Related rates *[20 marks]*

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

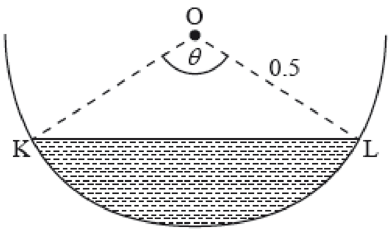


diagram not to scale

1a. Find an expression for the volume of water $V\ (\mathrm{m}^3)$ in the trough in terms of θ. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ ***M1A1***

$V = \text{area of segment} \times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ ***A1***

[3 marks]

The volume of water is increasing at a constant rate of $0.0008\mathrm{m}^3\mathrm{s}^{-1}$.

1b. Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. *[4 marks]*

Markscheme

METHOD 1

$\frac{dV}{dt} = \frac{5}{4}(1 - \cos\theta)\frac{d\theta}{dt}$ ***M1A1***

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{d\theta}{dt}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128\ (\text{rad}\ s^{-1})$ ***A1***

METHOD 2

$\frac{d\theta}{dt} = \frac{d\theta}{dV} \times \frac{dV}{dt}$ ***(M1)***

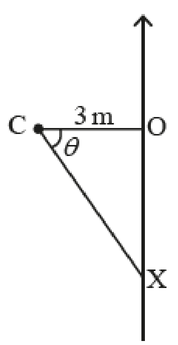
$\frac{dV}{d\theta} = \frac{5}{4}(1 - \cos\theta)$ ***A1***

$\frac{d\theta}{dt} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ ***(M1)***

$\frac{d\theta}{dt} = 0.00128\left(\frac{4}{3125}\right)(\text{rad}\ s^{-1})$ ***A1***

[4 marks]

2. A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at t = 0. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera. *[6 marks]*



A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let O$\hat{C}$X = θ.

Find $\frac{d\theta}{dt}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

let OX = x

METHOD 1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$3\tan\theta = x$ ***A1***

EITHER

$3\sec^2\theta = \frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{x}{3}\right)$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{3} \times \frac{1}{1+\frac{x^2}{9}}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 24 \times \frac{1}{3\left(1+\frac{x^2}{9}\right)}$

attempt to substitute for $x = 0$ into their differential equation ***M1***

THEN

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept −8 rad s^{-1}.

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept −8 rad s^{-1}.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d - 24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3}\left(= \frac{d}{3} - 8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{d}{3} - 8t\right)$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{8}{1+\left(\frac{d}{3}-8t\right)^2}$ ***A1***

at O, $t = \frac{d}{24}$ ***A1***

THEN

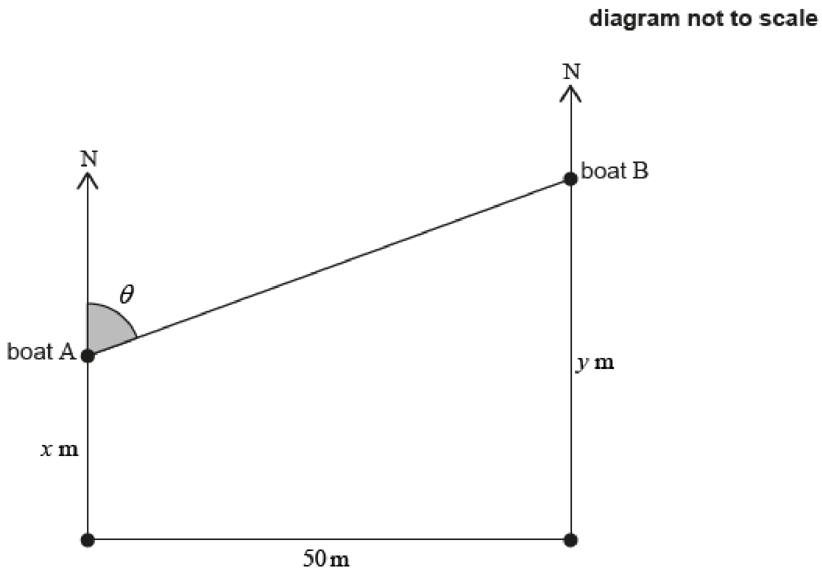
$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

[6 marks]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.



3a. Show that $y = x + 50 \cot \theta$. *[1 mark]*

Markscheme

$\tan \theta = \frac{50}{y-x}$ OR $\cot \theta = \frac{y-x}{50}$ ***A1***

$y = x + 50 \cot \theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

3b. At time T, the following conditions are true. *[6 marks]*

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

Find the speed of boat A at time T.

Markscheme

attempt to differentiate with respect to t ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

attempt to set speed of B equal to double the speed of A ***(M1)***

$2\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50(\operatorname{cosec}\theta)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots^\circ)$ OR

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{1}{5}\right)^2 = \frac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\frac{26}{25}\right) \times -0.1$

So the speed of boat A is $5.2\left(\mathrm{ms}^{-1}\right)$ ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]