***

continued…

Question 12 continued

$$=\left(-\frac{1}{4}\right)^{(k+1)-1}\frac{\left(2(k+1)-3\right)!}{((k+1)-2)!}(1+x)^{\frac{1}{2}-(k+1)}=\text{RHS}$$

THEN

since true for $n=2$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n\in\mathbb{Z}, n\geq 2$ by mathematical induction ***R1***

> **Note:** To obtain the final ***R1,*** at least four of the previous marks must have been awarded.

[9 marks]

(c) **METHOD 1**

$h(x)=\sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h'(x)$ ***(M1)***

$h'(x)=\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$ ***A1***

$h''(x)=m\left(\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right)+\frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx}-\frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$ ***A1***

substituting $x=0$ into $h''(x)$ ***M1***

$h''(0)=m^2+\frac{1}{2}m+\frac{1}{2}m-\frac{1}{4}\left(=m^2+m-\frac{1}{4}\right)$ ***A1***

$h(x)=h(0)+xh'(0)+\frac{x^2}{2!}h''(0)+...$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!}=\frac{7}{4}\quad\left(\Rightarrow h''(0)=\frac{7}{2}\right)$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 2

EITHER

attempt to find $f(0)$, $f'(0)$, $f''(0)$ ***(M1)***

$f(x) = (1+x)^{\frac{1}{2}}$ $f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $f''(0) = -\frac{1}{4}$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$$f(x) = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \ldots$$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

THEN

$g(x) = 1 + mx + \frac{m^2}{2}x^2 + \ldots$ ***(A1)***

$h(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + mx + \frac{m^2}{2}x^2 + \ldots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8} = \frac{7}{4}$

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 3

$g'(x) = me^{mx}$ and $g''(x) = m^2e^{mx}$ ***(A1)***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + \ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4}$ $\left(\Rightarrow h''(0) = \frac{7}{2}\right)$

using product rule to find $h'(x)$ and $h''(x)$ ***(M1)***

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h''(x) = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = f(0)g''(0) + 2g'(0)f'(0) + g(0)f''(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right)$ $\left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

Total [20 marks]