Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

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Mathematics:
analysis and approaches

Higher level

Paper 2

28 pages

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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg **M1, A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) (i) $a = 0.805084...$ and $b = 2.88135...$
$a = 0.805$ and $b = 2.88$ ***A1A1***

(ii) $r = 0.97777...$
$r = 0.978$ ***A1***

[3 marks]

(b) a represents the (average) increase in waiting time (0.805 mins) per additional customer (waiting to receive their coffee) ***R1***

[1 mark]

(c) attempt to substitute $x = 7$ into their equation (***M1)***

$8.51693...$
8.52 (mins) ***A1***

[2 marks]
Total [6 marks]

2. (a) attempt to use $u_1 + (n-1)d = 0$ ***(M1)***

$60 - 2.5(k-1) = 0$

$k = 25$ ***A1***

[2 marks]

(b) **METHOD 1**

attempting to express S_n in terms of n ***(M1)***

use of a graph or a table to attempt to find the maximum sum ***(M1)***

$= 750$ ***A1***

METHOD 2

EITHER

recognizing maximum occurs at $n = 25$ ***(M1)***

$S_{25} = \frac{25}{2}(60+0),\ S_{25} = \frac{25}{2}(2\times 60 + 24 \times -2.5)$ ***(A1)***

OR

attempting to calculate S_{24} ***(M1)***

$S_{24} = \frac{24}{2}(2\times 60 + 23 \times -2.5)$ ***(A1)***

THEN

$= 750$ ***A1***

[3 marks]

Total [5 marks]

3. (a) **EITHER**

$P(S)+P(T)+P(S'\cap T')-P(S\cap T)=1$ OR $P(S\cup T)=P((S'\cap T')')$ ***(M1)***

$0.7+0.2+0.18-P(S\cap T)=1$ ***OR*** $P(S\cup T)=1-0.18$

OR

a clearly labelled Venn diagram ***(M1)***

THEN

$P(S\cap T)=0.08$ (accept 8%) ***A1***

Note: To obtain the ***M1*** for the Venn diagram all labels must be correct and in the correct sections. For example, do not accept 0.7 in the area corresponding to $S\cap T'$.

[2 marks]

(b) **EITHER**

$P(T\cap S')=P(T)-P(T\cap S)(=0.2-0.08)$ OR

$P(T\cap S')=P(T\cup S)-P(S)(=0.82-0.7)$ ***(M1)***

OR

a clearly labelled Venn diagram including $P(S)$, $P(T)$ and $P(S\cap T)$ ***(M1)***

THEN

$=0.12$ (accept 12%) ***A1***

[2 marks]

(c) $P(G\cap T)=P(T|G)P(G)$ (0.25×0.48) ***(M1)***

$=0.12$ ***A1***

[2 marks]

continued…

Question 3 continued

(d) **METHOD 1**

$\mathrm{P}(G)\times\mathrm{P}(T)\left(=0.48\times0.2\right)=0.096$ ***A1***

$\mathrm{P}(G)\times\mathrm{P}(T)\neq\mathrm{P}(G\cap T)\Rightarrow$ G and T are not independent ***R1***

METHOD 2

$\mathrm{P}(T\mid G)=0.25$ ***A1***

$\mathrm{P}(T\mid G)\neq\mathrm{P}(T)\Rightarrow$ G and T are not independent ***R1***

Note: Do not award ***A0R1.***

[2 marks]
Total [8 marks]

4. (a) attempting to find the vertex ***(M1)***

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ ***A1***

[2 marks]

(b) **METHOD 1**

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \ \left(= -\left(6(x-1)^2 - 5\right) + c\right)$ ***(A1)***

EITHER

relating to the range of f OR attempting to find $g(-5)$ ***(M1)***

$5 + c \leq 0$ ***(A1)***

OR

attempting to find the discriminant of $(g \circ f)(x)$ ***(M1)***

$144 + 24(c-1) \leq 0 \ (120 + 24c \leq 0)$ ***(A1)***

THEN

$c \leq -5$ ***A1***

[4 marks]

METHOD 2

vertical reflection followed by vertical shift ***(M1)***

new vertex is $(1, 5+c)$ ***(A1)***

$5 + c \leq 0$ ***(A1)***

$c \leq -5$ ***A1***

[4 marks]

Total [6 marks]

5. (a) $100 = A_0 e^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

(b) correct substitution of values into exponential equation ***(M1)***

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln\frac{1}{2}$ ***A1***

$\ln\frac{1}{2} = -\ln 2$ OR $-\ln\frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if 25% of the carbon-14 has decayed, 75% remains ie, 75 units remain

$75 = 100e^{-\frac{\ln 2}{5730}t}$ ***(A1)***

EITHER

using an appropriate graph to attempt to solve for t ***(M1)***

OR

manipulating logs to attempt to solve for t ***(M1)***

$\ln 0.75 = -\frac{\ln 2}{5730}t$

$t = 2378.164...$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]

Total [7 marks]

6. (a) $\mathrm{E}(X) = (n+1)\int_0^1 x^{n+1}\mathrm{d}x$ ***M1***

$= (n+1)\left[\frac{x^{n+2}}{n+2}\right]_0^1$ ***A1***

leading to $\mathrm{E}(X) = \frac{n+1}{n+2}$ ***AG***

[2 marks]

(b) **METHOD 1**

use of $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ ***M1***

$$\mathrm{Var}(X) = (n+1)\int_0^1 x^{n+2}\mathrm{d}x - \left(\frac{n+1}{n+2}\right)^2$$

$$= (n+1)\left[\frac{1}{n+3}x^{n+3}\right]_0^1 - \left(\frac{n+1}{n+2}\right)^2$$

$= \frac{n+1}{n+3} - \left(\frac{n+1}{n+2}\right)^2$ ***A1***

$= \frac{(n+1)(n+2)^2 - (n+1)^2(n+3)}{(n+2)^2(n+3)}$ ***M1***

EITHER

$= \frac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$ ***A1***

OR

$= \frac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$ ***A1***

THEN

so $\mathrm{Var}(X) = \frac{n+1}{(n+2)^2(n+3)}$ ***AG***

continued…

Question 6 continued

METHOD 2

use of $\text{Var}(X) = \text{E}\left(X - \text{E}(X)\right)^2$ ***M1***

$$\text{Var}(X) = (n+1)\int_0^1 \left(x - \frac{n+1}{n+2}\right)^2 x^n \text{d}x$$

$$= (n+1)\left[\frac{1}{n+3}x^{n+3} - \frac{2(n+1)}{(n+2)^2}x^{n+2} + \frac{n+1}{(n+2)^2}x^{n+1}\right]_0^1$$

$$= \frac{n+1}{n+3} - \left(\frac{n+1}{n+2}\right)^2$$ ***A1***

$$= \frac{(n+1)\left((n+2)^2 - (n+1)(n+3)\right)}{(n+2)^2(n+3)}$$ ***M1***

EITHER

$$= \frac{(n+1)\left(n^2 + 4n + 4 - \left(n^2 + 4n + 3\right)\right)}{(n+2)^2(n+3)}$$ ***A1***

OR

$$= \frac{\left(n^3 + 5n^2 + 8n + 4\right) - \left(n^3 + 5n^2 + 7n + 3\right)}{(n+2)^2(n+3)}$$ ***A1***

THEN

so $\text{Var}(X) = \dfrac{n+1}{(n+2)^2(n+3)}$ ***AG***

[4 marks]
Total [6 marks]

7. (a) Jack and Andrea finish in that order (as a unit) so we are considering the arrangement of 7 objects ***(M1)***

$7!\ (=5040)$ ways ***A1***

[2 marks]

(b) **METHOD 1**

the number of ways that Andrea finishes in front of Jack is equal to the number of ways that Jack finishes in front of Andrea ***(M1)***

total number of ways is $8!$ ***(A1)***

$\frac{8!}{2}\ (=20160)$ ways ***A1***

[3 marks]

METHOD 2

the other six runners can finish in $6!\ (=720)$ ways ***(A1)***

when Andrea finishes first, Jack can finish in 7 different positions

when Andrea finishes second, Jack can finish in 6 different positions etc

$7+6+5+4+3+2+1\ (=28)$ ways ***(A1)***

hence there are $(7+6+5+4+3+2+1)\times 6!$ ways

$28\times 6!\ (=20160)$ ways ***A1***

[3 marks]
Total [5 marks]

8. $\frac{1+z}{1-z}=\frac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta}$

attempt to use the complex conjugate of their denominator ***M1***

$=\frac{(1+\cos\theta+\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}{(1-\cos\theta-\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}$ ***A1***

$\mathrm{Re}\left(\frac{1+z}{1-z}\right)=\frac{1-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta}\ \left(=\frac{1-\cos^2\theta-\sin^2\theta}{2-2\cos\theta}\right)$ ***M1A1***

Note: Award ***M1*** for expanding the numerator and ***A1*** for a correct numerator. Condone either an incorrect denominator or the absence of a denominator.

using $\cos^2\theta+\sin^2\theta=1$ to simplify the numerator ***(M1)***

$\mathrm{Re}\left(\frac{1+z}{1-z}\right)=0$ ***AG***

[5 marks]

9. (a) $1-t+t^2$ ***A1***

Note: Accept $1, -t$ and t^2.

[1 mark]

(b) $\sec x = \dfrac{1}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-\ldots)}$ $\left(=\left(1-\dfrac{x^2}{2!}+\left(\dfrac{x^4}{4!}(-\ldots)\right)\right)^{-1}\right)$ ***(M1)***

$t=\cos x-1$ or $\sec x = 1-(\cos x-1)+(\cos x-1)^2$ ***(M1)***

$=1-\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-\ldots)\right)+\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-\ldots)\right)^2$ ***A1***

$=1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}$ ***A1***

so the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}$ ***AG***

[4 marks]

Note: Condone the absence of '…'

(c) $\arctan 2x = 2x-\dfrac{(2x)^3}{3}+\ldots$

$\lim\limits_{x\to 0}\left(\dfrac{x\arctan 2x}{\sec x-1}\right)=\lim\limits_{x\to 0}\left(\dfrac{x\left(2x-\dfrac{(2x)^3}{3}+\ldots\right)}{\left(1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}\right)-1}\right)$ ***M1***

$=\lim\limits_{x\to 0}\left(\dfrac{2x^2-\dfrac{8x^4}{3}+\ldots}{\dfrac{x^2}{2}+\dfrac{5x^4}{24}}\right)$ ***A1***

$=\lim\limits_{x\to 0}\left(\dfrac{2x^2\left(1-\dfrac{4x^2}{3}\right)}{\dfrac{x^2}{2}\left(1+\dfrac{5x^2}{12}\right)}\right)$

$=4$ ***A1***

Note: Condone missing 'lim' and errors in higher derivatives.
Do not award ***M1*** unless *x* is replaced by 2*x* in arctan.

[3 marks]
Total [8 marks]

Section B

10. (a) use of inverse normal to find z-score ***(M1)***

$z = 2.0537...$

$2.0537... = \frac{82 - 75}{\sigma}$ ***(A1)***

$\sigma = 3.408401...$

$\sigma = 3.41$ ***A1***

[3 marks]

(b) evidence of identifying the correct area under the normal curve ***(M1)***

$\mathrm{P}(T > 80) = 0.071193...$

$\mathrm{P}(T > 80) = 0.0712$ ***A1***

[2 marks]

(c) recognition that $\mathrm{P}(80 < T < 82)$ is required ***(M1)***

$P(T < 82 | T > 80) = \frac{\mathrm{P}(80 < T < 82)}{\mathrm{P}(T > 80)} = \left(\frac{0.051193...}{0.071193...}\right)$ ***(M1)(A1)***

$= 0.719075...$

$= 0.719$ ***A1***

[4 marks]

continued…

Question 10 continued

(d) recognition of binomial probability ***(M1)***

$X \sim \mathrm{B}(64, 0.071193...)$ or $\mathrm{E}(X) = 64 \times 0.071193...$ ***(A1)***

$\mathrm{E}(X) = 4.556353...$

$\mathrm{E}(X) = 4.56$ (flights) ***A1***

[3 marks]

(e) $\mathrm{P}(X > 6) = P(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 1 - 0.83088...$ ***(A1)***

$= 0.1691196...$

$= 0.169$ ***A1***

[3 marks]

Total [15 marks]

11. (a) attempt to use $V = \pi\int_a^b (f(x))^2\,dx$ ***(M1)***

$V = \pi\int_0^{\ln 16}\left(\frac{ke^{\frac{x}{2}}}{1+e^x}\right)^2 dx \quad \left(V = k^2\pi\int_0^{\ln 16}\frac{e^x}{(1+e^x)^2}dx\right)$

EITHER

applying integration by recognition ***(M1)***

$= k^2\pi\left[-\frac{1}{1+e^x}\right]_0^{\ln 16}$ ***A3***

OR

$u = 1+e^x \Rightarrow du = e^x dx$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 2$ and when $x = \ln 16$, $u = 17$

$V = k^2\pi\int_2^{17}\frac{1}{u^2}du$ ***(A1)***

$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$ ***A1***

OR

$u = e^x \Rightarrow du = e^x dx$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 1$ and when $x = \ln 16$, $u = 16$

$V = k^2\pi\int_1^{16}\frac{1}{(1+u)^2}du$ ***(A1)***

$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

$= k^2\pi\left(\frac{1}{2}-\frac{1}{17}\right)$ ***A1***

so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

continued…

Question 11 continued

(b) a valid algebraic or graphical attempt to find k ***(M1)***

$k^2 = \frac{300 \times 34}{15\pi}$

$k = 14.7 \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ (as $k \in \mathbb{R}^+$) ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

(c) (i) attempting to find $\text{OA} = f(0) = \frac{k}{2}$

with $k = 14.712... \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{OA} = 7.36 \left(= \sqrt{\frac{170}{\pi}} \right)$ ***A1***

(ii) attempting to find $\text{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712... \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{BC} = 3.46 \left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}} \right)$ ***A1***

[4 marks]

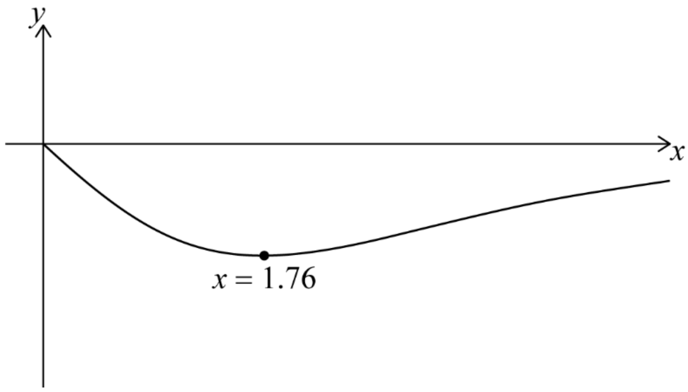
continued…

Question 11 continued

(d) (i) **EITHER**

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation. $f'(x) = \dfrac{k\mathrm{e}^{\frac{x}{2}}\left(1-\mathrm{e}^{x}\right)}{2\left(1+\mathrm{e}^{x}\right)^{2}}$



for $x > 0$ graph decreasing to the local minimum ***A1***

before increasing towards the x-axis ***A1***

continued…

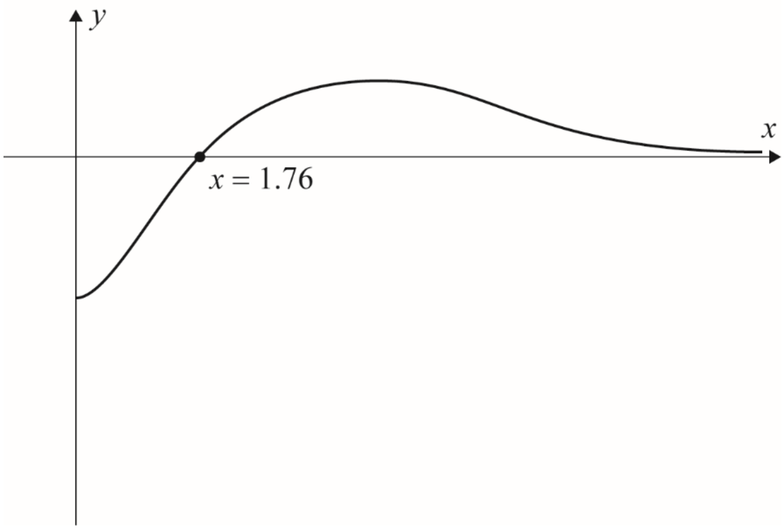
Question 11 continued

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f''(x) = \frac{k\mathrm{e}^{\frac{x}{2}}\left(\mathrm{e}^{2x} - 6\mathrm{e}^{x} + 1\right)}{4\left(1+\mathrm{e}^{x}\right)^{3}}$$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76 \left(= \ln\left(2\sqrt{2}+3\right)\right)$ ***A1***

Note: Only award ***A*** marks if either graph is seen.

(ii) attempting to find $f(1.76...)$ ***(M1)***

the cross-sectional radius at this point is $5.20 \left(\sqrt{\frac{85}{\pi}}\right)$ (cm) ***A1***

[6 marks]
Total [18 marks]

12. (a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

continued…

Question 12 continued

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\quad\left(=\frac{4x}{\left(x^2+1\right)^2}\right)$$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$$ ***M1***

$$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$$ ***A1***

$$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

Question 12 continued

(d) $x = \arcsin\left(\frac{y^2-1}{y^2+1}\right)$ ***M1***

$\sin x = \frac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ ***A1***

$y^2 = \frac{1+\sin x}{1-\sin x}$ ***A1***

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) ***R1***

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\frac{1+\sin x}{1-\sin x}}$ ***A1***

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

(e) domain is $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ ***A1***

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

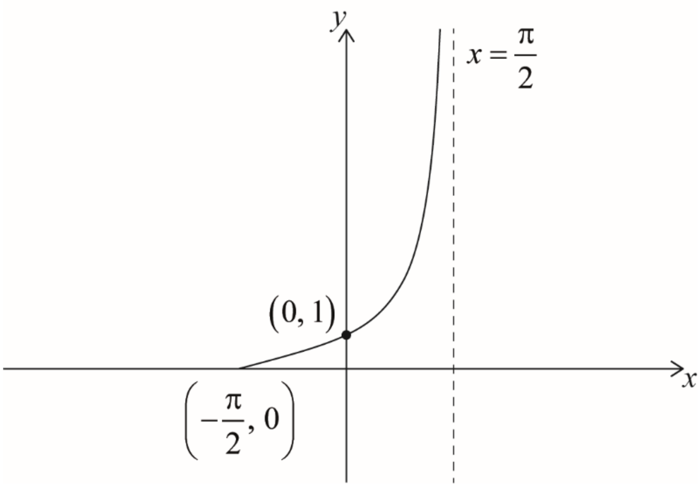
Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

continued…

Question 12 continued

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$

A1 for asymptotic behaviour $x \to \frac{\pi}{2}$

A1 for $x = \frac{\pi}{2}$

Coordinates are not required.

Do not accept $x = 1.57$ or other inexact values.

[3 marks]
Total [21 marks]